Armenian Journal of Mathematics
Volume 16, Number 6, 2024, 1–13
https://doi.org/10.52737/18291163-2024.16.6-1-13

Some Results on Perturbation of Duality of OPV-Frames

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Abstract. In this paper, we consider a perturbation of operator-valued frames (OPV-frames) and obtain conditions for their stability in terms of operators associated with the OPV-frames. Also, some duality relations of OPV-frames are discussed. Finally, some properties of the duals of OPV-frames are proven.

Key Words: Frames, Operator Valued Frames, Perturbation of Frames
Mathematics Subject Classification 2020: 42C15, 42A38

Introduction

As an extension of bases, the idea of frames has undergone rapid study and development in the last 10 years, largely because of its substantial applications in signal processing and coding theory. Frames were introduced by Duffin and Schaeffer [7] in 1952 and were reintroduced by Daubechies, Grossman, and Meyer [6] in 1986. Since then, other generalizations, including fusion frames and g-frames have been developed in [4,8,20]. Additionally, a broadening of the idea of a vector-valued frame that allows us to deal with the use of the operators in a Hilbert space as opposed to its elements was pioneered by L. Gavruta [9]. Further literature can be found in [3, 5, 14–17].

In quantum computing [12], packet encoding [2], and many other fields, the idea of OPV-frames offers a more generic method of series expansion of elements that is very similar to frame decomposition. Kaftal [13] proved various interesting results for OPV-frames. Poumai [18] recently developed the idea of OPV-frames showcasing its uses in quantum mechanics. Some interesting properties of OPV-frames and a result about a relationship between two Riesz OPV-frames were proved in [19]. For additional information about OPV-frames, see [1, 10, 11, 18].

This paper aims to prove a few more results on OPV-frames.

1 Preliminaries

Throughout the paper, the symbols $\mathcal{H}$ and $\mathcal{K}$ stand for Hilbert spaces, $\mathbb{N}$ for the collection of all natural numbers, and $\mathcal{I}$ for an index set. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$, while the set of all bounded operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H},\mathcal{K})$. We begin by the definition of an OPV-frame.

Definition 1 *A sequence* $\{\mathcal{F}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ *is called an operator valued-frame (OPV-frame) for* $\mathcal{H}$ *with range in* $\mathcal{K}$ *if there exist two constants* A *and* B *with* $0 < A \leq B < \infty$ *such that*

$$AI_{\mathcal{H}} \leq \sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n \leq BI_{\mathcal{H}},$$

or, equivalently,

$$A\|\mathfrak{h}\|^2 \leq \sum_{n\in\mathbb{N}} \|\mathcal{F}_n(\mathfrak{h})\|^2 \leq B\|\mathfrak{h}\|^2, \qquad \mathfrak{h} \in \mathcal{H}. \tag{1}$$

The positive constants A and B are known as lower and upper frame bounds for the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, respectively, and inequality (1) is termed as the *OPV-frame inequality*. Also, the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is said to be a *Bessel OPV-frame* for $\mathcal{H}$ with range in $\mathcal{K}$ if it satisfies the right hand side of inequality (1). If $\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n = I_{\mathcal{H}}$, the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is called a *Parseval OPV-frame.* In this case, $A = B = 1$.

If $\{e_n\}_{n\in\mathbb{N}}$ is the canonical orthonormal basis of ℓ^2, then for each $n \in \mathbb{N}$, one can define the partial isometry, that is, the map $|e_n\rangle_{\ell^2} : \mathcal{K} \to \mathcal{K} \otimes \ell^2$ such that

$$|e_n\rangle_{\ell^2}(k) = k \otimes e_n, \qquad k \in \mathcal{K}.$$

Its adjoint operator $|e_n\rangle_{\ell^2}^* := {}_{\ell^2}\langle e_n| : \mathcal{K} \otimes \ell^2 \to \mathcal{K}$ is given by

$${}_{\ell^2}\langle e_n|(k \otimes c) = c_n k, \qquad k \in \mathcal{K}, c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2.$$

Further, the *analysis operator* $\mathcal{T}_{\mathcal{F}} : \mathcal{H} \to \mathcal{K} \otimes \ell^2$ of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ with range in $\mathcal{K}$ is given by

$$\mathcal{T}_{\mathcal{F}}(\mathfrak{h}) = \sum_{n\in\mathbb{N}} \mathcal{F}_n(\mathfrak{h}) \otimes e_n, \qquad \mathfrak{h} \in \mathcal{H},$$

and its adjoint operator $\mathcal{T}_{\mathcal{F}}^* : \mathcal{K} \otimes \ell^2 \to \mathcal{K}$, called the *synthesis operator*, is defined as

$$\mathcal{T}_{\mathcal{F}}^*(k \otimes c) = \sum_{n\in\mathbb{N}} c_n \mathcal{F}_n^*(k), \qquad k \in \mathcal{K}, c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2.$$

The *frame operator* $\mathcal{S}_{\mathcal{F}} : \mathcal{H} \to \mathcal{H}$ of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is given by

$$\mathcal{S}_{\mathcal{F}} = \mathcal{T}_{\mathcal{F}}^*\mathcal{T}_{\mathcal{F}} = \sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{F}_n.$$

The frame operator for an OPV-frame is a bounded, self-adjoint, positive and invertible. For more details related to OPV-frames in quaternionic Hilbert spaces, we refer to [18].

OPV-frames can also be categorized as follows. Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is called

(i) a *Riesz OPV-frame* if the analysis operator $\mathcal{T}_{\mathcal{F}}$ is surjective, i.e., $\mathcal{T}_{\mathcal{F}}(\mathcal{H}) = \mathcal{K} \otimes \ell^2$;

(ii) an *orthonormal OPV-frame* if it is both Riesz OPV-frame and Parseval OPV-frame.

Remark 1 *Poumai [18] characterizes Riesz and orthonormal OPV-frames by means of the following result: an OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a Riesz OPV-frame if and only if $\mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{F}_m^* = \delta_{m,n}I_{\mathcal{K}}$, where $\mathcal{S}_{\mathcal{F}}$ is the frame operator of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.*

The remainder of the paper is organized as follows: In Section 2, we discuss the stability of OPV-frames under various kind of perturbations. We demonstrate that whenever there is a sufficiently small distance between two OPV-frames, they are both stable under perturbation. Additionally, we prove that an OPV-frame is formed whenever the analysis operator of a sequence of operators is sufficiently close to the analysis operator of an OPV-frame. In Section 3, we consider the duality of OPV-frames. We establish a relationship between the analysis operators of any OPV-frame and its dual. We also provide a different way to express the duals of OPV-frames in terms of a family of Bessel OPV-frames. Finally, we demonstrate that each Riesz OPV-frame has a unique dual Riesz OPV-frame.

2 Perturbation of operator valued-frames

One must test the stability of OPV-frames under various perturbation circumstances in order to use them in practical applications. In this section, we demonstrate the stability of OPV-frames under sufficiently small perturbations.

First of all, we prove a result stating that a sequence of operators forms an OPV-frame whenever its analysis operator is close enough to the analysis operator of a given OPV-frame.

Theorem 1 *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with lower and upper frame bounds A and B, respectively. Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ be a sequence of operators, and let there exist α, β and $\gamma > 0$ such that for* *all finite subsets* $\mathcal{I} \subset \mathbb{N}$ *and for each* $\mathfrak{h} \in \mathcal{H}$*,*

$$\Big\|\sum_{i\in\mathcal{I}}\Big(\mathcal{F}_i(\mathfrak{h})-\mathcal{G}_i(\mathfrak{h})\Big)\otimes e_i\Big\| \leq \alpha\Big\|\sum_{i\in\mathcal{I}}\mathcal{F}_i(\mathfrak{h})\otimes e_i\Big\| + \beta\Big\|\sum_{i\in\mathcal{I}}\mathcal{G}_i(\mathfrak{h})\otimes e_i\Big\| + \gamma\|\mathfrak{h}\|, \qquad (2)$$

where $\beta \neq 1$*. Then* $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ *forms an OPV-frame for* $\mathcal{H}$ *with range in* $\mathcal{K}$ *with lower and upper frame bounds* $(1-\alpha)A(\sqrt{B}^{-1}-\gamma)^2/(1+\beta)^2$ *and* $((1+\alpha)\sqrt{B}+\gamma)^2/(1-\beta)^2$*, respectively.*

Proof. Let $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{S}_{\mathcal{F}}$ be the analysis and frame operators, of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, respectively. Using condition (2), for $\mathfrak{h} \in \mathcal{H}$, we can write

$$\begin{aligned}\Big\|\sum_{i\in\mathcal{I}}\mathcal{G}_i(\mathfrak{h})\otimes e_i\Big\| &\leq \Big\|\sum_{i\in\mathcal{I}}\Big(\mathcal{F}_i(\mathfrak{h})-\mathcal{G}_i(\mathfrak{h})\Big)\otimes e_i\Big\| + \Big\|\sum_{i\in\mathcal{I}}\mathcal{F}_i(\mathfrak{h})\otimes e_i\Big\| \\ &\leq (1+\alpha)\Big\|\sum_{i\in\mathcal{I}}\mathcal{F}_i(\mathfrak{h})\otimes e_i\Big\| + \beta\Big\|\sum_{i\in\mathcal{I}}\mathcal{G}_i(\mathfrak{h})\otimes e_i\Big\| + \gamma\|\mathfrak{h}\|.\end{aligned}$$

This gives

$$\Big\|\sum_{i\in\mathcal{I}}\mathcal{G}_i(\mathfrak{h})\otimes e_i\Big\| \leq \frac{(1+\alpha)\|\mathcal{T}_{\mathcal{F}}(\mathfrak{h})\|+\gamma\|\mathfrak{h}\|}{1-\beta} \leq \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)\|\mathfrak{h}\|.$$

Now, define $\mathcal{T}_{\mathcal{G}} : \mathcal{H} \to \mathcal{K}\otimes\ell^2$ as

$$\mathcal{T}_{\mathcal{G}}(\mathfrak{h}) = \sum_{n\in\mathbb{N}}\mathcal{G}_n(\mathfrak{h})\otimes e_n, \qquad \mathfrak{h}\in\mathcal{H}.$$

Clearly, $\mathcal{T}_{\mathcal{G}}$ is a well-defined bounded operator satisfying

$$\|\mathcal{T}_{\mathcal{G}}\| \leq \frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta},$$

and hence,

$$\sum_{n\in\mathbb{N}}\|\mathcal{G}_n(\mathfrak{h})\|^2 = \|\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\|^2 \leq \left(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\right)^2\|\mathfrak{h}\|^2, \qquad \mathfrak{h}\in\mathcal{H}. \qquad (3)$$

Since

$$\|\mathcal{T}_{\mathcal{F}}(\mathfrak{h})-\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\| \leq \alpha\|\mathcal{T}_{\mathcal{F}}(\mathfrak{h})\| + \beta\|\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\| + \gamma\|\mathfrak{h}\|,$$

we obtain

$$\|\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\| \geq \frac{(1-\alpha)\|\mathcal{T}_{\mathcal{F}}(\mathfrak{h})\|-\gamma\|\mathfrak{h}\|}{1+\beta}, \qquad \mathfrak{h}\in\mathcal{H}. \qquad (4)$$

Also, for $\mathfrak{h} \in \mathcal{H}$, we can write

$$\begin{aligned} \|\mathfrak{h}\| &= \|\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{S}_{\mathcal{F}}(\mathfrak{h})\| \\ &\leq \|\mathcal{S}_{\mathcal{F}}^{-1}\|\|\mathcal{T}_{\mathcal{F}}^{*}\|\|\mathcal{T}_{\mathcal{F}}(\mathfrak{h})\| \\ &\leq A^{-1}\sqrt{B}\|\mathcal{T}_{\mathcal{F}}(\mathfrak{h})\|. \end{aligned} \tag{5}$$

Using the inequalities (4) and (5), we obtain

$$\sum_{n\in\mathbb{N}} \|\mathcal{G}_n(\mathfrak{h})\|^2 = \|\mathcal{T}_{\mathcal{G}}(\mathfrak{h})\|^2 \geq \left(\frac{(1-\alpha)A(\sqrt{B})^{-1} - \gamma}{1+\beta}\right)^2 \|\mathfrak{h}\|^2. \tag{6}$$

From (3) and (6), we conclude that $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with lower and upper frame bounds $(1-\alpha)A((\sqrt{B})^{-1} - \gamma)^2/(1+\beta)^2$ and $((1+\alpha)\sqrt{B}+\gamma)^2/(1-\beta)^2$, respectively. □

In the following result, we prove that whenever the difference between an OPV- frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and a sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is sufficiently small, then the sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is an OPV-frame provided it satisfy certain conditions.

Theorem 2 *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with lower and upper frame bounds A and B, respectively. Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(\mathcal{H},\mathcal{K})$ be a sequence of operators and let α, β and $\gamma > 0$ be such that for all finite subset $\mathcal{I} \subset \mathbb{N}$ and for each $\mathfrak{h} \in \mathcal{H}$*

$$\begin{aligned} \Big\|\sum_{i\in\mathcal{I}} \Big(\mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h}) - \mathcal{G}_i^*\mathcal{G}_i(\mathfrak{h})\Big)\Big\| &\leq \alpha\Big\|\sum_{i\in\mathcal{I}} \mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h})\Big\| + \beta\Big\|\sum_{i\in\mathcal{I}} \mathcal{G}_i^*\mathcal{G}_i(\mathfrak{h})\Big\| \\ &\quad + \gamma\Big(\sum_{i\in\mathcal{I}} \|\mathcal{F}_i(\mathfrak{h})\|^2\Big)^{\frac{1}{2}}, \end{aligned} \tag{7}$$

where $0 \leq \max\{\alpha + \gamma\sqrt{B}A^{-1}, \beta\} < 1$. Then $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with lower and upper frame bounds $(1-(\alpha + \gamma\sqrt{B}A^{-1})A)/(1+\beta)$ and $((1+\alpha)B + \gamma\sqrt{B})/(1-\beta)$, respectively.

Proof. Let $\mathcal{S}_{\mathcal{F}}$ be the frame operator of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Then using the condition (7), for $\mathfrak{h} \in \mathcal{H}$, we can write

$$\begin{aligned} \Big\|\sum_{i\in\mathcal{I}} \mathcal{G}_i^*\mathcal{G}_i(\mathfrak{h})\Big\| &\leq \Big\|\sum_{i\in\mathcal{I}} \Big(\mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h}) - \mathcal{G}_i^*\mathcal{G}_i(\mathfrak{h})\Big)\Big\| + \Big\|\sum_{i\in\mathcal{I}} \mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h})\Big\| \\ &\leq (1+\alpha)\Big\|\sum_{i\in\mathcal{I}} \mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h})\Big\| + \beta\Big\|\sum_{i\in\mathcal{I}} \mathcal{G}_i^*\mathcal{G}_i(\mathfrak{h})\Big\| \\ &\quad + \gamma\Big(\sum_{i\in\mathcal{I}} \|\mathcal{F}_i(\mathfrak{h})\|^2\Big)^{1/2}. \end{aligned}$$

Hence,

$$\Big\|\sum_{i\in\mathcal{I}}\mathcal{G}_i^*\mathcal{G}_i(\mathfrak{h})\Big\| \leq \frac{(1+\alpha)\|\sum_{i\in\mathcal{I}}\mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h})\|+\gamma\Big(\sum_{i\in\mathcal{I}}\|\mathcal{F}_i(\mathfrak{h})\|^2\Big)^{1/2}}{1-\beta}.$$

Also, we have

$$\begin{aligned}\Big\|\sum_{i\in\mathcal{I}}\mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h})\Big\| &= \sup_{\|\mathfrak{f}\|=1}\Big|\Big\langle\sum_{i\in\mathcal{I}}\mathcal{F}_i^*\mathcal{F}_i(\mathfrak{h})\Big|\mathfrak{f}\Big\rangle\Big|\\ &\leq \Big(\sum_{i\in\mathcal{I}}\|\mathcal{F}_i(\mathfrak{h})\|^2\Big)^{1/2}\sup_{\|\mathfrak{f}\|=1}\Big(\sum_{i\in\mathcal{I}}\|\mathcal{F}_i(\mathfrak{f})\|^2\Big)^{1/2}\\ &\leq \sqrt{B}\Big(\sum_{i\in\mathcal{I}}\|\mathcal{F}_i(\mathfrak{h})\|^2\Big)^{1/2}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\Big\|\sum_{i\in\mathcal{I}}\mathcal{G}_i^*\mathcal{G}_i(\mathfrak{h})\Big\| &\leq \Big(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\Big)\Big(\sum_{i\in\mathcal{I}}\|\mathcal{F}_i(\mathfrak{h})\|^2\Big)^{1/2}\\ &\leq \Big(\frac{(1+\alpha)\sqrt{B}+\gamma}{1-\beta}\Big)\sqrt{B}\|\mathfrak{h}\|\\ &\leq \Big(\frac{(1+\alpha)B+\gamma\sqrt{B}}{1-\beta}\Big)\|\mathfrak{h}\|.\end{aligned}$$

Now, define $\mathcal{S}_\mathcal{G}:\mathcal{H}\to\mathcal{H}$ by

$$\mathcal{S}_\mathcal{G}(\mathfrak{h})=\sum_{n\in\mathbb{N}}\mathcal{G}_n^*\mathcal{G}_n(\mathfrak{h}),\qquad \mathfrak{h}\in\mathcal{H}.$$

It is easy to verify that $\mathcal{S}_\mathcal{G}$ is a well-defined bounded operator satisfying $\|\mathcal{S}_\mathcal{G}\|\leq((1+\alpha)B+\gamma\sqrt{B})/(1-\beta)$. Therefore,

$$\begin{aligned}\sum_{n\in\mathbb{N}}\|\mathcal{G}_n(\mathfrak{h})\|^2 &= \langle\mathcal{S}_\mathcal{G}(\mathfrak{h})|\mathfrak{h}\rangle\\ &\leq \|\mathcal{S}_\mathcal{G}\|\|\mathfrak{h}\|^2\\ &\leq \Big(\frac{(1+\alpha)B+\gamma\sqrt{B}}{1-\beta}\Big)\|\mathfrak{h}\|^2. \qquad (8)\end{aligned}$$

Using (7), we obtain

$$\|\mathcal{S}_\mathcal{F}(\mathfrak{h})-\mathcal{S}_\mathcal{G}(\mathfrak{h})\|\leq\alpha\|\mathcal{S}_\mathcal{F}(\mathfrak{h})\|+\beta\|\mathcal{S}_\mathcal{G}(\mathfrak{h})\|+\gamma\Big(\sum_{n\in\mathbb{N}}\|\mathcal{F}_n(\mathfrak{h})\|^2\Big)^{1/2},\quad \mathfrak{h}\in\mathcal{H}.$$

Then

$$\begin{aligned}
\|\mathfrak{h}-\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\| &\leq \alpha\|\mathfrak{h}\|+\beta\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|+\gamma\Big(\sum_{n\in\mathbb{N}}\|\mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|^2\Big)^{1/2}\\
&\leq \alpha\|\mathfrak{h}\|+\beta\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|+\gamma\Big(B\|\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|^2\Big)^{1/2}\\
&\leq \alpha\|\mathfrak{h}\|+\beta\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|+\gamma\Big(B\|\mathcal{S}_{\mathcal{F}}^{-1}\|^2\|\mathfrak{h}\|^2\Big)^{1/2}\\
&\leq \alpha\|\mathfrak{h}\|+\beta\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|+\gamma\Big(BA^{-2}\|\mathfrak{h}\|^2\Big)^{1/2}\\
&\leq (\alpha+\gamma\sqrt{B}A^{-1})\|\mathfrak{h}\|+\beta\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|\\
&< \|\mathfrak{h}\|+\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|.
\end{aligned}$$

Thus, $\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}$ is invertible, and hence, the operator $\mathcal{S}_{\mathcal{G}}$ is invertible. Moreover, we have

$$\begin{aligned}
\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\| &= \|\mathfrak{h}-(\mathfrak{h}-\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h}))\|\\
&\geq \|\mathfrak{h}\|-\|\mathfrak{h}-\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|\\
&\geq \|\mathfrak{h}\|-(\alpha+\gamma\sqrt{B}A^{-1})\|\mathfrak{h}\|-\beta\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\|.
\end{aligned}$$

This gives

$$\|\mathcal{S}_{\mathcal{G}}\mathcal{S}_{\mathcal{F}}^{-1}(\mathfrak{h})\| \geq \left(\frac{1-(\alpha+\gamma\sqrt{B}A^{-1})}{1+\beta}\right)\|\mathfrak{h}\|, \qquad \mathfrak{h}\in\mathcal{H},$$

and

$$\|\mathcal{S}_{\mathcal{F}}\mathcal{S}_{\mathcal{G}}^{-1}\| \leq \frac{1+\beta}{1-(\alpha+\gamma\sqrt{B}A^{-1})}.$$

Thus,

$$\|\mathcal{S}_{\mathcal{G}}^{-1}\| = \|\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{S}_{\mathcal{F}}\mathcal{S}_{\mathcal{G}}^{-1}\| \leq \frac{1+\beta}{A(1-(\alpha+\gamma\sqrt{B}A^{-1}))},$$

which yields

$$\left(\frac{(1-(\alpha+\gamma\sqrt{B}A^{-1}))A}{1+\beta}\right)I_{\mathcal{H}} \leq \mathcal{S}_{\mathcal{G}}.$$

Therefore, we obtain

$$\left(\frac{(1-(\alpha+\gamma\sqrt{B}A^{-1}))A}{1+\beta}\right)\|\mathfrak{h}\|^2 \leq \sum_{n\in\mathbb{N}}\|\mathcal{G}_n(\mathfrak{h})\|^2. \tag{9}$$

Hence, according to (8) and (9), $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and with lower and upper frame bounds $(1-(\alpha+\gamma\sqrt{B}A^{-1})A)/(1+\beta)$ and $((1+\alpha)B+\gamma\sqrt{B})/(1-\beta)$, respectively. $\square$

3 Dual operator valued frames

In this section, we discuss about the dual OPV-frames and prove some results related to their existence. We begin with the definition of a dual OPV-frame.

Definition 2 *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ with the frame operator $\mathcal{S}_{\mathcal{F}}$. Then $\{\mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\}_{n\in\mathbb{N}}$ is called a canonical dual OPV-frame of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. An OPV-frame $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is called a dual OPV-frame of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ if it satisfies*

$$\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{G}_n = I_{\mathcal{H}}.$$

Note that the canonical dual OPV-frame of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is also a dual OPV-frame of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

From Definition 3.1, one can easily obtain the relation between the analysis operators of an OPV-frame and its dual.

Lemma 1 *Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a dual OPV-frame of the OPV-frame $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, and let $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{G}}$ be the analysis operators of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$, respectively. Then*

(i) $\mathcal{T}_{\mathcal{G}}^*\mathcal{T}_{\mathcal{F}} = I_{\mathcal{H}}$ if and only if $\mathcal{T}_{\mathcal{F}}^*\mathcal{T}_{\mathcal{G}} = I_{\mathcal{H}}$.

(ii) $\sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{F}_n = I_{\mathcal{H}}$ if and only if $\sum_{n\in\mathbb{N}} \mathcal{F}_n^*\mathcal{G}_n = I_{\mathcal{H}}$.

Now, we present the precise form of the family of duals of an OPV-frame.

Lemma 2 *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ and analysis operator $\mathcal{T}_{\mathcal{F}}$. Then the dual OPV-frames of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ are precisely of the form $\{{}_{\ell^2}\langle e_n|\mathcal{V}\}_{n\in\mathbb{N}}$, where $\mathcal{V} : \mathcal{H} \to \mathcal{K} \otimes \ell^2$ is a bounded operator such that $\mathcal{V}^*\mathcal{T}_{\mathcal{F}} = I_{\mathcal{H}}$.*

Proof. Let $\mathcal{V} : \mathcal{H} \to \mathcal{K} \otimes \ell^2$ be a bounded operator such that $\mathcal{V}^*\mathcal{T}_{\mathcal{F}} = I_{\mathcal{H}}$. For $n \in \mathbb{N}$, let $\mathcal{G}_n := {}_{\ell^2}\langle e_n|\mathcal{V}$. Then, for $\mathfrak{h} \in \mathcal{H}$, we have

$$\begin{aligned}\mathfrak{h} = \mathcal{V}^*\mathcal{T}_{\mathcal{F}}(\mathfrak{h}) &= \mathcal{V}^*\Big(\sum_{n\in\mathbb{N}} \mathcal{F}_n(\mathfrak{h}) \otimes e_n\Big) \\ &= \sum_{n\in\mathbb{N}} \mathcal{V}^*|e_n\rangle_{\ell^2}\mathcal{F}_n(\mathfrak{h}) \\ &= \sum_{n\in\mathbb{N}} \mathcal{G}_n^*\mathcal{F}_n(\mathfrak{h}).\end{aligned}$$

Thus, $\{{}_{\ell^2}\langle e_n|\mathcal{V}\}_{n\in\mathbb{N}}$ forms a dual OPV-frame of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ for any bounded operator $\mathcal{V}$.

Conversely, let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a dual OPV-frame of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ and $\mathcal{T}_{\mathcal{G}}$ be its analysis operator. Then $\mathcal{G}_n = {}_{\ell^2}\langle e_n|\mathcal{T}_{\mathcal{G}}$, and using Lemma 2, we obtain $\mathcal{T}_{\mathcal{G}}^*\mathcal{T}_{\mathcal{F}} = I_{\mathcal{H}}$. □

Next, we establish the precise form of the left-inverses of the analysis operator of an OPV-frame.

Lemma 3 *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$, range in $\mathcal{K}$ with analysis operator $\mathcal{T}_\mathcal{F}$, and frame operators $\mathcal{S}_\mathcal{F}$. Then the left-inverses of $\mathcal{T}_\mathcal{F}$ are precisely of the form $\mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^* + \mathcal{V}(I_{\mathcal{K}\otimes\ell^2} - \mathcal{T}_\mathcal{F}\mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^*)$, where $\mathcal{V}:\mathcal{K}\otimes\ell^2\to\mathcal{H}$ is a bounded operator.*

Proof. Let $\mathcal{V}:\mathcal{K}\otimes\ell^2\to\mathcal{H}$ is a bounded operator. Then

$$\begin{aligned}\Big(\mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^* + \mathcal{V}(I_{\mathcal{K}\otimes\ell^2} - \mathcal{T}_\mathcal{F}\mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^*)\Big)\mathcal{T}_\mathcal{F} &= \mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^*\mathcal{T}_\mathcal{F} + \mathcal{V}\mathcal{T}_\mathcal{F} - \mathcal{V}\mathcal{T}_\mathcal{F}\mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^*\mathcal{T}_\mathcal{F}\\ &= I_\mathcal{H}.\end{aligned}$$

Conversely, if $\mathcal{U}$ is a left-inverse of $\mathcal{T}_\mathcal{F}$, i.e., $\mathcal{U}\mathcal{T}_\mathcal{F} = I_\mathcal{H}$, then

$$\mathcal{U} = \mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^* + \mathcal{U}(I_{\mathcal{K}\otimes\ell^2} - \mathcal{T}_\mathcal{F}\mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^*).$$

□

In the following result, we give another form of the duals of an OPV-frame in terms of a family of Bessel OPV-frames.

Theorem 3 *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$, analysis operator $\mathcal{T}_\mathcal{F}$, and frame operators $\mathcal{S}_\mathcal{F}$. Then the dual OPV-frames of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ are precisely of the form $\{\mathcal{F}_n\mathcal{S}_\mathcal{F}^{-1} + \mathcal{E}_n - \sum_{i\in\mathbb{N}}\mathcal{F}_n\mathcal{S}_\mathcal{F}^{-1}\mathcal{F}_i^*\mathcal{E}_i\}_{n\in\mathbb{N}}$, where $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is a Bessel OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$.*

Proof. Let $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ be a Bessel OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. For $n\in\mathbb{N}$, write $\mathcal{G}_n := \mathcal{F}_n\mathcal{S}_\mathcal{F}^{-1} + \mathcal{E}_n - \sum_{i\in\mathbb{N}}\mathcal{F}_n\mathcal{S}_\mathcal{F}^{-1}\mathcal{F}_i^*\mathcal{E}_i$. Then

$$\sum_{n\in\mathbb{N}}\mathcal{F}_n^*\mathcal{G}_n = \sum_{n\in\mathbb{N}}\Big(\mathcal{F}_n^*\mathcal{F}_n\mathcal{S}_\mathcal{F}^{-1} + \mathcal{F}_n^*\mathcal{E}_n - \sum_{i\in\mathbb{N}}\mathcal{F}_n^*\mathcal{F}_n\mathcal{S}_\mathcal{F}^{-1}\mathcal{F}_i^*\mathcal{E}_i\Big) = I_\mathcal{H}.$$

Conversely, let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a dual OPV-frame of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Then, by Theorem 2, there exists some bounded operator $\mathcal{V}:\mathcal{H}\to\mathcal{K}\otimes\ell^2$ such that for each $n\in\mathbb{N}$, $\mathcal{G}_n = {}_{\ell^2}\langle e_n|\mathcal{V}$ and $\mathcal{V}^*\mathcal{T}_\mathcal{F} = I_\mathcal{H}$. This implies that $\mathcal{V}^*$ is a left-inverse of $\mathcal{T}_\mathcal{F}$ and hence by Lemma 3, there exists some bounded operator $\mathcal{W}:\mathcal{K}\otimes\ell^2\to\mathcal{H}$ such that

$$\mathcal{V}^* = \mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^* + \mathcal{W} - \mathcal{W}\mathcal{T}_\mathcal{F}\mathcal{S}_\mathcal{F}^{-1}\mathcal{T}_\mathcal{F}^*.$$

For each $n\in\mathbb{N}$, let $\mathcal{E}_n = {}_{\ell^2}\langle e_n|\mathcal{W}^*$. Then it is easy to verify that

$$\sum_{n\in\mathbb{N}}\mathcal{E}_n^*\mathcal{E}_n = \mathcal{W}\Big(\sum_{n\in\mathbb{N}}|e_n\rangle_{\ell^2\,\ell^2}\langle e_n|\Big)\mathcal{W}^* = \mathcal{W}\mathcal{W}^*.$$

Thus, $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ forms a Bessel OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. For $n \in \mathbb{N}$, we can write

$$\begin{aligned}\mathcal{G}_n = {}_{\ell^2}\langle e_n|\mathcal{V} &= {}_{\ell^2}\langle e_n|(\mathcal{T}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}}^{-1} + \mathcal{W}^* - \mathcal{T}_{\mathcal{F}}\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{T}_{\mathcal{F}}^*\mathcal{W}^*)\\ &= \mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1} + \mathcal{E}_n - \mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{T}_{\mathcal{F}}^*\mathcal{W}^*. \end{aligned} \tag{10}$$

Also, we have

$$\mathcal{T}_{\mathcal{F}}^*\mathcal{W}^*(\mathfrak{h}) = \mathcal{T}_{\mathcal{F}}^*\Big(\sum_{n\in\mathbb{N}}\mathcal{E}_n(\mathfrak{h})\otimes e_n\Big) = \sum_{n\in\mathbb{N}}\mathcal{F}_n^*\mathcal{E}_n(\mathfrak{h}), \qquad \mathfrak{h}\in\mathcal{H}.$$

Hence, using (10), we obtain

$$\mathcal{G}_n = \mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1} + \mathcal{E}_n - \sum_{i\in\mathbb{N}}\mathcal{F}_n\mathcal{S}_{\mathcal{F}}^{-1}\mathcal{F}_i^*\mathcal{E}_i, \qquad n\in\mathbb{N}.$$

□

Finally, we prove that for every Riesz OPV-frame, there exists a unique Riesz OPV-frame as its dual.

Theorem 4 *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$. Then, there exists a unique Riesz OPV-frame $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ such that*

$$\sum_{n\in\mathbb{N}}\mathcal{G}_n^*\mathcal{F}_n = I_{\mathcal{H}}.$$

Proof. By hypotheses, there exists a bounded invertible operator $\mathcal{U}$ on $\mathcal{H}$ and an orthonormal OPV-frame $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$, $\mathcal{F}_n = \mathcal{E}_n\mathcal{U}$. Then

$$\begin{aligned} I_{\mathcal{H}} = \mathcal{U}^{-1}\mathcal{U} &= \mathcal{U}^{-1}\Big(\sum_{n\in\mathbb{N}}\mathcal{E}_n^*\mathcal{E}_n\Big)\mathcal{U}\\ &= \sum_{n\in\mathbb{N}}\mathcal{U}^{-1}\mathcal{E}_n^*\mathcal{F}_n\\ &= \sum_{n\in\mathbb{N}}(\mathcal{E}_n(\mathcal{U}^{-1})^*)^*\mathcal{F}_n. \end{aligned}$$

For each $n\in\mathbb{N}$, putting $\mathcal{G}_n := \mathcal{E}_n(\mathcal{U}^{-1})^*$, we obtain

$$\sum_{n\in\mathbb{N}}\mathcal{G}_n^*\mathcal{G}_n = \mathcal{U}^{-1}\Big(\sum_{n\in\mathbb{N}}\mathcal{E}_n^*\mathcal{E}_n\Big)(\mathcal{U}^{-1})^* = \mathcal{U}^{-1}(\mathcal{U}^{-1})^*.$$

Define $\mathcal{S}_{\mathcal{G}} := \mathcal{U}^{-1}(\mathcal{U}^{-1})^*$. Then one may verify that $\mathcal{S}_{\mathcal{G}}$ is a bounded invertible operator, and for $n, m\in\mathbb{N}$, we have

$$\mathcal{G}_n\mathcal{S}_{\mathcal{G}}^{-1}\mathcal{G}_m^* = \mathcal{E}_n(\mathcal{U}^{-1})^*(\mathcal{U}^*\mathcal{U})\mathcal{U}^{-1}\mathcal{E}_m^* = \mathcal{E}_n\mathcal{E}_m^* = \delta_{m,n}I_{\mathcal{K}}.$$

Hence, $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ forms a Riesz OPV-frame for $\mathcal{H}$ with range in $\mathcal{K}$ such that $\sum_{n\in\mathbb{N}}\mathcal{G}_n^*\mathcal{F}_n = I_{\mathcal{H}}$. □

4 Conclusion

In many branches of mathematics, the theory of perturbations is an essential tool. In reality, examining a system's stability in the face of numerous disturbances is necessary. Keeping this in mind, we discuss the stability of OPV-frames under various perturbations and show that they are both perturbation-resistant whenever there is an adequate gap in size between two OPV-frames. Additionally, it is shown that whenever the analysis operator of one OPV-frame and the analysis operator of another OPV-frame are sufficiently close to one another, an OPV-frame is created. The frame elements and its dual frame serve as the foundation for the potential of reconstructing every element in the space.

The duality of OPV-frames is discussed along with the relationship between an OPV-frame's analysis operators and its dual. Also, a different way to express an OPV-frame's duals in terms of a family of Bessel OPV-frames is given. Finally, we demonstrate that every Riesz OPV-frame has a unique dual Riesz OPV-frame.

Acknowledgements. The author sincerely acknowledges the efforts of the anonymous referee(s) and the editing department of the journal for carefully reading the manuscript and providing fruitful suggestions for the improvement of the manuscript.

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Please, cite to this paper as published in
Armen. J. Math., V. **16**, N. 6(2024), pp. 1–13
https://doi.org/10.52737/18291163-2024.16.6-1-13