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An Embedding Theorem for Medial n-ary Groupoid with Cancellation

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Abstract. In 1949 M. Sholander proved that every medial cancellation groupoid can be embedded into a medial quasigroup. In this paper we prove that it is also true for n-ary cancellation groupoids, namely, every medial cancellation n-ary groupoid can be embedded into a medial n-ary quasigroup.

Key Words: n-ary groupoid, cancellation n-ary groupoid, medial n-ary groupoid, n-ary quasigroup
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Introduction

We use notations and preliminary results from the works [1, 3] (see also [2]).

The sequence $x_n, x_{n+1}, \ldots, x_m$ is denoted by x_n^m, where n, m are natural numbers, $n \leqslant m$. If $n = m$, then x_n^m is the element x_n. The sequence $a, a, \ldots, a$ (m times) is denoted by $\overset{m}{a}$. The n-ary operation on the set Q is denoted by (a_1^n). A nonempty set Q with an n-ary operation is called n-ary groupoid or shortly n-groupoid.

Let $Q(\ )$ be an n-groupoid. Denote by $\bar{a}$ a sequece $a_1^n \in Q^n$. Let $L_i(\bar{a})$, $i = 1, ..., n$, be the map from Q to Q defined for all $x \in Q$ as

$$L_i(\bar{a})x = (a_1 \cdots a_{i-1} x a_{i+1} \cdots a_n) = \left(a_1^{i-1} x a_{i+1}^n\right).$$

The map $L_i(\bar{a})$ is called i-translation with respect to $\bar{a}$.

An n-groupoid $Q(\ )$ is called cancellation n-groupoid if every $L_i(\bar{a})$ is an injection for all $\bar{a} \in Q^n$ and $i = 1, \ldots, n$. An n-groupoid $Q(\ )$ is an n-ary i-quasigroup if $L_i(\bar{a})$ is a bijection for all $\bar{a} \in Q^n$. An n-groupoid $Q(\ )$ is an n-ary quasigroup if it is an n-ary i-quasigroup for every $i = 1, \ldots, n$.

The n-groupoid $Q(\ )$ is said to be medial, if it satisfies the identity of mediality:

$$((x_{11} \cdots x_{1n}) \cdots (x_{n1} \cdots x_{nn})) = ((x_{11} \cdots x_{n1}) \cdots (x_{1n} \cdots x_{nn})). \quad (1)$$

It should be noted that the medial identity are called by various names: abelian, alternation, bi-commutative, bisymmetric, entropic, surcommutative.

In [6] M. Sholander showed that every medial cancellation groupoid can be embedded into a medial quasigroup. J. Ježek and T. Kepka in [4, 5] presented a description of the equational theory of the class of cancellation medial groupoids. Based on it, they provided a new proof of the Sholander's theorem. In this paper we prove that every medial cancellation n-ary groupoid can be embedded into a medial n-ary quasigroup.

1 Preliminary Results

Let $Q(\ )$ be an n-groupoid and n-groupoid $Q_1(\ )$ be the n^{th} direct product of $Q(\ )$, where $Q_1 = Q^n = \underbrace{Q\times\cdots\times Q}_{n}$. Define the binary relations α_i $(i = 1,\dots,n)$ on the set Q_1 by the following way: $\langle a_1^n, b_1^n\rangle \in \alpha_i$ if

$$\Big(\big(\overset{i-1}{a_1}\, a_1\, \overset{n-i}{a_1}\big)\cdots\big(\overset{i-1}{a_1}\, a_{i-1}\, \overset{n-i}{a_1}\big)\underbrace{b_n}_{i}\big(\overset{i-1}{a_1}\, a_i\, \overset{n-i}{a_1}\big)\cdots\big(\overset{i-1}{a_1}\, a_{n-1}\, \overset{n-i}{a_1}\big)\Big) =$$
$$\Big(\big(\overset{i-1}{a_1}\, b_1\, \overset{n-i}{a_1}\big)\cdots\big(\overset{i-1}{a_1}\, b_{i-1}\, \overset{n-i}{a_1}\big)\underbrace{a_n}_{i}\big(\overset{i-1}{a_1}\, b_i\, \overset{n-i}{a_1}\big)\cdots\big(\overset{i-1}{a_1}\, b_{n-1}\, \overset{n-i}{a_1}\big)\Big). \quad (2)$$

Lemma 1 *If $Q(\ )$ is a cancellation medial n-groupoid, then $\langle a_1^n, b_1^n\rangle \in \alpha_i$ if and only if*

$$\Big(\big(x_1^{i-1} a_1 x_{i+1}^n\big)\cdots\big(y_1^{i-1} a_{i-1} y_{i+1}^n\big)\underbrace{b_n}_{i}\big(z_1^{i-1} a_i z_{i+1}^n\big)\cdots\big(w_1^{i-1} a_{n-1} w_{i+1}^n\big)\Big) =$$
$$\Big(\big(x_1^{i-1} b_1 x_{i+1}^n\big)\cdots\big(y_1^{i-1} b_{i-1} y_{i+1}^n\big)\underbrace{a_n}_{i}\big(z_1^{i-1} b_i z_{i+1}^n\big)\cdots\big(w_1^{i-1} b_{n-1} w_{i+1}^n\big)\Big), \quad (3)$$

for all $x_1^n,\dots,y_1^n, z_1^n,\dots,w_1^n \in Q^n$.

Proof. If we have (2), then

$$\Big(\Big(\big(\overset{n}{a_1}\big)\cdots\overbrace{a_1}^{i}\cdots\big(\overset{n}{a_1}\big)\Big)\Big(\big(\overset{n}{a_1}\big)\cdots\overbrace{a_1}^{i}\cdots\big(\overset{n}{a_1}\big)\Big)\cdots$$
$$\cdots\underbrace{\Big(\big(\overset{i-1}{x_1}\, b_1 x_{i+1}^n\big)\big(\overset{i-1}{y_1}\, b_2 y_{i+1}^n\big)\cdots\overbrace{a_n}^{i}\cdots\big(\overset{i-1}{z_1}\, b_{n-1} z_{i+1}^n\big)\Big)}_{i}\cdots\Big(\big(\overset{n}{a_1}\big)\cdots\overbrace{a_1}^{i}\cdots\big(\overset{n}{a_1}\big)\Big)\Big) \overset{(1)}{=}$$
$$\overset{(1)}{=}\Big(\Big(\big(\overset{n}{a_1}\big)\cdots\underbrace{\big(x_1^{i-1} b_1 x_{i+1}^n\big)}_{i}\cdots\big(\overset{n}{a_1}\big)\Big)\Big(\big(\overset{n}{a_1}\big)\cdots\underbrace{\big(y_1^{i-1} b_2 y_{i+1}^n\big)}_{i}\cdots\big(\overset{n}{a_1}\big)\Big)\cdots$$

$$\cdots \underbrace{\left(a_1^{i-1} a_n a_{i+1}^n\right)}_{i} \cdots \left(\left(\overset{n}{a_1}\right) \cdots \underbrace{\left(z_1^{i-1} b_{n-1} z_{i+1}^n\right)}_{i} \cdots \left(\overset{n}{a_1}\right)\right)\right) \overset{(1)}{=}$$

$$\overset{(1)}{=} \left(\left(\left(\overset{i-1}{a_1}\, x_1 \overset{n-i}{a_1}\right) \cdots \underbrace{\left(\overset{i-1}{a_1}\, b_1 \overset{n-i}{a_1}\right)}_{i} \cdots \left(\overset{i-1}{a_1}\, x_n \overset{n-i}{a_1}\right)\right)\left(\left(\overset{i-1}{a_1}\, y_1 \overset{n-i}{a_1}\right) \cdots\right.\right.$$

$$\cdots \underbrace{\left(\overset{i-1}{a_1}\, b_2 \overset{n-i}{a_1}\right)}_{i} \cdots \left(\overset{i-1}{a_1}\, y_n \overset{n-i}{a_1}\right)\right) \cdots$$

$$\left.\left.\cdots \underbrace{\left(\overset{i-1}{a_1}\, a_n \overset{n-i}{a_1}\right)}_{i} \cdots \left(\left(\overset{i-1}{a_1}\, z_1 \overset{n-i}{a_1}\right) \cdots \underbrace{\left(\overset{i-1}{a_1}\, b_{n-1} \overset{n-i}{a_1}\right)}_{i} \cdots \left(\overset{i-1}{a_1}\, z_n \overset{n-i}{a_1}\right)\right)\right)\right) \overset{(1)}{=}$$

$$\overset{(1)}{=} \left(\left(\left(\overset{i-1}{a_1}\, x_1 \overset{n-i}{a_1}\right)\left(\overset{i-1}{a_1}\, y_1 \overset{n-i}{a_1}\right) \cdots \overbrace{a_1}^{i} \cdots \left(\overset{i-1}{a_1}\, z_1 \overset{n-i}{a_1}\right)\right) \cdots\right.$$

$$\cdots \underbrace{\left(\left(\overset{i-1}{a_1}\, b_1 \overset{n-i}{a_1}\right)\left(\overset{i-1}{a_1}\, b_2 \overset{n-i}{a_1}\right) \cdots \overbrace{a_n}^{i} \cdots \left(\overset{i-1}{a_1}\, b_{n-1} \overset{n-i}{a_1}\right)\right)}_{i} \cdots$$

$$\left.\cdots \left(\left(\overset{i-1}{a_1}\, x_n \overset{n-i}{a_1}\right)\left(\overset{i-1}{a_1}\, y_n \overset{n-i}{a_1}\right) \cdots \overbrace{a_1}^{i} \cdots \left(\overset{i-1}{a_1}\, z_n \overset{n-i}{a_1}\right)\right)\right).$$

By the same calculations we get

$$\left(\left(\left(\overset{i-1}{a_1}\, x_1 \overset{n-i}{a_1}\right)\left(\overset{i-1}{a_1}\, y_1 \overset{n-i}{a_1}\right) \cdots\cdots \overbrace{a_1}^{i} \cdots \left(\overset{i-1}{a_1}\, z_1 \overset{n-i}{a_1}\right)\right) \cdots\right.$$

$$\cdots \underbrace{\left(\left(\overset{i-1}{a_1}\, a_1 \overset{n-i}{a_1}\right)\left(\overset{i-1}{a_1}\, a_2 \overset{n-i}{a_1}\right) \cdots \overbrace{b_n}^{i} \cdots \left(\overset{i-1}{a_1}\, a_{n-1} \overset{n-i}{a_1}\right)\right)}_{i} \cdots$$

$$\left.\cdots \left(\left(\overset{i-1}{a_1}\, x_n \overset{n-i}{a_1}\right)\left(\overset{i-1}{a_1}\, y_n \overset{n-i}{a_1}\right) \cdots \overbrace{a_1}^{i} \cdots \left(\overset{i-1}{a_1}\, z_n \overset{n-i}{a_1}\right)\right)\right) =$$

$$\left(\left(\left(\overset{i-1}{a_1}\, a_1 \overset{n-i}{a_1}\right) \cdots \overbrace{a_1}^{i} \cdots \left(\overset{i-1}{a_1}\, a_1 \overset{n-i}{a_1}\right)\right)\left(\left(\overset{i-1}{a_1}\, a_1 \overset{n-i}{a_1}\right) \cdots \overbrace{a_1}^{i} \cdots \left(\overset{i-1}{a_1}\, a_1 \overset{n-i}{a_1}\right)\right) \cdots\right.$$

$$\cdots \underbrace{\left(\left(x_1^{i-1} a_1 x_{i+1}^n\right)\left(y_1^{i-1} a_2 y_{i+1}^n\right) \cdots \overbrace{b_n}^{i} \cdots \left(z_1^{i-1} a_{n-1} z_{i+1}^n\right)\right)}_{i} \cdots$$

$$\left.\cdots \left(\left(\overset{i-1}{a_1}\, a_1 \overset{n-i}{a_1}\right) \cdots \overbrace{a_1}^{i} \cdots \left(\overset{i-1}{a_1}\, a_1 \overset{n-i}{a_1}\right)\right)\right).$$

Because of (2), both terms are equal and if we use the cancellation law, we get (3). Conversely, (2) follows straigtforward from (3). □

Lemma 2 *If $Q(\ )$ is a cancellation medial n-groupoid, then α_i is an equivalence relation on Q_1 for any $i = 1, \ldots, n$.*

Proof. The proof is a simple consequence of Lemma 1. □

Lemma 3 *If $Q(\ )$ is a cancellation medial n-groupoid, then α_k is a congruence of $Q_1(\ )$ for any $k = 1, \ldots, n$.*

Proof. Let $\langle\left(a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}\right), \left(b_1^{(i)}, b_2^{(i)}, \ldots, b_n^{(i)}\right)\rangle \in \alpha_k$, i.e.

$$\Big(\big(a_1^{(i)}\cdots\overbrace{a_1^{(i)}}^{k}\cdots a_1^{(i)}\big)\big(a_1^{(i)}\cdots\overbrace{a_2^{(i)}}^{k}\cdots a_1^{(i)}\big)\cdots\overbrace{b_n^{(i)}}^{k}\cdots \cdots\big(a_1^{(i)}\cdots\overbrace{a_{n-1}^{(i)}}^{k}\cdots a_1^{(i)}\big)\Big) = \Big(\big(a_1^{(i)}\cdots\overbrace{b_1^{(i)}}^{k}\cdots a_1^{(i)}\big)\big(a_1^{(i)}\cdots\overbrace{b_2^{(i)}}^{k}\cdots a_1^{(i)}\big)\cdots \cdots\overbrace{a_n^{(i)}}^{k}\cdots\big(a_1^{(i)}\cdots\overbrace{b_{n-1}^{(i)}}^{k}\cdots a_1^{(i)}\big)\Big), \quad (4)$$

where $i = 1, 2, \ldots, n$. We need to prove that

$$\langle\Big(\big(a_1^{(1)}, \ldots, a_1^{(n)}\big), \big(a_2^{(1)}, \ldots, a_2^{(n)}\big), \ldots, \big(a_n^{(1)}, \ldots, a_n^{(n)}\big)\Big), \Big(\big(b_1^{(1)}, \ldots, b_1^{(n)}\big), \big(b_2^{(1)}, \ldots, b_2^{(n)}\big), \ldots, \big(b_n^{(1)}, \ldots, b_n^{(n)}\big)\Big)\rangle \in \alpha_k. \quad (5)$$

Indeed,

$$\Big(\Big(\big(a_1^{(1)}\cdots a_1^{(n)}\big)\cdots\overbrace{\big(a_1^{(1)}\cdots a_1^{(n)}\big)}^{k}\cdots\big(a_1^{(1)}\cdots a_1^{(n)}\big)\Big)\Big(\big(a_1^{(1)}\cdots a_1^{(n)}\big)\cdots \cdots\overbrace{\big(a_2^{(1)}\cdots a_2^{(n)}\big)}^{k}\cdots\big(a_1^{(1)}\cdots a_1^{(n)}\big)\Big)\cdots \cdots\underbrace{\big(b_n^{(1)}\cdots b_n^{(n)}\big)}_{k}\cdots\Big(\big(a_1^{(1)}\cdots a_1^{(n)}\big)\cdots\overbrace{\big(a_{n-1}^{(1)}\cdots a_{n-1}^{(n)}\big)}^{k}\cdots\big(a_1^{(1)}\cdots a_1^{(n)}\big)\Big)\Big) \overset{(1)}{=}$$

$$\Big(\Big(\big(a_1^{(1)}\cdots\overbrace{a_1^{(1)}}^{k}\cdots a_1^{(1)}\big)\cdots\underbrace{\big(a_1^{(k)}\cdots\overbrace{a_1^{(k)}}^{k}\cdots a_1^{(k)}\big)}_{k}\cdots\big(a_1^{(n)}\cdots\overbrace{a_1^{(n)}}^{k}\cdots a_1^{(n)}\big)\Big) \Big(\big(a_1^{(1)}\cdots\overbrace{a_2^{(1)}}^{k}\cdots a_1^{(1)}\big)\cdots\underbrace{\big(a_1^{(k)}\cdots\overbrace{a_2^{(k)}}^{k}\cdots a_1^{(k)}\big)}_{k}\cdots\big(a_1^{(n)}\cdots\overbrace{a_2^{(n)}}^{k}\cdots a_1^{(n)}\big)\Big)\cdots$$

$$\cdots \underbrace{\left(b_n^{(1)} \cdots b_n^{(n)}\right)}_{k} \cdots \left(\left(a_1^{(1)} \cdots \overbrace{a_{n-1}^{(1)}}^{k} \cdots a_1^{(1)}\right) \cdots \underbrace{\left(a_1^{(k)} \cdots \overbrace{a_{n-1}^{(k)}}^{k} \cdots a_1^{(k)}\right)}_{k} \cdots\right.$$

$$\left.\cdots \left(a_1^{(n)} \cdots \overbrace{a_{n-1}^{(n)}}^{k} \cdots a_1^{(n)}\right)\right)\Big) \overset{(1)}{=} \Bigg(\bigg(\Big(a_1^{(1)} \cdots \overbrace{a_1^{(1)}}^{k} \cdots a_1^{(1)}\Big)$$

$$\Big(a_1^{(1)} \cdots \overbrace{a_2^{(1)}}^{k} \cdots a_1^{(1)}\Big) \cdots \underbrace{b_n^{(1)}}_{k} \cdots \Big(a_1^{(1)} \cdots \overbrace{a_{n-1}^{(1)}}^{k} \cdots a_1^{(1)}\Big)\bigg)$$

$$\underbrace{\bigg(\Big(a_1^{(2)} \cdots \overbrace{a_1^{(2)}}^{k} \cdots a_1^{(2)}\Big)\Big(a_1^{(2)} \cdots \overbrace{a_2^{(2)}}^{k} \cdots a_1^{(2)}\Big) \cdots \underbrace{b_n^{(2)}}_{k} \cdots \Big(a_1^{(2)} \cdots \overbrace{a_{n-1}^{(2)}}^{k} \cdots a_1^{(2)}\Big)\bigg) \cdots \bigg(\Big(a_1^{(k)} \cdots \overbrace{a_1^{(k)}}^{k} \cdots a_1^{(k)}\Big)\Big(a_1^{(k)} \cdots \overbrace{a_2^{(k)}}^{k} \cdots a_1^{(k)}\Big) \cdots \underbrace{b_n^{(k)}}_{k} \cdots \Big(a_1^{(k)} \cdots \overbrace{a_{n-1}^{(k)}}^{k} \cdots a_1^{(k)}\Big)\bigg)}_{k} \cdots$$

$$\bigg(\Big(a_1^{(n)} \cdots \overbrace{a_1^{(n)}}^{k} \cdots a_1^{(n)}\Big)\Big(a_1^{(n)} \cdots \overbrace{a_2^{(n)}}^{k} \cdots a_1^{(n)}\Big) \cdots \underbrace{b_n^{(n)}}_{k} \cdots \Big(a_1^{(n)} \cdots \overbrace{a_{n-1}^{(n)}}^{k} \cdots a_1^{(n)}\Big)\bigg)\Bigg)$$

By the same calculations we get

$$\Bigg(\bigg(\Big(a_1^{(1)} \cdots \overbrace{b_1^{(1)}}^{k} \cdots a_1^{(1)}\Big)\Big(a_1^{(1)} \cdots \overbrace{b_2^{(1)}}^{k} \cdots a_1^{(1)}\Big) \cdots \underbrace{a_n^{(1)}}_{k} \cdots \Big(a_1^{(1)} \cdots \overbrace{b_{n-1}^{(1)}}^{k} \cdots a_1^{(1)}\Big)\bigg)$$

$$\underbrace{\bigg(\Big(a_1^{(2)} \cdots \overbrace{b_1^{(2)}}^{k} \cdots a_1^{(2)}\Big)\Big(a_1^{(2)} \cdots \overbrace{b_2^{(2)}}^{k} \cdots a_1^{(2)}\Big) \cdots \underbrace{a_n^{(2)}}_{k} \cdots \Big(a_1^{(2)} \cdots \overbrace{b_{n-1}^{(2)}}^{k} \cdots a_1^{(2)}\Big)\bigg) \cdots \bigg(\Big(a_1^{(k)} \cdots \overbrace{b_1^{(k)}}^{k} \cdots a_1^{(k)}\Big)\Big(a_1^{(k)} \cdots \overbrace{b_2^{(k)}}^{k} \cdots a_1^{(k)}\Big) \cdots \underbrace{a_n^{(k)}}_{k} \cdots \Big(a_1^{(k)} \cdots \overbrace{b_{n-1}^{(k)}}^{k} \cdots a_1^{(k)}\Big)\bigg)}_{k} \cdots$$

$$\bigg(\Big(a_1^{(n)} \cdots \overbrace{b_1^{(n)}}^{k} \cdots a_1^{(n)}\Big)\Big(a_1^{(n)} \cdots \overbrace{b_2^{(n)}}^{k} \cdots a_1^{(n)}\Big) \cdots \underbrace{a_n^{(n)}}_{k} \cdots \Big(a_1^{(n)} \cdots \overbrace{b_{n-1}^{(n)}}^{k} \cdots a_1^{(n)}\Big)\bigg)\Bigg) =$$

$$\Bigg(\bigg(\Big(a_1^{(1)} \cdots a_1^{(n)}\Big) \cdots \underbrace{\Big(b_1^{(1)} \cdots b_1^{(n)}\Big)}_{k} \cdots \Big(a_1^{(1)} \cdots a_1^{(n)}\Big)\bigg)\bigg(\Big(a_1^{(1)} \cdots a_1^{(n)}\Big) \cdots$$

$$\cdots \underbrace{\Big(b_2^{(1)} \cdots b_2^{(n)}\Big)}_{k}\Big(a_1^{(1)} \cdots a_1^{(n)}\Big)\bigg) \cdots \underbrace{\Big(a_n^{(1)} \cdots a_n^{(n)}\Big)}_{k} \cdots$$

$$\bigg(\Big(a_1^{(1)} \cdots a_1^{(n)}\Big) \cdots \underbrace{\Big(b_{n-1}^{(1)} \cdots b_{n-1}^{(n)}\Big)}_{k} \cdots \Big(a_1^{(1)} \cdots a_1^{(n)}\Big)\bigg)\Bigg).$$

Taking into account the equality (4) we obtain (5). $\square$

For all $i = 1, \ldots, n$, Q_1^i denotes the factor groupoid of Q_1 with respect to α_i.

Lemma 4 *Let Q_1 be a medial cancellation n-groupoid. Then the factor groupoid $Q_1^i = Q_1/\alpha_i$ is a medial cancellation n-groupoid for any $i = 1, \ldots, n$.*

Proof. It is sufficient to prove that Q_1^i is a cancellation n-groupoid. Consider the following equation in Q_1^i:

$$\Big(\Big[a_1^{(1)}, \ldots, a_n^{(1)}\Big] \cdots \underbrace{\Big[x_1, \ldots, x_n\Big]}_{j} \cdots \Big[a_1^{(n)}, \ldots, a_n^{(n)}\Big]\Big) = \Big(\Big[a_1^{(1)}, \ldots, a_n^{(1)}\Big] \cdots \underbrace{\Big[y_1, \ldots, y_n\Big]}_{j} \cdots \Big[a_1^{(n)}, \ldots, a_n^{(n)}\Big]\Big),$$

$\Big[a_1^{(k)}, \ldots, a_n^{(k)}\Big]$ is the congruence class of $\Big(a_1^{(k)}, \ldots, a_n^{(k)}\Big)$ and $a_1^{(k)}, \ldots, a_n^{(k)} \in Q$ $(k = 1, \ldots, j-1, j+1, \ldots, n)$, $x_1^n, y_1^n \in Q^n$. Hence, we have

$$\langle \Big(\Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big), \Big(a_2^{(1)} \cdots \overbrace{x_2}^{j} \cdots a_2^{(n)}\Big), \cdots \Big(a_n^{(1)} \cdots \overbrace{x_n}^{j} \cdots a_n^{(n)}\Big)\Big), \Big(\Big(a_1^{(1)} \cdots \overbrace{y_1}^{j} \cdots a_1^{(n)}\Big), \Big(a_2^{(1)} \cdots \overbrace{y_2}^{j} \cdots a_2^{(n)}\Big), \cdots \Big(a_n^{(1)} \cdots \overbrace{y_n}^{j} \cdots a_n^{(n)}\Big)\Big)\rangle \in \alpha_i,$$

or

$$\Big(\Big(\Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big) \cdots \underbrace{\Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big)}_{i} \cdots \Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big)\Big)$$

$$\Big(\Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big) \cdots \underbrace{\Big(a_2^{(1)} \cdots \overbrace{x_2}^{j} \cdots a_2^{(n)}\Big)}_{i} \cdots \Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big)\Big) \cdots$$

$$\cdots \underbrace{\Big(a_n^{(1)} \cdots \overbrace{y_n}^{j} \cdots a_n^{(n)}\Big)}_{i} \cdots \Big(\Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big) \cdots \underbrace{\Big(a_{n-1}^{(1)} \cdots \overbrace{x_{n-1}}^{j} \cdots a_{n-1}^{(n)}\Big)}_{i} \cdots$$

$$\cdots \Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big)\Big)\Big) = \Big(\Big(\Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big) \cdots \underbrace{\Big(a_1^{(1)} \cdots \overbrace{y_1}^{j} \cdots a_1^{(n)}\Big)}_{i} \cdots$$

$$\cdots \Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big)\Big)\Big(\Big(a_1^{(1)} \cdots \overbrace{x_1}^{j} \cdots a_1^{(n)}\Big) \cdots \underbrace{\Big(a_2^{(1)} \cdots \overbrace{y_2}^{j} \cdots a_2^{(n)}\Big)}_{i} \cdots$$

$$\cdots\Big(a_1^{(1)}\cdots\overbrace{x_1}^{j}\cdots a_1^{(n)}\Big)\Big)\cdots\underbrace{\Big(a_n^{(1)}\cdots\overbrace{x_n}^{j}\cdots a_n^{(n)}\Big)}_{i}\cdots\Big(\Big(a_1^{(1)}\cdots\overbrace{x_1}^{j}\cdots a_1^{(n)}\Big)\cdots$$

$$\cdots\underbrace{\Big(a_{n-1}^{(1)}\cdots\overbrace{y_{n-1}}^{j}\cdots a_{n-1}^{(n)}\Big)}_{i}\cdots\Big(a_1^{(1)}\cdots\overbrace{x_1}^{j}\cdots a_1^{(n)}\Big)\Big)\Big).$$

Without loss of generality it can be assumed that $i < j$. Using twice the identity (1), we get from the last equality

$$\Big(\Big(\Big(a_1^{(1)}\cdots\overbrace{a_1^{(1)}}^{i}\cdots a_1^{(1)}\Big)\cdots\underbrace{a_n^{(1)}}_{i}\cdots\Big(a_1^{(1)}\cdots\overbrace{a_{n-1}^{(1)}}^{i}\cdots a_1^{(1)}\Big)\Big)$$

$$\Big(\Big(a_1^{(2)}\cdots\overbrace{a_1^{(2)}}^{i}\cdots a_1^{(2)}\Big)\cdots\underbrace{a_n^{(2)}}_{i}\cdots\Big(a_1^{(2)}\cdots\overbrace{a_{n-1}^{(2)}}^{i}\cdots a_1^{(2)}\Big)\Big)\cdots$$

$$\cdots\underbrace{\Big(\Big(a_1^{(i)}\cdots\overbrace{a_1^{(i)}}^{i}\cdots a_1^{(i)}\Big)\cdots\underbrace{a_n^{(i)}}_{i}\cdots\Big(a_1^{(i)}\cdots\overbrace{a_{n-1}^{(i)}}^{i}\cdots a_1^{(i)}\Big)\Big)}_{i}\cdots$$

$$\cdots\underbrace{\Big(\Big(x_1\cdots\overbrace{x_1}^{i}\cdots x_1\Big)\cdots\underbrace{y_n}_{i}\cdots\Big(x_1\cdots\overbrace{x_{n-1}}^{i}\cdots x_1\Big)\Big)}_{j}\cdots$$

$$\cdots\Big(\Big(a_1^{(n)}\cdots\overbrace{a_1^{(n)}}^{i}\cdots a_1^{(n)}\Big)\cdots\underbrace{a_n^{(n)}}_{i}\cdots\Big(a_1^{(n)}\cdots\overbrace{a_{n-1}^{(n)}}^{i}\cdots a_1^{(n)}\Big)\Big)\Big)=$$

$$=\Big(\Big(\Big(a_1^{(1)}\cdots\overbrace{a_1^{(1)}}^{i}\cdots a_1^{(1)}\Big)\cdots\underbrace{a_n^{(1)}}_{i}\cdots\Big(a_1^{(1)}\cdots\overbrace{a_{n-1}^{(1)}}^{i}\cdots a_1^{(1)}\Big)\Big)$$

$$\Big(\Big(a_1^{(2)}\cdots\overbrace{a_1^{(2)}}^{i}\cdots a_1^{(2)}\Big)\cdots\underbrace{a_n^{(2)}}_{i}\cdots\Big(a_1^{(2)}\cdots\overbrace{a_{n-1}^{(2)}}^{i}\cdots a_1^{(2)}\Big)\Big)\cdots$$

$$\cdots\underbrace{\Big(\Big(a_1^{(i)}\cdots\overbrace{a_1^{(i)}}^{i}\cdots a_1^{(i)}\Big)\cdots\underbrace{a_n^{(i)}}_{i}\cdots\Big(a_1^{(i)}\cdots\overbrace{a_{n-1}^{(i)}}^{i}\cdots a_1^{(i)}\Big)\Big)}_{i}\cdots$$

$$\cdots\underbrace{\Big(\Big(x_1\cdots\overbrace{y_1}^{i}\cdots x_1\Big)\cdots\underbrace{x_n}_{i}\cdots\Big(x_1\cdots\overbrace{y_{n-1}}^{i}\cdots x_1\Big)\Big)}_{j}\cdots$$

$$\cdots\Big(\Big(a_1^{(n)}\cdots\overbrace{a_1^{(n)}}^{i}\cdots a_1^{(n)}\Big)\cdots\underbrace{a_n^{(n)}}_{i}\cdots\Big(a_1^{(n)}\cdots\overbrace{a_{n-1}^{(n)}}^{i}\cdots a_1^{(n)}\Big)\Big)\Big).$$

Since $Q(\ )$ is a cancellation n-groupoid, we get from the last equality

$$\Big(\Big(\overset{i-1}{x_1}\,x_1\,\overset{n-i}{x_1}\Big)\cdots\underbrace{y_n}_{i}\cdots\Big(\overset{i-1}{x_1}\,x_{n-1}\,\overset{n-i}{x_1}\Big)\Big)=\Big(\Big(\overset{i-1}{x_1}\,y_1\,\overset{n-i}{x_1}\Big)\cdots\underbrace{x_n}_{i}\cdots\Big(\overset{i-1}{x_1}\,y_{n-1}\,\overset{n-i}{x_1}\Big)\Big).$$

Thus, $\langle x_1^n, y_1^n\rangle \in \alpha_i$, i.e. $[x_1,\dots,x_n]=[y_1,\dots,y_n]$. □

Lemma 5 $\langle\big(a_1,\dots,a_{n-1},\big(a_1^{i-1}xa_i^{n-1}\big)\big),\big(b_1,\dots,b_{n-1},\big(b_1^{i-1}yb_i^{n-1}\big)\big)\rangle\in\alpha_i$ *if and only if* $x=y$.

Proof. Let $\langle\big(a_1,\dots,a_{n-1},\big(a_1^{i-1}xa_i^{n-1}\big)\big),\big(b_1,\dots,b_{n-1},\big(b_1^{i-1}yb_i^{n-1}\big)\big)\rangle\in\alpha_i$, then we have

$$\Big(\Big(\overset{i-1}{a_1}\,a_1\,\overset{n-i}{a_1}\Big)\cdots\underbrace{\Big(b_1^{i-1}yb_i^{n-1}\Big)}_{i}\cdots\Big(\overset{i-1}{a_1}\,a_{n-1}\,\overset{n-i}{a_1}\Big)\Big)=$$

$$=\Big(\Big(\overset{i-1}{a_1}\,b_1\,\overset{n-i}{a_1}\Big)\cdots\underbrace{\Big(a_1^{i-1}xa_i^{n-1}\Big)}_{i}\cdots\Big(\overset{i-1}{a_1}\,b_{n-1}\,\overset{n-i}{a_1}\Big)\Big).$$

According to (1) we obtain

$$\Big(\Big(\overset{i-1}{a_1}\,b_1\,\overset{n-i}{a_1}\Big)\cdots\underbrace{\Big(a_1^{i-1}ya_i^{n-1}\Big)}_{i}\cdots\Big(\overset{i-1}{a_1}\,b_{n-1}\,\overset{n-i}{a_1}\Big)\Big)=$$

$$=\Big(\Big(\overset{i-1}{a_1}\,b_1\,\overset{n-i}{a_1}\Big)\cdots\underbrace{\Big(a_1^{i-1}xa_i^{n-1}\Big)}_{i}\cdots\Big(\overset{i-1}{a_1}\,b_{n-1}\,\overset{n-i}{a_1}\Big)\Big)$$

and using twice the cancellation law we get $x=y$.

The converse assertion is obvious. □

Let φ_i $(i=1,\dots,n)$ be the map from Q to Q_1^i such that $\varphi_i(x)=\Big[e_1,\dots,e_{n-1},\Big(e_1^{i-1}xe_i^{n-1}\Big)\Big]$ for all $x\in Q$ and some $e_1^{n-1}\in Q^{n-1}$. It follows from Lemma 5 that φ_i is well defined and injective.

Lemma 6 *The injective map* φ_i *is an embedding for any* $i=1,\dots,n$.

Proof. It follows from Lemma 5. □

Thus, we can identify Q with $\varphi_i(Q)\subseteq Q_1^i$.

Lemma 7 *Equation* $\big(a_1^{i-1}xa_i^{n-1}\big)=a_n$ *has a solution in* Q_1^i *for all* $a_1^n\in Q^n$.

Proof. We have

$$\Big(\Big(e,\ldots,e,\Big(\overset{i-1}{e}\,a_1\,\overset{n-i}{e}\Big)\Big)\cdots\Big(e,\ldots,e,\Big(\overset{i-1}{e}\,a_{i-1}\,\overset{n-i}{e}\Big)\Big)\underbrace{\big(a_1^n\big)}_{i}\Big(e,\ldots,e,\Big(\overset{i-1}{e}\,a_i\,\overset{n-i}{e}\Big)\Big)\cdots$$
$$\cdots\Big(e,\ldots,e,\Big(\overset{i-1}{e}\,a_{n-1}\,\overset{n-i}{e}\Big)\Big)\Big)=\Big(\Big(\overset{i-1}{e}\,a_1\,\overset{n-i}{e}\Big),\ldots,\Big(\overset{i-1}{e}\,a_{n-1}\,\overset{n-i}{e}\Big),\Big(\Big(\overset{i-1}{e}\,a_1\,\overset{n-i}{e}\Big)\cdots$$
$$\cdots\Big(\overset{i-1}{e}\,a_{i-1}\,\overset{n-i}{e}\Big)\underbrace{a_n}_{i}\Big(\overset{i-1}{e}\,a_i\,\overset{n-i}{e}\Big)\cdots\Big(\overset{i-1}{e}\,a_{n-1}\,\overset{n-i}{e}\Big)\Big)\Big),$$

where e is some element of Q. Thus, by Lemma 5, we get

$$\langle\Big(\Big(\overset{i-1}{e}\,a_1\,\overset{n-i}{e}\Big),\ldots,\Big(\overset{i-1}{e}\,a_{n-1}\,\overset{n-i}{e}\Big),\Big(\Big(\overset{i-1}{e}\,a_1\,\overset{n-i}{e}\Big)\cdots$$
$$\cdots\underbrace{a_n}_{i}\cdots\Big(\overset{i-1}{e}\,a_{n-1}\,\overset{n-i}{e}\Big)\Big)\Big),\Big(e,\ldots,e,\Big(\overset{i-1}{e}\,a_n\,\overset{n-i}{e}\Big)\Big)\rangle\in\alpha_i.$$

Then, the congruence class $[a_1^n]$ is a solution of $\big(a_1^{i-1}xa_i^{n-1}\big)=a_n$. □

We say that Q_1^i is the first i-extension of Q.

2 General Extension OF Q

Let us define the sequence $Q_0^i, Q_1^i, \ldots, Q_k^i, \ldots$ of i-extensions of Q as follows: $Q_0^i = Q$, $Q_k^i = \Big(Q_{k-1}^i\Big)_1^i$ where $k = 1, 2, \ldots$. Then we have the direct family of groupoids

$$Q \xrightarrow{\varphi_i^{(1)}} Q_1^i \xrightarrow{\varphi_i^{(2)}} Q_2^i \xrightarrow{\varphi_i^{(3)}} \ldots \tag{6}$$

Let Q_∞^i be the direct limit of the direct family of groupoids (6). It is clear that Q_∞^i is a cancellation medial n-groupoid. The following result is obvious.

Theorem 1 *Q_∞^i is a medial i-quasigroup for any $i = 1, \ldots, n$.*

Lemma 8 *$(Q_1^i)_1^j$ is isomorphic to $\big(Q_1^j\big)_1^i$ for all $i, j = 1, \ldots, n$.*

Proof. Let us define a map from $(Q_1^i)_1^j$ to $\big(Q_1^j\big)_1^i$. Take an arbitrary element x of $(Q_1^i)_1^j$. Since the elements of $(Q_1^i)_1^j$ are solutions of the equations $\big(\beta_1^{j-1}x\beta_j^{n-1}\big) = \beta_n$, where $\beta_1^n \in (Q_1^i)^n$, then there exist elements $a_1^n \in (Q_1^i)^n$ such that:

$$\big(a_1^{j-1}xa_j^{n-1}\big) = a_n. \tag{7}$$

On the other hand, for this elements there exist $b_1^{(k)}, \ldots, b_n^{(k)} \in Q$, where $k = 1, \ldots, n$, such that

$$\Big(b_1^{(k)}\cdots b_{i-1}^{(k)}a_k b_i^{(k)}\cdots b_{n-1}^{(k)}\Big) = b_n^{(k)}. \tag{8}$$

Also, there exist elements $c_1^n \in (Q_1^j)^n$ such that

$$\left(b_k^{(1)} \cdots b_k^{(j-1)} c_k b_k^{(j)} \cdots b_k^{(n-1)}\right) = b_k^{(n)} \tag{9}$$

for all $k = 1, \ldots, n$. Finally, there exists an element $y \in \left(Q_1^j\right)_1^i$ such that

$$\left(c_1^{i-1} y c_i^{n-1}\right) = c_n. \tag{10}$$

The element y is assigned to x. We claim that the defined map is an isomorphism from $(Q_1^i)_1^j$ to $\left(Q_1^j\right)_1^i$. This can be proved by direct calculations using equalities (7)-(10) and the medial identity. □

Lemma 9 *$(Q_n^i)_m^j$ is isomorphic to $(Q_m^j)_n^i$ for $n, m = 0, 1, 2, \ldots$, and $i, j = 1, 2, \ldots, n$.*

Proof. Clearly, the lemma is true if either m or n equal to 0. We assume $m \neq 0$ and $n \neq 0$, then the fact can be deduced from Lemma 8. If $(Q_n^i)_1^j \cong \left(Q_1^j\right)_n^i$, then $\left(Q_{n+1}^i\right)_1^j \cong \left((Q_n^i)_1^i\right)_1^j \cong \left((Q_n^i)_1^j\right)_1^i \cong \left(\left(Q_1^j\right)_n^i\right)_1^i \cong \left(Q_1^j\right)_{n+1}^i$.

If we have $(Q_n^i)_m^j \cong (Q_m^i)_n^j$, then $(Q_n^i)_{m+1}^j \cong \left((Q_n^i)_m^j\right)_1^j \cong \left((Q_m^j)_n^i\right)_1^j \cong \left((Q_m^j)_1^j\right)_n^i \cong \left(Q_{m+1}^j\right)_n^i$. □

Let us define the general extension $Q^{(i_1,\ldots,i_n)}$ of Q as follows: $Q^{(i_1)} = Q_\infty^{i_1}$, $Q^{(i_1,i_2)} = (Q_\infty^{i_1})_\infty^{i_2}$, …, $Q^{(i_1,\ldots,i_n)} = \left(Q^{(i_1,\ldots,i_{n-1})}\right)_\infty^{i_n}$.

Lemma 10 *$Q^{(i_1,\ldots,i_n)}$ is isomorphic to $Q^{(j_1,\ldots,j_n)}$ for all perutations $(i_1, \ldots, i_n)$, $(j_1, \ldots, j_n)$ of the numbers $1, 2, \ldots, n$.*

Proof. It follows from Lemma 9. □

Theorem 2 *A medial cancellation n-groupoid can be embedded into a medial n-ary quasigroup. This quasigroup is unique up to isomorphism.*

Proof. It follows from Theorem 1 and Lemma 10. □

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