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Notes on Ergodic Theory in Infinite Measure Spaces

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Abstract. This article is concerned with ergodic theory for transformations which preserve an infinite measure. In the first part we present an overview of the invertible case with a focus on weakly wandering sequences and their applications to number theory as it has developed over the last fifty years. The second part presents a very preliminary investigation into extending weakly wandering sequences to the non-invertible case. This consists primarily of a few examples which illustrate the complexities which arise in the non-invertible case.

Key Words: infinite measure space, ergodic, exact, full measure preserving irreversible (many-to-one) transformation, weakly wandering, exhaustive set and sequence
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Introduction

Let $(X, \mathcal{B}, m)$ be a σ-finite Lebesgue measure space. A subset $A \subset X$ is *measurable* if $A \in \mathcal{B}$. $(X, \mathcal{B}, m)$ is said to be a *finite* or *infinite* measure space if $m(X) < \infty$ or $m(X) = \infty$ respectively. In case $m(X) = \infty$ we require X to be the countable union of sets of finite measure. For simplicity one may consider X to be the unit interval or the whole real line with ordinary Lebesgue measure defined on them. All the sets we consider are assumed to be measurable, and statements are meant to be understood as modulo sets of measure zero even when explicitly not stated.

A *measure preserving* transformation T is a (not necessarily invertible) map defined on X into itself such that $m(T^{-1}A) = m(A)$ for all sets $A \in \mathcal{B}$. In this case we also say m is an invariant measure for T. Two measures m and

μ defined on a measurable space $(X, \mathcal{B})$ are *equivalent* $(m \sim \mu)$ if $m(A) = 0$ if and only if $\mu(A) = 0$.

Ergodic theory is the study of properties of measure preserving transformations T defined on a measure space $(X, \mathcal{B}, m)$. A subset $E \subset X$ is an *invariant* set for T if $T^{-1}E = E$. It is clear that if the space X is the union of two disjoint invariant subsets of positive measure then the study of the properties of a transformation T on the space X reduces to the study of its properties on these invariant subsets. The significant transformations then are the ones defined on spaces that do not decompose into two disjoint invariant subsets of positive measure. These transformations are usually called ergodic. We define the following:

T is an *ergodic* transformation if the only T-invariant sets are trivial, i.e.
$$T^{-1}A = A \implies m(A) = 0 \text{ or } m(X - A) = 0.$$

Birkhoff's Individual Ergodic Theorem is a well known and fundamental theorem. It was published in 1932 and was instrumental in the development of ergodic theory. We state it below without proof, see [4].

Theorem 1 *(Individual Ergodic Theorem) Let T be a measure preserving (not necessarily invertible) transformation defined on the σ-finite measure space $(X, \mathcal{B}, m)$, and let $f \in L_1(X, \mathcal{B}, m)$ be an integrable function, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} f(T^i x) = f^*(x)$$

exists a.e. The limit function f^* is integrable $(f^* \in L_1(X, \mathcal{B}, m))$, and invariant $(f^*(Tx) = f^*(x)$ a.e.). If $m(X) < \infty$ then $\int_X f^* dm(x) = \int_X f\, dm(x)$.

In the next section we discuss invertible (1-to-1 and onto) ergodic transformations. We present two examples of ergodic transformations that preserve a finite measure then concentrate on ergodic transformations that preserve an infinite measure. In the last section of this article we discuss the complications that are present when the transformations are not invertible.

1 Invertible Ergodic Transformations

In this section we consider invertible ergodic transformations and discuss properties of ergodic transformations that preserve an infinite measure.
An important property of ergodic transformations is the following:

Suppose $m \sim \mu$ are two equivalent and invariant measures for an ergodic transformation T then $m = c\,\mu$ for some constant $c \neq 0$.

The proof is an immediate consequence of the definition of T being ergodic and the fact that the Radon-Nikodym derivative of m with respect to μ is an invariant function of T.

Let us call an ergodic transformation that preserves a finite measure a *finite ergodic* transformation and one that preserves an infinite measure an *infinite ergodic* transformation. From the above follows that the collection of finite ergodic transformations and the infinite ones are mutually exclusive.

For ergodic transformations since the limit function f^* in theorem [1] is an integrable and invariant function it follows that it is a constant.

For finite ergodic transformations applying the above to the characteristic functions of the sets A and B we get:

$$\text{for any two sets } \ A, B \in \mathcal{B} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}A \cap B) = m(A)m(B). \tag{1}$$

For infinite ergodic transformations since f^* is a constant and an integrable function with $m(X) = \infty$ it follows that $f^* = 0$. Applying this to the characteristic functions of sets A and B of finite measure and using the dominated convergence theorem we conclude:

$$m(A), m(B) < \infty \implies \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}A \cap B) = 0. \tag{2}$$

The theory of finite ergodic transformations is well developed with many applications to dynamical systems, coding theory and additive number theory. When the invariant measure is finite a number of powerful techniques from functional analysis and probability theory apply. The literature is rich with many of the important and powerful results that have been obtained during the past several decades.

First we mention a few properties of finite ergodic transformations and present two important examples. Then we discuss briefly properties of infinite ergodic transformations. These properties are treated more fully and presented in more detail in [6].

For finite ergodic transformations property (1) is a significant property. It implies that finite ergodic transformations are "mixing" on the average. This was an exciting result when it was noticed, and quickly property (1) was strengthened to:

$$A, B \in \mathcal{B} \implies \lim_{n\to\infty} m(T^n A \cap B) = m(A)m(B). \tag{3}$$

Transformations satisfying property (3) were called "*strongly mixing*" or at times simply"*mixing*" transformations. Soon several important examples and many significant results followed. We present two important examples of finite ergodic transformations.

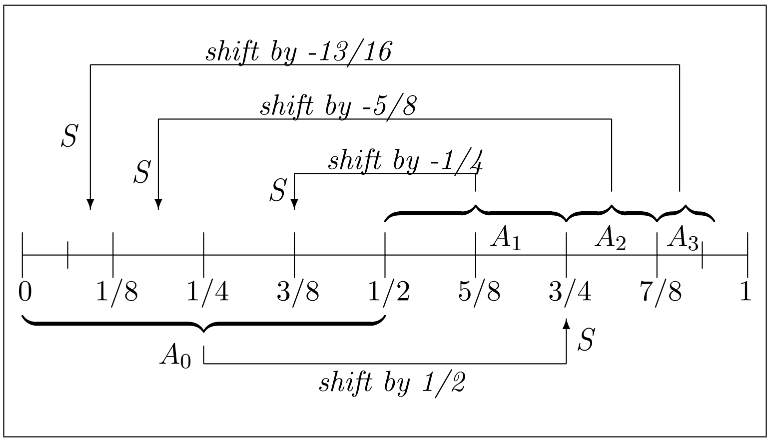


Figure 1: The transformation S of Example 1 on the space X_0.

Example 1 *Let* $(X_0, \mathcal{B}_0, m_0)$ *be a finite measure space where*

$$X_0 = \{x \in \mathbb{R},\ 0 < x < 1,\ x \neq \text{dyadic rational}\},$$

$\mathcal{B}_0$ *is the* σ*-field of all Lebesgue measurable subsets of* X_0 *and* m_0 *is the ordinary Lebesgue measure on* $\mathcal{B}_0$ *with* $m_0(X_0) = 1$*. A dyadic rational in (0,1] is a rational number of the form* $k/2^n$ *for* $n = 1, 2, \cdots$ *and* $0 \leq k \leq n$*. For* $n \geq 0$ *we let*

$$A_n = \{x \in X_0 :\ 1 - 1/2^n < x < 1 - 1/2^{n+1}\}$$

be the dyadic interval of length $1/2^{n+1}$*. We define the transformation* S *on* X_0 *onto itself as follows:*

$$Sx = x - \left(1 - 3/2^{n+1}\right) \quad if \quad x \in A_n, \quad n \geq 0\ .$$

Then S *is an ergodic measure preserving transformation defined on the measure space* $(X_0, \mathcal{B}_0, m_0)$ *with* $m_0(X_0) = 1$ *. In the literature this transformation is usually referred to as the Von Neumann adding machine transformation (or the dyadic odometer). Figure 1 describes the transformation* S *acting on* X_0*.*

Example 2 *We let* $I = [0, 1)$ *be the unit interval and* S *the measure preserving transformation defined on the unit square* $I^2 = I \times I$ *by*

$$S(x, y) = \begin{cases} (2x, y/2) & \text{if } \ 0 \leq x < \frac{1}{2}, \\ (2x - 1, (y + 1)/2) & \text{if } \ \frac{1}{2} \leq x < 1\,. \end{cases}$$

Often the transformation S *is referred to as the baker's transformation defined on the unit square. It is a finite ergodic transformation satisfying property (3), in other words it is strongly mixing.*

We introduce the following

Definition 1 Let T be a measurable transformation defined on $(X, \mathcal{B}, m)$.

- A set $A \in \mathcal{B}$ is a *wandering* set (for T) if $m(A) > 0$ and $T^i A \cap T^j A = \emptyset$ for $i, j \in \mathbb{Z},\ i \neq j$.
- A set $A \in \mathcal{B}$ is a *weakly wandering* (*ww*) set (for T) if $m(A) > 0$ and for an infinite set of integers $\{n_i : i \geq 0\}$ $T^{n_i} A \cap T^{n_j} A = \emptyset$ for $i \neq j$.
- A set $A \in \mathcal{B}$ is an *exhaustive weakly wandering* (*eww*) set (for T) if $m(A) > 0$ and for an infinite set of integers $\{n_i : i \geq 0\}$ we have $X = \bigcup_{i=0}^{\infty} T^{n_i} A(\text{disj})$.

  Often we say A is a *ww, or eww* set (for T) (with the sequence $\{n_i\}$), or at times say $\{n_i\}$ is a *ww or eww* sequence (for T) (with the set A).

As we saw above the collection of infinite ergodic transformations are disjoint from the set of finite ones. In contrast to the finite case the theory of infinite ergodic transformations is in a primitive state, and hardly any of the properties of finite ergodic transformations generalize to the infinite ones. We introduce several sequences associated with infinite ergodic transformations and discuss several important features of infinite ergodic transformations that show how they differ in fundamental ways from the finite ones.

An important property that describes an ergodic transformation T in general is the following:

$$m(A) > 0,\ m(B) > 0 \implies \exists\ n > 0 \text{ such that } m(T^n A \cap B) > 0.$$

This says that given two sets A and B of positive measure there exists an integer $n > 0$ such that the n-th iterate of the images of the set A under the action of T intersects B in a set of positive measure. It is not difficult to show that all ergodic transformations defined on a non-atomic measure space do not possess wandering sets. In fact they also possess a slightly stronger property than recurrence. Namely, for any set A of positive measure and for almost every point $x \in X$ there exist infinitely many integers $n > 0$ such that $T^n x \in A$. All of these features misled several mathematicians working in ergodic theory to a faulty conclusion that ergodic transformations in general possess some mixing feature that precludes the existence of *ww* sequences for any ergodic transformation in general, see [13]. However once it is realized that *ww* sequences exist for all infinite ergodic transformations it is a simple exercise to show their existence. Actually a bit more follows readily from property (2) as seen in the next proposition.

Proposition 1 *Let T be an infinite ergodic transformation, and let C be a measurable set with $0 < m(C) < \infty$. Then for any $0 < \epsilon < m(C)$ there exists a subset $C' \subset C$, $m(C') < \epsilon$, and the set $W = C \setminus C'$ is a ww set.*

Proof. For an infinite ergodic transformation T we use property (2) and conclude:

$$m(A) < \infty,\ m(B) < \infty \implies \liminf_{n\to\infty} m(T^n A \cap B) = 0. \tag{4}$$

For $i = 1, 2, \cdots$ we let $\epsilon_i = \epsilon/2^i$. We let $n_0 = 0$, and using property (4) choose an integer $n_1 > 0$ such that $m\big(T^{n_1}[T^{-n_0}C] \cap C\big) < \epsilon_1$.
We note that $m(T^{-n_0}C \cup T^{-n_1}C) < \infty$ and using property (4) choose an integer $n_2 > n_1$ such that $m\big(T^{n_2}[T^{-n_0}C \cup T^{-n_1}C] \cap C\big) < \epsilon_2$. We continue by induction.
Having chosen the integers $0 = n_0 < n_1 < \cdots < n_{i-1}$ we use property (4) and choose an integer $n_i > n_{i-1}$ such that $m\big(T^{n_i}[\bigcup_{j=0}^{i-1} T^{-n_j}C] \cap C\big) < \epsilon_i$.

Let $C' = \big(\bigcup_{i=1}^{\infty}\bigcup_{j=0}^{i-1} T^{n_i-n_j}C\big) \cap C$ and $W = C \setminus C'$.

Then $m(C') \leq \sum_{i=1}^{\infty} \epsilon_i = \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon$, and $m(W) \geq m(C) - \epsilon > 0$.
We have $W \subset C$ and, for $i > j$, $T^{n_i-n_j}W \cap C \subset C'$. This says:
$T^{n_i-n_j}W \cap W = \emptyset$ for $i > j$, or $T^{n_i}W \cap T^{n_j}W = \emptyset$ for $i > j$. □

The above says that for any infinite ergodic transformation a small piece can be removed from any set of finite measure and the rest becomes a *ww* set. This implies an abundance of *ww* sets for every infinite ergodic transformation. Our next task was to actually see an example of a *ww* set for some infinite ergodic transformation. There exist several examples of infinite ergodic transformations in the literature. Yet all the attempts to locate a concrete example of a *ww* set for any of these transformations failed.

Next we present an infinite ergodic transformation and a specific *ww* set for it. This example first appeared in [11].

Example 3 *We consider the finite ergodic transformation S defined on the measure space $(X_0, \mathcal{B}_0, m_0)$ of Example 1.*
For $n \geq 0$ we recall the dyadic intervals

$$A_n = \{x : x \in X_0,\ 1 - 1/2^n < x < 1 - 1/2^{n+1}\}$$

of length $1/2^{n+1}$.
Let us put $f(0) = 0$ and $f(n) = 2 + 2^3 + \cdots + 2^{2n-1}$ for $n \geq 1$.
For $n \geq 1$ we consider $f(n) + 1$ mutually disjoint copies of A_n; namely $A_n^0 = A_n,\ A_n^1,\ \ldots,\ A_n^{f(n)}$. We denote by the same letter R all the following isomorphisms

$$RA_n^i = A_n^{i+1},\quad i = 0, 1, \ldots, f(n) - 1;\ n \geq 1.$$

Next we consider the measure space $(X, \mathcal{B}, m)$

$$X = \bigcup_{n=0}^{\infty} \bigcup_{i=0}^{f(n)} A_n^i (\text{disj}),$$

where $\mathcal{B}$ *is the* σ*-field of all "Lebesgue measurable" subsets of* X*, and* m *is the corresponding measure on* $(X, \mathcal{B})$ *.*
We now define the transformation T *from* X *onto itself as follows:*

$$Tx = \begin{cases} Rx & \text{if } \ x \in \mathrm{A_n^i} \quad \text{for } \ \mathrm{i} = 0, 1, \ldots, \mathrm{f(n)} - 1 \text{ and } \mathrm{n} \geq 0, \\ SR^{-f(n)}x & \text{if } \ x \in \mathrm{A_n^{f(n)}} \quad \text{for } \ \mathrm{n} \geq 0. \end{cases} \tag{5}$$

It follows that T *is an ergodic measure preserving transformation defined on the* σ*-finite measure space* $(X, \mathcal{B}, m)$ *with* $m(X) = \infty$.
Next we consider the following subsets of X*:* $B_0 = X_0$,

$$B_j = \bigcup_{n=j}^{\infty} \bigcup_{i=f(j-1)+1}^{f(j)} A_n^i (\text{disj}) \text{ for } j \geq 1, \text{and } \ \mathrm{C_k} = \bigcup_{\mathrm{j=0}}^{\mathrm{k}} \mathrm{B_j}(\text{disj}) \text{ for } \mathrm{k} \geq 0\,.$$

We observe that the sets B_j *and* C_k *have the following properties:*

$$m(B_j) = [f(j) - f(j-1)] \sum_{n=j}^{\infty} 1/2^{n+1} = 2^{j-1} \quad \text{for} \quad j \geq 1,$$

$$m(C_k) = 1 + \sum_{j=1}^{k} m(B_j) = 2^k \text{ for } \ k \geq 0, \quad X = \bigcup_{j=0}^{\infty} B_j\, (\text{disj}), \text{and}$$

$$C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_k \subset \cdots \quad \longrightarrow X\,. \tag{6}$$

Also from the above definition (5) of T *we get*

$$T^n(C_k) \subset C_{k+l} \quad \textit{for} \quad 0 \leq n \leq f(k+l) - f(k), \ k \geq 0, \ l \geq 0\,. \tag{7}$$

We put $n_0 = 0$ *and define for* $i = 1, 2, \ldots$

$$n_i = \epsilon_0 2^1 + \epsilon_1 2^3 + \cdots + \epsilon_k 2^{2k+1} \quad \textit{if} \quad i = \epsilon_0 2^0 + \epsilon_1 2^1 + \cdots + \epsilon_k 2^k, \tag{8}$$

where $\epsilon_j = 0$ *or* 1 *for* $0 \leq j \leq k$*. The members of the above sequence are*

$$\mathbb{A} = 0,\ 2,\ 8,\ 10,\ 32,\ 34,\ 40,\ 42,\ \ldots. \tag{9}$$

We shall return to this sequence in the next Section (see 2.3.1).
We observe the following important relations:

$$B_j = \bigcup_{i=2^{j-1}}^{2^j-1} T^{n_i} X_0\, (\text{disj}) \quad \text{for} \quad \mathrm{j} \geq 1.$$

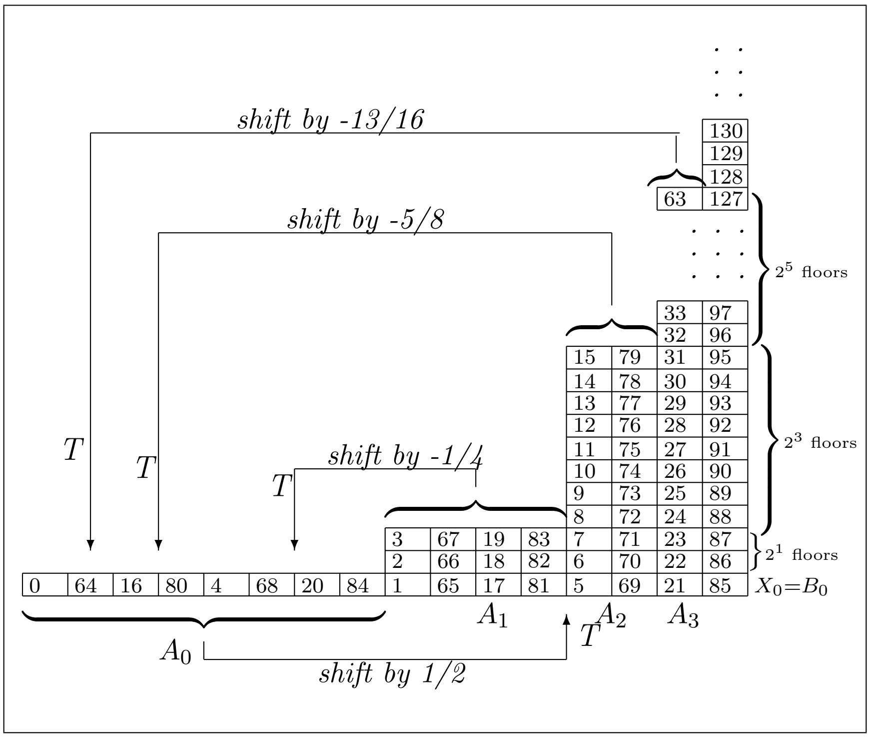


Figure 2: The transformation T of Example 3 on the space X.

$$\textit{Hence}\quad C_k = \bigcup_{i=0}^{2^k-1} T^{n_i} X_0 \text{ (disj) } \textit{for } k \geq 0, \quad \textit{and} \quad X = \bigcup_{i=0}^{\infty} T^{n_i} X_0 \text{ (disj)}. \quad (10)$$

From the above discussion follows that for the infinite ergodic transformation T defined by (5) relation (10) holds for the set X_0 of measure 1 with the sequence $\{n_i : i \geq 0\}$ defined by (8). In other words, $\{n_i : i \geq 0\}$ is an eww sequence for the transformation T with the set X_0 of measure 1.
Figure 2 describes the transformation T acting on X.

The presence of *ww* sets was first noticed for all infinite ergodic transformations while studying conditions for the existence of a finite invariant measure in [8] and [10]. This was a surprising event at the time, and the fact that an infinite ergodic transformation may possess an *eww* sequence was equally surprising. The existence of an *eww* set of finite measure for any infinite ergodic transformation on the other hand was unexpected. It was in [11], Example 3 above, that an *eww* set of finite measure first appeared.

Let us call an infinite ergodic transformation which possesses an *eww* set of finite measure a transformation of *finite type*, see [7]. In time it was shown that these transformations possess several interesting properties.
For example, if T is an ergodic transformation of finite type then the centralizer of T,

$$\mathcal{C}(T) = \{\text{nonsingular transformations } S : ST = TS\},$$

consists of measure preserving transformations only. The proof follows from what we saw earlier. Namely, two equivalent measures invariant for the same ergodic transformation are constant multiples of each other. We complete the discussion of transformations of finite type with the following proposition.

Proposition 2 *Let T be an ergodic transformation of finite type. Let $\{n_i : i \geq 1\}$ be an eww sequence for T with a set W of finite measure. For a set $V \in \mathcal{B}$ let us consider the following three statements:*

1. *V is an exhaustive set under $\{n_i\}$ $\quad (X = \bigcup_{i=1}^{\infty} T^{n_i} V)$.*
2. *V is a ww set under $\{n_i\}$.*
3. *$m(V) = m(W)$.*

Then any two of the above statements together imply the third.

The proof of the above proposition is not complicated and we leave it as an interesting exercise for the reader, see also [7]. There are examples of infinite ergodic transformations that are not of finite type. They too have interesting properties and are discussed in [9] in more detail.

Next we mention a few interesting properties of *ww* and *eww* sequences.

Let us denote the cardinality of a set A by $|A|$ (= the number of elements in A). For a subset $\mathbb{E} \subset \mathbb{Z}$ let us denote by $\mathrm{Orb_{T,\mathbb{E}}(x)}$ the set $\{T^n x : n \in \mathbb{E}\}$. It was briefly mentioned earlier that an ergodic transformation T in general possesses the following property:

For any set A of positive measure and $|\mathrm{Orb_{T,\mathbb{Z}}(x) \cap A}| = \infty$ for a.a. $x \in X$.

In other words, almost all point $x \in X$ visit every set of positive measure infinitely often under images of T.

Let us call an infinite set of integers $\mathbb{D} = \{n_i\}$ a *dissipative* sequence for a transformation T if for every set A of finite measure $|\mathrm{Orb_{T,\mathbb{D}}(x) \cap A}| < \infty$ for almost all points $x \in X$. This says that almost every point $x \in X$ visits every set of finite measure only finitely often under the n-th images of T for $n \in \mathbb{D}$. It is an interesting fact that every *ww* sequence for an infinite ergodic transformation happens to be a dissipative sequence. The proof is simple and uses properties of the characteristic function of the set A of finite measure.

Related to the *ww* sequences of a transformation there are other interesting sequences of integers that may (or may not) exist for some infinite ergodic transformations. We introduce the following:

Infinite set of integers $\{r_i\}$ *is a* recurrent *sequence for an infinite ergodic transformation* T *if* $|\{r_i\} \cap \{n_i\}| < \infty$ *for all* ww *sequence* $\{n_i\}$ *of* T.

Using similar arguments as in the proof of Proposition 1 it is possible to give a description of recurrent sequences without making reference to the *ww* sequences of a transformation T. Namely,

an infinite sequence of integers $\{r_i\}$ *is a recurrent sequence for a transformation* T *if and only if there exists a set* A *of finite measure such that* $\liminf_{i\to\infty} m(T^{r_i}A \cap A) > 0$.

We note that the infinite ergodic transformation of example 3 possesses recurrent sequences. It is not difficult to observe that the sequence of integers $\{r_i = 2^{2i} : i = 0, 1, 2, \cdots\}$ satisfies $m(T^{r_i}X_0 \cap X_0) = \frac{1}{2}$ for all $i \geq 0$, and this implies that the sequence $\{r_i\}$ is a recurrent sequence for T. In fact, for this example it is possible to describe all the recurrent sequences for it, see [11].

We present our final example of this section. This is an example of an infinite ergodic transformation without recurrent sequences.

Example 4 *We let* $I = \{x : 0 \leq x < 1\}$ *be the unit interval and consider the measure preserving transformation* S *of example 2 defined on* $I \times I$. *It is also possible to represent* S *as follows:*
First we express the points $x \in I$ *and* $y \in I$ *by their binary representations* $x = .\epsilon_0\epsilon_1\epsilon_2\cdots$ *and* $y = .\epsilon_{-1}\epsilon_{-2}\cdots$ *where we choose to write the* $\epsilon_i = \pm 1$ *instead of the usual choice of* $\epsilon_i = 0$ *or* 1. *Next we represent the point* $(x, y) \in I \times I$ *as a two sided point* $z = (\cdots\epsilon_{-2}\epsilon_{-1} \,.\, \epsilon_0\epsilon_1\epsilon_2\cdots)$ *in the product space* $Z = \prod_{-\infty}^{\infty}\{-1, 1\}$. *The dot "." is for centralizing the representation of* z. *The transformation* S *is then represented as the shift transformation on* Z *as follows:*
for $z = (\cdots\epsilon_{-2}\epsilon_{-1} \,.\, \epsilon_0\epsilon_1\epsilon_2\cdots)$ *we let* $Sz = (\cdots\epsilon_{-2}\epsilon_{-1}\epsilon_0 \,.\, \epsilon_1\epsilon_2\cdots)$.
We consider the space $X = \bigcup_{n=-\infty}^{\infty}(n, Z)$ *and define the transformation* T *on* X *as follows:*

$$\text{for a point } (n, z) \in X,\ n \in \mathbb{Z} \text{ and } z = (\cdots\epsilon_{-2}\epsilon_{-1} \,.\, \epsilon_0\epsilon_1\epsilon_2\cdots) \in Z \text{ we let } T(n, z) = (n + \epsilon_0, Sz).$$

The transformation T *may be considered a random walk on the integers* $\mathbb{Z}$. *Using standard arguments it is possible to show that* T *actually is an infinite ergodic transformation defined on* X.
We let $A = (0, I \times I)$. *Using mathematical induction it can be shown that* $m(T^nA \cap A) < \frac{1}{\sqrt{n}}$ *for all* $n > 0$. *This says* $\lim_{n\to\infty} m(T^nA \cap A) = 0$, *which implies* T *is an infinite ergodic transformation without recurrent sequences.*

Let us call an *eww* sequence $\{n_i\}$ a *hereditary eww* sequence for a transformation T if every infinite subset $\{n'_i\} \subset \{n_i\}$ is an *eww* sequence for

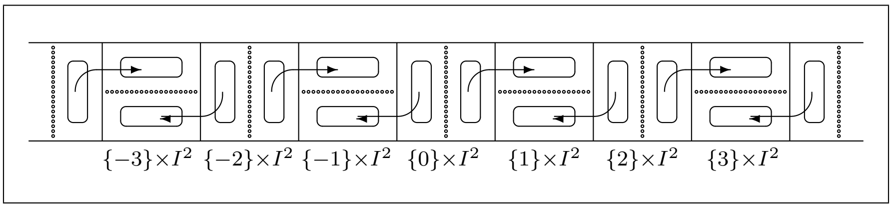


Figure 3: The transformation T of Example 4 on the space $\mathbb{Z} \times I^2$, displayed for even numbered boxes $\{-2\} \times I^2$, $\{0\} \times I^2$, $\{2\} \times I^2$ only.

the same T. For some mathematicians working in ergodic theory it was not easy to predict the existence of *ww* sequences for any ergodic transformation. However, as we saw in Proposition 1, once the existence of *ww* sequences is suspected it is not too difficult to prove their existence. The situation for *eww* sequences is similar. In fact using property (2) the existence of hereditary *eww* sequences can be proven directly. The proof is a bit more intricate, and we refer to chapter 2 of [6] for a complete proof. Using the same methods in proving the existence of *eww* sequences it is also possible to prove the following fact which is far from being intuitive.

Proposition 3 *If* $\{n_i\}$ *is a* ww sequence for an egodic transformation T and $\{2n_i\}$ is also a ww sequence for T then there exists a subset $\{n_i'\} \subset \{n_i\}$ which is a hereditary eww sequence for the same transformation T.

We state next the following theorem about the existence of *ww* and *eww* sequences for infinite ergodic transformations without recurrent sequences.

Theorem 2 *Let T be an infinite ergodic transformation without recurrent sequences. Then there exists an increasing sequence $\{0 < N_1 < N_2 \cdots\}$ of integers satisfying the following:*
Let $\{0 < n_1 < n_2 \cdots\}$ be an increasing sequence of positive integers. Then

(I) $n_i - n_{i-1} \geq N_i$ *for* $i \geq 1 \implies \{n_i\}$ *is a* ww *sequence for* T.

(II) $n_i - 2n_{i-1} \geq N_i$ *for* $i \geq 1 \implies \{n_i\}$ *is an* eww *sequence for* T.

The proof of theorem 2 follows along similar lines to the proof of the existence of *ww* and *eww* sequences with a few additional arguments. For a complete proof see Chapter 3 of [6].

We complete this section with an application of Theorem 2 to a result in additive number theory. For any two subset $\mathbb{E}$ and $\mathbb{F}$ of $\mathbb{Z}$ let us denote by

$$\mathbb{E} + \mathbb{F} = \{a + b : a \in \mathbb{E}, \text{ and } b \in \mathbb{F}\},$$

and by

$$\mathbb{E} \oplus \mathbb{F} = \{a + b : a \in \mathbb{E},\ b \in \mathbb{F} \text{ such that if } a + b = a' + b' \text{ for } a, a' \in \mathbb{E} \text{ and } b, b' \in \mathbb{F}, \text{ then } a = a' \text{ and } b = b'\}.$$

We say the subset $\mathbb{E} \subset \mathbb{Z}$ *tiles* $\mathbb{Z}$ if there exists another subset $\mathbb{F} \subset \mathbb{Z}$ such that $\mathbb{E} \oplus \mathbb{F} = \mathbb{Z}$. Next we prove the following:

Every eww sequence $\{n_i\}$ for an infinite ergodic transformation tiles $\mathbb{Z}$.

Let $\mathbb{E} = \{n_i\}$ be an *eww* sequence for T with the set W. Let $n \in \mathbb{Z}$ be any integer and $x \in W$ any point. Let $\mathbb{F}(x) = \{n \in \mathbb{Z} : \ T^n x \in W\}$. Then $T^n x \in T^{n_i} W$ for a unique integer $n_i \in \mathbb{E}$ and $T^{n-n_i} x \in W$. This implies $n - n_i = b$ for an integer $b \in \mathbb{F}(x)$, which says $n = n_i + b$ with $n_i \in \mathbb{E}$, and $b \in \mathbb{F}(x)$. The uniqueness of the representation of n as $n_i + b$ follows from the fact that T is 1-to-1.

From Proposition 2 follows that if an increasing sequence of positive integers $\{n_i\}$ increases fast enough to ∞ then it tiles the integers $\mathbb{Z}$. The speed with which the sequence $\{n_i\}$ tends to infinity clearly depends on the transformation T. However this result prompts us to conjecture the following theorem for which there exists a proof in [5] that follows from ergodic theoretic arguments. We present a short algebraic proof that was communicated to us by James Schmerl [18].

Theorem 3 *Let $\mathbb{E} = \{0 = n_0 < n_1 < n_2 < \cdots\}$ be an increasing sequence of positive integers. If $\lim_{i\to\infty}(a_{n_i} - 2a_{n_{i-1}}) = \infty$ then A tiles $\mathbb{Z}$.*

Proof. Let us enumerate $\mathbb{Z} = \{z_0, z_1, z_2, \cdots\}$ and inductively define a sequence $\{0\} = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \mathbb{F}_2 \subset \cdots$ of finite subsets of $\mathbb{Z}$ as follows:
Assume $\mathbb{F}_i$ is defined. If $z_i \in \mathbb{E} + \mathbb{F}_i$ then let $\mathbb{F}_{i+1} = \mathbb{F}_i$. If $z_i \notin \mathbb{E} + \mathbb{F}_i$ then choose n_i large enough so that for all $j \geq n_i$, $\quad a_j - 2a_{j-1} > (|z_i| + \max(|b| : b \in \mathbb{F}_i)$, and let $\mathbb{F}_{i+1} = \mathbb{F}_i \cup \{z_i - a_{n_i}\}$.

Let $\mathbb{F} = \bigcup_{i \geq 0} \mathbb{F}_i$. Clearly $z_i = a_{n_1} + (z_i - a_{n_i}) \in \mathbb{E} + \mathbb{F}_{i+1}$, so $\mathbb{E} + \mathbb{F} = \mathbb{Z}$. We show $\mathbb{E} \oplus \mathbb{F} = \mathbb{Z}$ using induction on i. That is, for all $i \in \mathbb{N}$ no $z \in \mathbb{Z}$ can be written in two ways as $a + b$ for $a \in \mathbb{E}$ and $b \in \mathbb{F}$.

It is trivial for $i = 0$.

Suppose $\mathbb{E} + \mathbb{F}_i = \mathbb{E} \oplus \mathbb{F}_i$.
If $\mathbb{F}_{i+1} = \mathbb{F}_i$ then $\mathbb{E} + \mathbb{F}_{i+1} = \mathbb{E} \oplus \mathbb{F}_{i+1}$.
If $\mathbb{F}_{i+1} \neq \mathbb{F}_i$, $z_i \notin \mathbb{E} + \mathbb{F}_i$, then n_i exists and $\mathbb{F}_{i+1} = \mathbb{F}_i \cup \{z_i - a_{n_i}\}$.

Assume $a \in \mathbb{Z}$, $b \in \mathbb{F}_i$, and $j \in \mathbb{N}$ are such that $a + b = a_j + (z_i - a_{n_i})$ trying to show $a \notin \mathbb{E}$, or $a_j + (z_i - a_{n_i}) \neq a + b$ with $a \in \mathbb{E}$ and $b \in \mathbb{F}$.

Suppose $j = n_i$. Then $a + b = z_i$ but $z_i \notin \mathbb{E} + \mathbb{F}_i$ and $b \in \mathbb{F}$.
Therefore $a \notin \mathbb{E}$.

Suppose $j < n_i$. Then $a = a_j + (z_i - a_{n_i}) - b \leq a_{n_{i-1}} + z_i - a_{n_i} + \max(\mathbb{F}_i) < 0$. Therefore $a \notin \mathbb{E}$.

Suppose $j > n_i$.
Then $a \leq a_j + (z_i - a_{n_i} + \max(\mathbb{F}) < 0$, $a \geq a_j + z_i - a_{j-1} > a_{j-1}$.
Thus $a_{j-1} < a < a_j$. Therefore $a \notin \mathbb{E}$. $\square$

2 Non-invertible Ergodic Transformations

In this section we examine three non-invertible ergodic transformations (see Examples 5, 6, 7) on infinite σ-finite Lebesgue spaces with an eye toward weakly wandering sequences. A great deal is known about infinite ergodic transformations as well as a wide range of examples - we refer the reader to J. Aaronson's book and references therein [1].

However, very little is known about weakly wandering sequences for the many examples presented in Aaronson's book. These include Markov maps (see Chapter 4), inner functions (see Chapter 6), the geodesic flow on hyperbolic surfaces (see Chapter 7), to name a few.

2.1 Assumptions

We restrict ourselves in this section to non-invertible transformations T which are finite-to-one (the pre-image of each point is finite and we assume greater than 1) on $(X, \mathcal{B}, m)$ a σ-finite, non-atomic Lebesgue space.

The fundamental papers of Rohlin [16], [17] apply in this case and we make free use of all results there. We include some of the definitions for completeness. In particular, if $A \in \mathcal{B}$ then $TA \in \mathcal{B}$.

We point out, as will be seen in the examples, that even when T is measure preserving, *i.e.* $m(T^{-1}A = m(A)$ it does not follow that $m(TA) = A$ and moreover T need not be expanding with respect to any equivalent finite measure.

Unless otherwise indicated all transformations will be *conservative* by construction or assumption that is for any set A with positive measure, $m(A) > 0$, there exists a positive integer k such that $m(A \cap T^{-n}A) > 0$.

The main tool as usual is Birkhoff's Ergodic Theorem presented in the Introduction (Theorem 1). We remind the reader that an immediate consequence of the theorem is:

$$For\ all\ sets\ \ A, B\ \ of finite\ measure\ \ \liminf_{n\to\infty} m(T^{-n}B \cap A) = 0. \qquad (11)$$

When T is invertible the corresponding result for forward images also holds.

$$For\ all\ sets\ \ A, B\ \ of finite\ measure\ \ \liminf_{n\to\infty} m(T^{n}B \cap A) = 0. \qquad (12)$$

However, in the sequel we show by example that this second result need not be true in the non-invertible case. Specifically, it does not hold for the Harmonic Series example (see 7), see Theorem 4.

This gives an immediate difference with the invertible case. For invertible transformations there exist weakly wandering sequences consisting of positive images of T. But there are non-invertible transformations that do not have positive image weakly wandering sequences.

2.2 Definitions

A transformation T is *totally ergodic* if T^k is ergodic for $k > 0$. This definition applies in all cases, *i.e.* invertible and non-invertible cases and for finite and infinite measure spaces.

The set $\tau(T) = \cap_{i=0}^{\infty} T^{-i}\mathcal{B}$ is called the *tail* σ-field of the transformation T.

Definition 2 *A transformation T is called* exact *if its tail σ-field is trivial,*

$$\tau(T) = \{\emptyset, X\}.$$

This notion holds equally in finite and infinite measure spaces. However, invertible transformations cannot be exact as the tail σ-algebra always corresponds with $\mathcal{B}$.
Rohlin introduced the notion of exact in [17] where he proved that exactness implied ergodicity for the finite measure case.

Definition 3 *A set $A \in \mathcal{B}$ is called* full *if*

$$\lim_{n\to\infty} \mu(T^n A) = \mu(X)$$

for a finite measure μ equivalent to the measure m. A transformation T is called full *if each set of positive measure is full.*

The definition makes sense since

for two equivalent probability measures either the transformation is full for both measures or not full for each of them.

In the finite measure case that means the forward T-images of every set of positive measure converges to the entire space. However, in the infinite measure case, the complement of $T^n A$ may still be infinite.
The notion of full was also introduced by Rohlin in [17] although he did not give it a name. He showed that in the finite measure preserving case exact and full are equivalent. This is no longer true for the infinite measure case - see the 2-to-1 and 3-to-1 random walk transformations (Example 5 and Example 6)

An immediate consequence of the definition is that a full set can have at most a finite number of disjoint forward images.

The following definition for a set is stronger than fullness. This was also introduced by Rohlin but not given a name by him.

Definition 4 *A set A is* finite-full *if there exists an integer $n = n(A) > 0$ such that $T^n(A) = X$ and T^n restricted to A is 1-to-1.*

Rohlin showed that finite-full sets always exist for non-invertible transformations.

Since not all sets may be finite-full, it may still be possible to construct a weakly wandering set whose sequence consists of positive integers (other than n_0 of course). This will be illustrated by constructing a weakly wandering set and sequence for 2-to-1 and 3-to-1 Random walk transformations (see Examples 5, 6.

Theorem 4 *If B is a finite-full set for a transformation T then*

$$\liminf_{n\to\infty} m(T^n B \cap A) = m(A) \tag{13}$$

for all A, $m(A) < \infty$.

There is a notion which is weaker than the fullness.

Definition 5 *A transformation T is called* limsup-full *if*

$$\limsup_{n\to\infty} \mu(T^n(A)) = 1 \tag{14}$$

for a finite measure μ equivalent to the measure m and each set A of positive measure.

Again, this definition is well defined since

> *for two equivalent probability measures either a transformation is* lim sup-*full for both measures or not* lim sup-*full for neither.*

The following result due to J. Barnes ([3]) shows that limsup-fullness is stronger that exactness.

Theorem 5 *A limsup-full transformation is exact.*

Corollary 1 *Each full transformation is exact.*

As mentioned Rohlin showed that properties full and exact coincide in the finite measure preserving case. In the infinite measure preserving case a transformation may be exact but not full (see, e.g., the 3-to-1 random walk R_3, Example 6).

The properties of fullness and limsup-fullness are clearly related to the local expansion of the transformation. This may be understood using the *Jacobian* function J define by W Parry in [15] which is an invariant of the transformation and expresses the local expansion at each point. Two examples illustrate the significance of the Jacobian.

The 2-to-1 Random Walk (Example 5) has a Jacobian function identically 2 - which immediately means no set of finite measure can be finite full.

The Harmonic Series Transformation 7 has an unbounded Jacobian function and its finite full sets generate the measure algebra.

2.3 Wandering sets and sequences

As we have seen in the first section, *ww* and *eww* sequences are known to exist for all invertible ergodic infinite measure preserving transformations. In point of fact, in the invertible case, the *ww* and *eww* sequences can be backward (strictly negative excluding $n_0 = 0$) or forward (strictly positive excluding $n_0 = 0$) or two-sided sequences. However in the non-invertible case, which has not been studied in any detail to date, the situation is quite different and the unilateral (forward and backward) and bilateral sequences need to be examined separately.

2.3.1 Weakly Wandering sets.

Comments on negative powers. Using property (11) which is derived from Birkhoff's Ergodic Theorem, the existence of *ww* sets whose sequences are backward can be shown (the proof is similar to the invertible case):

Theorem 6 *For any transformation T there exists a measurable set A with $0 < \mu(A) < \infty$ and an infinite sequence of integers $\{0 = n_0 < n_1 < n_2 \cdots\}$ such that the sets $T^{-n_i}A, i = 1, 2, \ldots$ are pairwise disjoint.*

However negative exhaustive weakly wandering do not always exist.

Theorem 7 *If T is an exact transformation then there does not exist any negative-imaged* eww *set, i.e. a set A such that $X = \cup_{i=0}^{\infty} T^{-n_i}A(\text{disj})$, for $0 = n_0 < n_1 < n_2 \ldots$.*

This is an immediate corollary of the following fact.

Proposition 4 *If $X = \bigcup_{i=0}^{\infty} T^{-n_i}A$* (disj) with $0 = n_0 < n_1 < \ldots$, for a transformation T then A is a tail set, $A \in \tau(\mathcal{B})$.

A slight weakening of exactness is still able to prevent the existence of an negative-imaged exhaustive weakly wandering set and sequence, The 2-to-1 Random Walk (Example 5) is not exact yet still does not possess an exhaustive weakly wandering sequence with negative powers.

Remark 1 *If the tail field is large enough, then the* sww *construction (see* Section 1*) can be applied to get an* eww *set and sequence.*

Comments on positive powers. In the invertible case there always exist weakly wandering sequences with positive powers, *i.e.* $T^{n_i}A$ are disjoint for $0 = n_0 < n_1 < n_2, \ldots$. However, as mentioned previously a full set can not have more than a finite number of disjoint forward images.

Theorem 8 *If a transformation T is full then there are no positive* ww *sequences.*

Note that the 2-to-1 Random Walk Transformation R_2 is not full, and has positive-imaged *ww* sets.

As a corollary of the Proposition 4, theorem 7 and the previous one we have the following.

Theorem 9 *If a transformation T is full then there are no* eww *sequences (positive, negative or two-sided).*

Comments on shifting sequences. In the invertible case, if $\{n_i\}$ is an *eww* sequence for a transformation then for each integer k, positive and negative, the shifted sequence $\{k+n_i\}$ is also *eww*. In the noninvertible case a problem arises because forward images of disjoint sets need not be disjoint. However we still have the following.

Proposition 5 *Given $0 = n_0 < n_1 < n_2 \ldots$, iff $X = \cup_{i=0}^{\infty} T^{-n_i} W$ and $k > 0$ then $X = \cup_{i=0}^{\infty} T^{-k} \circ T^{-n_i} W$* (disj).

For positive sequences a partial result is still possible.

Comments on Positive sequence with arithmetic properties. Recall the sequence $\mathbb{A} = \{0, 2, 8, 10, \cdots\}$ which occur (see (9) as an *eww* sequence for the Hajian-Kakutani (HK) example (Example 3 of the previous section). This can occur as an *eww* sequence for a 2-to-1 transformation IHK (see 8) which is the direct product of the HK-transformation with the finite measure preserving shift σ_2 (see Example 5).

This has some curious arithmetic properties.

Returning to the issues raised at the end of the previous subsubsection, we mention one more result.

Theorem 10 *Suppose a transformation T has $\mathbb{A}$ as an* eww *sequence,* $X = \cup_{i=0}^{\infty} T^{a_i} A$ (disj), $a_i \in \mathbb{A}$. Then for $k > 0$ the forward images are still disjoint, that is

$$X = \cup_{i=0}^{\infty} T^k (T^{a_i} A) \text{ (disj)}.$$

We have also

Theorem 11 *Let T have $\mathbb{A}$ as an* eww *sequence with the set W (positive powers). Then W is a tail set.*

Corollary 2 *If T is exact then the sequence $\mathbb{A}$ cannot be* eww *sequence (positive powers).*

In the invertible case, there exist *eww* sequences with algebraic properties which allow for shifting positively (for example, just mentioned sequence $\mathbb{A}$). This works because the sequence $\mathbb{A}$ has some "nice" arithmetic properties.

3 Examples.

We gather here the examples mentioned earlier in the paper illustrating the properties concerning non-invertible infinite measure preserving transformations.

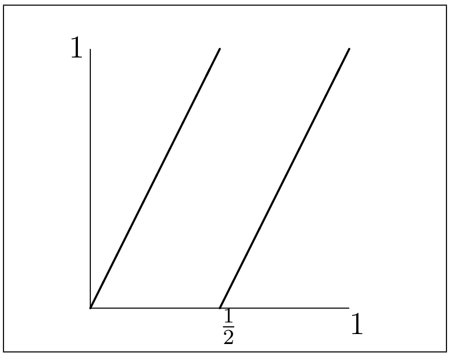
Example 5 (The 2-to-1 Random walk R_2) *Our first non-invertible example is constructed as a skew product. We recall the classic finite measure preserving one-sided Bernoulli shift σ_2 and remind the reader of it and its various isomorphic realizations.*

$\Omega = \prod_0^\infty \{-1, 1\}$ product space

$\omega = (\omega_0, \omega_1, \omega_2, \cdots)$

$\lambda = \prod_0^\infty \{\frac{1}{2}, \frac{1}{2}\}$, product measure

$\sigma_2 : \omega \to (\omega_1, \omega_2, , \cdots)$ (left-shift)



The transformation σ_2 is isomorphic to the following two maps:

(i) $x \to 2x \pmod 1$ on $[0, 1)$ with Lebesgue measure

(ii) $x \to x^2$ on the unit circle with normed Lebesgue arc measure.

It is well known that the transformation σ_2 is a finite measure preserving map. It is easy to see it is full, which implies it is exact which in turn implies it is ergodic.

Let η denote the counting measure on $\mathbb{Z}$.

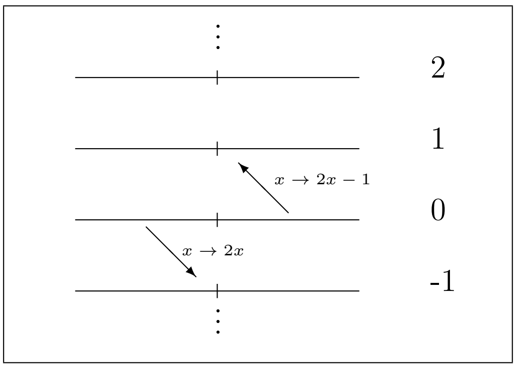
We define the 2-to-1 random walk transformation $\boldsymbol{R_2}$ on $X = \Omega \times \mathbb{Z}$ as follows.

$X = \Omega \times \mathbb{Z}$

$(\omega, n) = (\omega_0 \omega_1, \cdots, n)$

$\mu = \lambda \times \eta$

$\mathrm{R}_2(\omega, n) = (\sigma_2(\omega), n + \omega_0)$



This transformation R_2 is infinite measure preserving and ergodic. However it is not totally ergodic nor is it exact (the tail field is not trivial). Therefore it cannot be full nor can it be limsup-full. These results follow by the construction of a *ww* set with an associated weakly wandering sequence which consists of strictly nonnegative integers (we omit the construction in this survey).

This is related to W. Parry's Jacobian function J which for R_2 satsifies $J = 2$. Hence if $m(A) < \infty$ then $m(\mathrm{R}_2 A) < \infty$. Thus no set of finite measure can be finite-full (though finite-full sets of infinite measure are easily constructed).

This example shows that exactness can be weakened and yet the same (as for exact transformations) conclusion can be reached: the transformation R_2 has no an exhaustive weakly wandering sequence with negative powers. Note that there exists a positive *ww* set for R_2.

The next example is only a slight modification of the previous. However where R_2 is not exact or totally ergodic R_3 is.

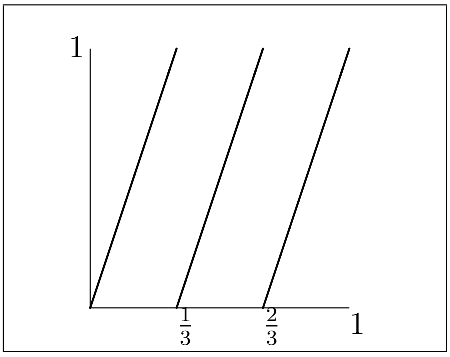
Example 6 (Random walk 3-to-1 $\boldsymbol{R_3}$) *Now recall the finite measure preserving one-sided Bernoulli 3-to-1 shift σ_3. It is defined similarly to σ_2 and also has various isomorphic realizations.*

$\Omega = \prod_0^\infty \{-1, 0, 1\}$ product space

$$\omega = (\omega_0, \omega_1, \omega_2, \cdots)$$

$\lambda = \prod_0^\infty \{\frac{1}{3}, \frac{1}{3}\frac{1}{3}, \}$, product measure

$\sigma_3 : \omega \to (\omega_1, \omega_2, , \cdots)$ (left-shift)



Other isomorphic realizations:

(i) $x \to 3x \pmod 1$ on $[0, 1)$ with Lebesgue measure

(ii) $x \to x^3$ on the unit circle with normed Lebesgue arc measure.

The transformation σ_3 is full, exact and ergodic.

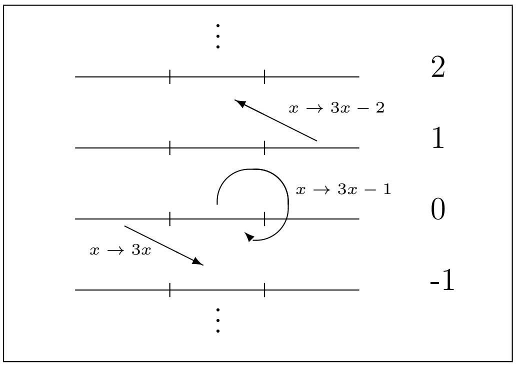
The analysis of the random walk 3-to-1 transformation R_3 is the same as R_2 but improves on the results. We assume again that η is the counting measure on $\mathbb{Z}$ and define the transformation R_3 by the following way.

$$X = \Omega \times \mathbb{Z}$$

$$(\omega, n) = (\omega_0\omega_1, \cdots, n)$$

$$\mu = \lambda \times \eta$$

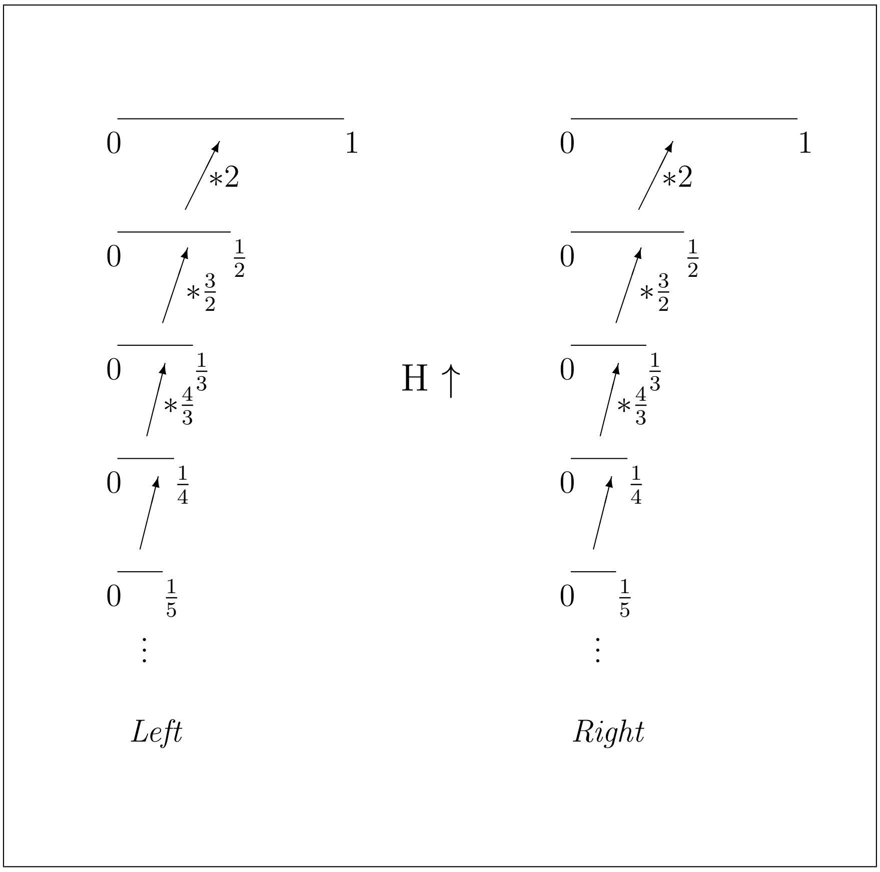
$$R_3(\omega, n) = (\sigma_3(\omega), n + \omega_0)$$



As in the case of R_2, it is possible to construct for R_3 a set A of positive measure which is *ww* under a positive sequence $n_i > 0$ for all $i > 0$. Thus R_3 is not full. However, in contrast to the 2-to-1 random walk this transformation is limsup-full, and exact.

Example 7 (Harmonic series transformation.)
Given any divergent series with positive members $1 = a_1 > a_2 > \ldots$, we can construct a 2-to-1 infinite measure preserving transformation.

We illustrate this with the Harmonic series $\{1, 1/2, 1/3, \ldots\}$.

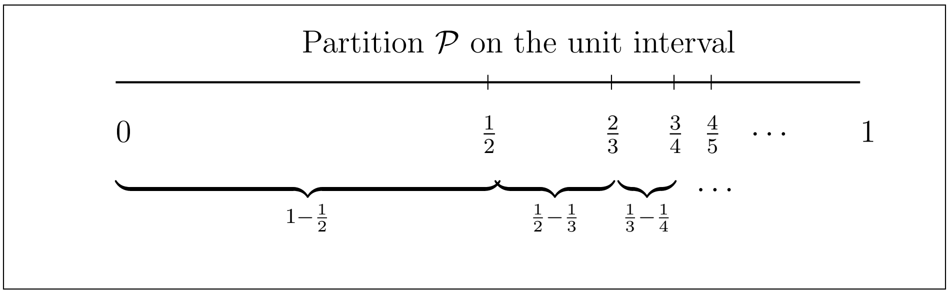


Description. There are two descending columns, denoted *Left* and *Right*. Each column consists of the intervals $[0,1)$, $[0,\frac{1}{2})$, $[0,\frac{1}{3})$, $[0,\frac{1}{4})$, $\ldots$ $[0,\frac{1}{n})$ $\ldots$.

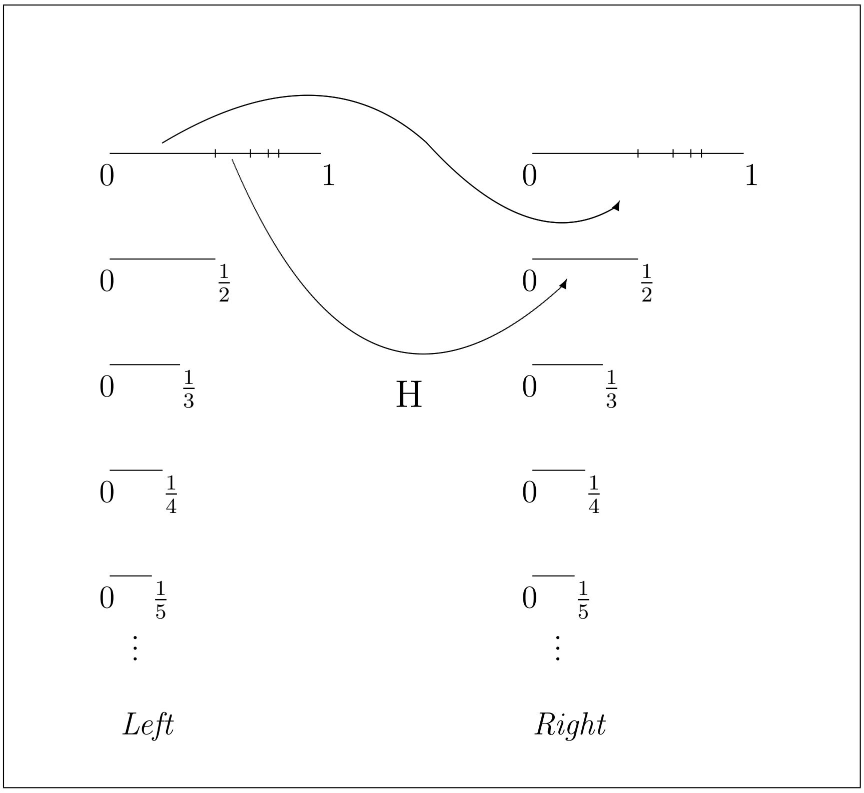
The transformation H is defined to move up linear.

To define H on the tops, we first create a partition $\mathcal{P}$ of the Unit Interval by taking consecutive differences of the Harmonic series.

The tops are similarly partitioned.



The transformation is defined on the top levels by each element of the partition maps to the corresponding level of the other column.



We mention the following properties of this transformation:

(i) H is measure preserving 2-to-1 limsup-full transformation (therefore exact and ergodic).

(ii) H has sets of finite measure which map to sets of infinite measure.

(iii) The top level of each column is finite-full in 2 steps.

(iv) Each level of either column is finite-full. Level n fills the entire space in $n+1$ steps.

(v) The collection of disjoint unions of finite full sets is dense in the measure algebra (with respect to a finite equivalent measure.

(v) The Jacobian for H is unbounded.

It remains an open question whether H is full or not.

Example 8 Irreversible Hajian-Kakutani (IHK) transformation. *We consider the direct product of the 2-to-1 Bernoulli one-sided shift* σ_2 *and the Hajian-Kakutani (HK) transformation described in the first section (Example 3).*

The transformation IHK obviously inherits all the *ww* and *eww* sequences of the HK-transformation.

Conjecture 1 *The transformation IHK has no* additional ww *sequences. That is, any HK*-ww *sequence is already* ww *for IHK-transformation.*

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