Armenian Journal of Mathematics
Volume 16, Number 10, 2024, 1–7
https://doi.org/10.52737/18291163-2024.16.10-1-7

On the Total Dominating Set of 3/2-Generated Groups

V. S. Atabekyan and H. T. Aslanyan

Abstract. A subset S of a group G is called a total dominating set of G if for any nontrivial element $x \in G$ there is an element $y \in S$ such that $G = \langle x, y \rangle$. Tarski monsters, constructed by Olshanskii, are infinite simple groups, any pair of non-commuting elements of which is a total dominating set. In this paper, we construct an infinite non-cyclic and non-simple group having a total dominating set from two elements. This gives a positive answer to Donoven and Harper's question about the existence of infinite groups (other than Tarski monsters) having a finite total dominating set. In addition, our examples have an infinite uniform spread.

Key Words: Spred of Group, Total Dominating Set, 3/2-generated Group, Tarski Monster
Mathematics Subject Classification 2020: 20F05, 20F06, 20E32

Introduction

Let G be a 2-generated group. The spread of a group G is denoted by $s(G)$ and is the greatest k such that for any non-trivial elements $x_1, ..., x_k$ of G there exists $y \in G$ satisfying the conditions $G = \langle x_i, y \rangle$ for all $i = 1, 2, ..., k$. This notion was first introduced in [1] in 1975. The uniform spread of G is denoted by $u(G)$ and is the greatest k such that there is a conjugacy class C of G with the property that for any nontrivial elements $x_1, ..., x_k$ there exists $y \in C$ satisfying the condition $G = \langle x_i, y \rangle$ for all $i = 1, 2, ..., k$. If such a largest number does not exist in the first or second case, then we will write $s(G) = \infty$ or $u(G) = \infty$, respectively. It is clear that $u(G) \leq s(G)$. In general, the concepts of spread and uniform spread are not the same. For example, the uniform spread of the group $SL_3(2)$ is 3, while its spread is 4 (see [3]).

A group is said to be 3/2-generated (or one and a half generated (see, for example, [8])), if any of its non-trivial elements is a part of a generating pair of G. This property is stronger than being 2-generated but is weaker than being cyclic. Obviously, the condition $s(G) \geq 1$ is equivalent to the fact that G is 3/2-generated. In 2000, it was proven [7] that every finite simple group is 3/2-generated. Later, in 2008, it was shown [2] that $u(G) \geq 2$ for any non-abelian finite simple group G. In the same work, it was conjectured that a finite group is a 3/2-generated group if and only if all of its proper quotients is cyclic. This conjecture was proven in 2021 [4]. For infinite groups, the situation is different. Recently, Cox [5] constructed an example of an infinite 2-generated group, each of whose proper quotient groups is cyclic and which is not 3/2-generated. Tarski monsters constructed by A.Yu. Olshanskii [9] (see also [10]) are examples of infinite and simple 3/2-groups. Moreover, it can be shown that for these groups $u(G) = \infty$.

A subset $S \subset G$ is called a total dominating set of G if for any nontrivial element $x \in G$ there is an element $y \in S$ such that $G = \langle x, y \rangle$ (see [6]). For example, any generator of a cyclic group forms its total dominating set. Tarski monsters constructed by Olshanskii are infinite simple groups, any pair of non-commuting elements of which is a total dominating set.

In 2020, Donoven and Harper [6] proved that $s(G) \geq 1$ for all Higman–Thompson and Brin–Thompson groups G. These are the first examples of finitely defined 3/2-generated groups. In [6], the authors posed the following question: *Is there a non-cyclic infinite group (distinct from Tarski monsters) which has a finite total dominating set* (see Question 5, [6]). This problem was also emphasized in [5] (see p. 222, [5]). Work [6] also poses the following question: *Is there a non-cyclic infinite group (distinct from Tarski monsters) for which* $s(G) = \infty$ (see Question 3, [6]).

In this paper, we will answer both of these questions positively. By modifying Olshanskii's method, we will construct infinite 2-generated groups that contain subgroups of index 2 that are isomorphic to a Tarski monster (and thus are not simple), allowing us to answer the above questions.

Theorem 1 *There is an infinite non-cyclic and non-simple torsion free group that has total dominating set of two elements.*

Theorem 2 *There is an infinite non-cyclic and non-simple torsion free group for which* $u(G) = \infty$.

From Theorem 2 and the inequality $u(G) \leq s(G)$, it follows

Corollary 1 *There is an infinite non-abelian and non-simple torsion free group for which* $s(G) = \infty$.

1 Defining relations

Consider the free group F_2 with two free generators a, b. It is easy to show that the elements a, bab^{-1} and b^2 in F_2 freely generate a free subgroup of index 2. We will construct a group with two generators in which these elements generate the subgroup H of index 2 isomorphic to the Tarski's monster without torsion and having some additional properties. We will follow the original scheme for constructing Tarski's monsters, proposed by Olshansky in the monograph [10] (see Paragraphs 25, 27), suggesting a slight modification in this scheme that is necessary for us.

We fix sufficiently large odd numbers n, h, d according to Paragraph 19 [10] (see p. 218) and require that $h \equiv 3(\mod 4)$. As in Paragraph 27 [10], we denote by $G(0) = F_2$ the free group with basis a, b and set $\mathcal{R}_0 = \emptyset$. Suppose, by induction, that we have defined the set of relators $\mathcal{R}_{i-1}$ and the group $G(i-1) = \langle a, b \,\|\, r = 1, r \in \mathcal{R}_{i-1}\rangle$.

A non-empty word A over $\{a, b\}$ is called a simple word in rank $i-1$ if it is not conjugate in the group $G(i-1)$ to a power of a shorter word and is not conjugate in $G(i-1)$ to a power of a period of rank $k \leq i-1$. Denote by $\mathcal{X}_i$ a maximal set of simple words of length i which are simple in rank $i-1$ and with the condition that $A, B \in \mathcal{X}_i$ and $A \not\equiv B$ implies that A is not conjugate in rank $i-1$ to B or B^{-1}. The words from $\mathcal{X}_i$ are called periods of rank i. We note right away that we declare all periods from $\mathcal{X}_i$ to be periods of the second type in the sense of §25.1 [10].

For each period $A \in \mathcal{X}_i$ we fix some maximal subset of words $\mathcal{Y}_A$ such that:

1. if $T \in \mathcal{Y}_A$, then $1 \leq |T| < d|A|$;

2. each double coset of the form $\langle A\rangle\, g\, \langle A\rangle$ of subgroups of $G(i-1)$ contains at most one word in $\mathcal{Y}_A$, and this word is of minimal length among the words representing this double coset.

For an arbitrary $A \in \mathcal{X}_i$, if a is not contained in the subgroup $\langle A\rangle$ of $G(i-1)$, then for each word $T \in \mathcal{Y}_A$ which is not contained in the double cosets $\langle A\rangle\, bab^{-1}\, \langle A\rangle$ and $\langle A\rangle\, b\, \langle A\rangle$ of the group $G(i-1)$, we introduce a defining relation

$$aA^nTA^{n+5}T\cdots TA^{n+5h-5}, \tag{1}$$

and for each word $T \in \mathcal{Y}_A$ which is not contained in $\langle A\rangle\, a\, \langle A\rangle$ and $\langle A\rangle\, b\, \langle A\rangle$ in $G(i-1)$, we introduce a relation

$$bab^{-1}A^{n+2}TA^{n+7}T\cdots TA^{n+5h-3}, \tag{2}$$

and if b is not contained in the subgroup $\langle A\rangle$ of the group $G(i-1)$, then for every word $T \in \mathcal{Y}_A$ which is not contained in $\langle A\rangle\, bab^{-1}\, \langle A\rangle$ and $\langle A\rangle\, a\, \langle A\rangle$ in $G(i-1)$, we introduce a relation

$$b^2A^{n+4}TA^{n+9}T\cdots TA^{n+5h-1}. \tag{3}$$

The set of relations of the form (1), (2), (3) is denoted by $\mathcal{P}_i$. Finally, we set $\mathcal{R}_i = \mathcal{R}_{i-1} \bigcup \mathcal{P}_i$,

$$G(i) = \langle a,\, b \,|\, R = 1, R \in \mathcal{R}_i \rangle \tag{4}$$

and

$$G(\infty) = \left\langle a,\, b \,|\, R = 1, R \in \bigcup_{i=1}^{\infty} \mathcal{R}_i \right\rangle. \tag{5}$$

2 Auxiliary lemmas

Lemma 1 *The element b is not contained in the subgroup generated by the elements a, bab^{-1} and b^2 in $G(\infty)$.*

Proof. From the definition of words (1), (2) and (3), it follows that in each of these relations, the word T appears $h - 1$ times, i.e., an even number of times. In turn, the word A appears in the defining relation (1) exactly $nh + \frac{5h(h-1)}{2}$ times, which is an even number by the choice of odd numbers n and h ($h \equiv 3 (\mod 4)$). Similarly, the word A is included in the defining relations (2) and (3), respectively $nh + \frac{5h(h-1)}{2} + 2h$ and $nh + \frac{5h(h-1)}{2} + 4h$ times, i.e., again an even number of times. From this it follows that in each of the relations (1), (2), (3) of the group $G(\infty)$, the letter b appears an even number of times and, in particular, in $G(\infty)$, the element b is not contained in the subgroup generated by the elements a, bab^{-1} and b^2. □

Let us denote by N the subgroup of the group $G(\infty)$ generated by the images of the elements a, bab^{-1}, b^2:

$$N = \left\langle a, bab^{-1}, b^2 \right\rangle.$$

Since $a, b^2 \in N$ and $b \notin N$, then $(G(\infty) : N) = 2$.

The following lemma is an analogue of Lemma 27.1 in [10] and is proved in a similar way.

Lemma 2 *Let VV_1 and VV_1' be cyclic shifts of a word $R^{\pm 1}$, where $R \in \mathcal{P}_i$, and V contains a subword of the form $A^c T_1 A^m T_2 A^c$, where $c = d+3$, $m \geq n$, and the words T_1 and T_2 are a, bab^{-1}, or b^2, or belong to the set $\mathcal{Y}_A$. Then $VV_1 \equiv VV_1'$. The word R is not a proper power in F_2.*

In the Paragraph 25 [10], the conditions R1–R7 are defined.

Lemma 3 *Presentation (4) of the group $G(i)$ satisfies conditions $R1 - R7$.*

Proof. Condition $R5$ follows from Lemma 2, and conditions $R1 - R4$, $R6$, $R7$ are verified exactly as in Lemma 27.2 [10]. □

Lemma 4 *(Lemma 27.3 [10]) Suppose that a presentation of a group G satisfies the conditions $R1 - R7$, and let H be a non-abelian subgroup of G. Then there is a period F of some rank $i \geq 1$ and a word T not commuting with F in G such that $|T| < 3|F|$, and the subgroup $\langle F, T\rangle$ is contained in a subgroup conjugate to H in G.*

Lemma 5 *Any non-abelian subgroup of $G(\infty)$ contains N.*

Proof. Let H be a non-abelian subgroup. Some subgroup H_1 conjugate to H contains elements F and T satisfying the conclusion of Lemma 4. Since $|T| < 3|F| < d|F|$, it follows from the definition of relations (1), (2), (3) that $a, bab^{-1}, b^2 \in H_1$, that is, $N \subset H_1$. Since N is a normal subgroup, then $N \subset H$. $\square$

From Theorem 26.5 [10], follows

Lemma 6 *The centralizer of a non-trivial element $X \in G(\infty)$ is cyclic. Any abelian subgroup of $G(\infty)$ is cyclic.*

From Theorem 26.4(1) [10], follows

Lemma 7 *The group $G(\infty)$ is torsion-free.*

By analogy with Theorem 28.3 [10], it is proved

Lemma 8 *The subgroup N of $G(\infty)$ is simple and any of its proper subgroups is infinite cyclic.*

3 Proofs of main theorems

Proof of Theorem 1 From Lemma 6 and Lemma 8, it follows that $G(\infty)$ is an infinite non-abelian group. It is not a simple group because of $(G(\infty) : N) = 2$. Let us prove that the set $\{b, aba^{-1}\}$ is a total dominating set for $G(\infty)$.

Choose a non-trivial element $X \in G(\infty)$. Since $ab \neq ba$, then $baba^{-1} \neq aba^{-1}b$ by Lemma 25.14 [10]. Therefore, by Lemma 6 one of the subgroups $\langle b, X\rangle$ and $\langle aba^{-1}, X\rangle$ is non-abelian. Indeed, otherwise the elements b and aba^{-1} would belong to the centralizer of X, which is a cyclic group. Hence, $N \subset \langle b, X\rangle$ or $N \subset \langle aba^{-1}, X\rangle$ by Lemma 5. Since $a \in N$, this means that either $G(\infty) = \langle b, X\rangle$ or $G(\infty) = \langle aba^{-1}, X\rangle$. $\square$

Proof of Theorem 2 As it was mentioned above, from Lemma 6 and Lemma 8, it follows that $ab \neq ba$. Choose an arbitrary finite sequence of non-trivial elements $X_1, X_2, \cdots, X_k$ from $G(\infty)$. First, let us prove that $a^t ba^{-t} a^s ba^{-s} \neq a^s ba^{-s} a^t ba^{-t}$ for $t \neq s$. Indeed, otherwise $a^{t-s} ba^{s-t} b = ba^{t-s} ba^{s-t}$, which by

Lemma 25.14 [10] means that $ba^{t-s} = a^{t-s}b$. Thus, by Lemma 6, a and b belong to the centralizer of a^{t-s}, which is only possible if $t = s$. Consequently, as in the proof of Theorem 1, for some s and for any i, the subgroups $\langle a^s ba^{-s}, X_i\rangle$ are non-abelian. Then $N \subset \langle a^s ba^{-s}, X_i\rangle$ by Lemma 5, that is, $G(\infty) = \langle a^s ba^{-s}, X_i\rangle$ for any $i = 1, ..., k$ due to $a \in N$. We obtain that the conjugacy class b provides the equality $u(G(\infty)) = \infty$. □

Acknowledgments. This work is partially supported by the SC of Republic of Armenia research project 21T-1A213.

The authors thank the anonymous reviewers for carefully reading the manuscript and for helpful comments and remarks that helped improve the paper.

References

[1] J.L. Brenner and J. Wiegold, Two-generator groups. I. Michigan Math. J., **22** (1975), no. 1, pp. 53–64. https://doi.org/10.1307/mmj/1029001421

[2] T. Breuer, R.M. Guralnick and W.M. Kantor, Probabilistic generation of finite simple groups, II. J. Algebra, **320** (2008), no. 2, pp. 443–494. https://doi.org/10.1016/j.jalgebra.2007.10.028

[3] T.C. Burness and S. Guest, On the uniform spread of almost simple linear groups. Nagoya Math. J., **209** (2013), pp. 35–109. https://doi.org/10.1017/s0027763000010680

[4] T. Burness, R. Guralnick and S. Harper, The spread of a finite group. Ann. of Math., **193** (2021), no. 2, pp. 619–687. https://doi.org/10.4007/annals.2021.193.2.5

[5] Ch.G. Cox, On the spread of infinite groups. Proceedings of the Edinburgh Mathematical Society, **65** (2022), pp. 214–228. https://doi.org/10.1017/S0013091522000037

[6] C. Donoven and S. Harper, Infinite 3/2-generated groups. Bull. London Math. Soc., **52** (2020), no. 4, pp. 657–673. https://doi.org/10.1112/blms.12356

[7] R.M. Guralnick and W.M. Kantor, Probabilistic generation of finite simple groups. J. Algebra, **234** (2000), no. 2, pp. 743–792. https://doi.org/10.1006/jabr.2000.8357

[8] R.M. Guralnick, E. Plotkin and A. Shalev, Burnside-type problems related to solvability. Internat. J. Algebra Comput., **17** (2007), no. 5-6, pp. 1033–1048. https://doi.org/10.1142/s0218196707003962

[9] A.Yu. Olshanskii, Groups of bounded period with subgroups of prime order. Algebra Logika, **21** (1982), no. 5, pp. 553–618.

[10] A.Yu. Olshanskii, *The Geometry of Defining Relations in Groups*, Kluwer Academic Publishers, 1991. https://doi.org/10.1007/978-94-011-3618-1

Varujan S. Atabekyan
Yerevan State University,
Alex Manoogian str. 1, 0025 Yerevan, Armenia.
avarujan@ysu.am

Haik T. Aslanyan
American University of Armenia,
Marshal Bagramian ave. 40, 0019 Yerevan, Armenia.
haikaslanyan@gmail.com

Please, cite to this paper as published in
Armen. J. Math., V. **16**, N. 10(2024), pp. 1–7
https://doi.org/10.52737/18291163-2024.16.10-1-7