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$M_K(T)$-Spaces: Some Order-Theoretical Properties and Applications to Distributive Logics

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Abstract. We present some results involving several ordered structures related to the construction of $M_K(T)$-spaces. These are special closure spaces defined on the basis of a previously given one K and were developed by Fernández and Brunetta, 2023. Specifically, we show that for every $T \in K$, the poset $\mathbb{RE}_K(T)$ of weak-relative closure spaces is a sublattice of $\mathbb{CSP}(X)$ (the family of all the spaces with support X). On the other hand, it will be showed that the poset $\mathbb{M}(K)$ of all the $M_K(T)$-spaces does not verify this property, but it is a complete lattice itself. Also, we show in which way some order-theoretic properties are related to the recovery of closure spaces. Finally, we show some applications of $M_K(T)$-spaces to the class of distributive logics.

Key Words: Lattices of Closure Spaces, Recovery of $M_K(T)$-Spaces, Distributive Logics
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Introduction

Closure spaces (CSP) have been extensively studied in the literature, from different points of view (see [17] for a historical survey). For instance, when they are defined by means of *closure systems*, they can be understood as a natural generalization of the families of closed sets of the topological spaces. In addition, they can be used in several branches of Mathematics, since the definition of several generated substructures (generated subgroups, rings, vector spaces, filters, and so on) determines *closure operators*. In addition, they can be applied to Abstract Logic considering that an abstract logic is, simply, a closure space whose support is an abstract algebra. Moreover, CSP are often studied by their own right (see [15] or [16] as examples of recent works in the area).

In [9], we defined CSP that are *determined by intersections*, as a generalization of the notion of a *relative* closure space. Informally, given a fixed CSP (X, K) (being K the family of the closed sets of such space) and given $T \in K$, it is defined $M_K(T) := \{A \in X : A \cap T \in K\}$. It is easy to see that the pair $(X, M_K(T))$ is a CSP, too. This definition generalizes the notion of the closure space relative to T (which will be remembered later) in the following sense: the support of $M_K(T)$ is the whole set X, instead of T, the standard support set of the relative closure spaces.

The definition of CSP of the form $(X, M_K(T))$ (informally called $M_K(T)$-spaces here) is related to several interesting order-theoretical questions. For instance, it suggests the definition of a more general family of closure spaces, which was called $RE_K(T)$. It was proven in [9] that $\mathbb{RE}_K(T){:=}(RE_K(T), \subseteq)$ is a poset with greatest element $M_K(T)$. However, some order-theoretical questions remained open. Among them, the question if $RE_K(T)$ is a sublattice of $\mathbb{CSP}(X)$ (the family of *all the* CSP *with support* X, ordered by inclusion), was unanswered. The first goal of this paper is, precisely, the demonstration that $\mathbb{RE}_K(T)$ is a sublattice of $\mathbb{CSP}(X)$.

This result suggested some questions of a similar nature (which, by the way, were very natural in the context of the $M_K(T)$-spaces, as it will be shown along this paper). For instance, it can be defined the set $M(K) := \{M_K(T) : T \in K\}$. It was already shown that $T_2 \subseteq T_1$ implies $M_K(T_1) \subseteq M_K(T_2)$. Thus, $M_K(X) = K$ is the first element of the bounded poset $\mathbb{M}(K) := (M(K), \subseteq)$. However, there were some additional open questions about the order-theoretical behavior of this poset. In this work, we will show that $\mathbb{M}(K)$ is a lattice itself, but it is not a sublattice of $\mathbb{CSP}(X)$. By the way, the technique applied to prove this result is based on the definition of a certain equivalence relation on K, as we will show here.

Also, this paper studies the *recovery* of $M_K(T)$-spaces. Informally, the recovery of a pair (T, K), with $T \in K \in CSP(X)$ is the family of all the CSP H such that $M_H(T) = K$. Bearing this in mind, some results referred to this notion will be obtained here. Among them, we relate the families $\mathcal{F}_{(K,T)}$ with the sets $RE_K(T)$ already mentioned. In addition, we will show some specific results for the cases where $\mathcal{F}_{(K,T)}$ is a singleton.

Finally, we show some results of $M_K(T)$-spaces in the context of Abstract Logic. For that, note first that the "process" of obtention of an $M_K(T)$-space can be understood as a "transformation" $K \longrightarrow M_K(T)$ determined by $T \in K$. Thus, it is usual to ask whether certain properties of K can be transferred to $M_K(T)$. For instance, the initial study of $M_K(T)$-spaces in [9] was motivated by problems related to the transference of *finitariness* and of *structurality*, being these properties important in the context of Abstract Logic. Continuing with the application of $M_K(T)$-spaces to the study of logical properties, we will show here some conditions that are necessary for the preservation of the main properties of *distributive logics* (which will be defined later). Moreover, these results can be related to some order-theoretical aspects of the involved logics, as we shall see.

In order to develop all the results mentioned above, we will only consider the notion of closure space, by the moment: other definitions will be explained all along the paper.

Definition 1 *Let X be a non-empty set.*
(a) A closure system on X (c.s.) is a family $K \subseteq \wp(X)$ closed by arbitrary intersections (note that, by vacuity, $X \in K$ for every c.s.).
(b) A closure operator on X (c.o.) is a map $Cl : \wp(X) \longrightarrow \wp(X)$ satisfying for every $B, D \subseteq X$,
b.1) Extensiveness: $D \subseteq Cl(B)$.
b.2) Idempotency: $Cl(Cl(B)) \subseteq Cl(B)$.
b.3) Monotonicity: if $B \subseteq D$, then $Cl(B) \subseteq Cl(D)$.

The previous notions can be interdefined, as it is well-known.

Proposition 1 *If Cl is a c.o. on X, then $K_{Cl} := \{B \subseteq X : B = Cl(B)\}$ is a c.s. on X. Reciprocally, for every closure system $K \subseteq \wp(X)$, the map $Cl_K : \wp(X) \longrightarrow \wp(X)$ defined as $Cl_K(B) = \bigcap_{D \in \mathcal{F}_B^K} D$ (where $\mathcal{F}_B^K := \{D \in K : B \subseteq D\}$) is a c.o. on X. Moreover, $K_{Cl_K} = K$ and $Cl_{K_{Cl}} = Cl$.*

From this, the following definition makes sense.

Definition 2 *A closure space (CSP) is a pair (X, K) being K a c.s. on X. Equivalently, we can say that a CSP is a pair (X, Cl) (with Cl a c.o.) without risk of confusion.*

Remark 1 As we said before, the definition of closure space simply generalizes the well-known families of closed sets *of topological spaces* (T.S.) of the form (X, τ), where $\tau \subseteq \wp(X)$, additionally verifying:
− $\{\emptyset, X\} \subseteq \tau$,
− τ is closed under arbitrary unions and under finite intersections.
In this case the family $\kappa_\tau := \{V^c : V \in \tau\}\}$ is a c.s. (with additional properties), and therefore, (X, κ_τ) is a CSP. Note that there are other ways of generalization of topological spaces. We will mention here, among others:
• Generalized topologies (see [7])]: pairs of the form (X, μ) such that μ is closed under arbitrary unions (this implies that $\emptyset \in \mu$; on the other hand, there are not requirements on X).
• Supra-topological spaces (see [1, 14]): roughly speaking, generalized topologies with $X \in \mu$.
• Infra-topological spaces (see [18]): pairs (X, τ_i) such that τ_i is closed *only* under finite intersections and $\{\emptyset, X\} \subseteq \tau_i$.

All these notions (which are focused on open sets) also determine closed sets, in the obvious way. From this, the notion of $M_K(T)$-spaces that we will develop in this paper can be adapted and studied in all the above generalizations of T.S. We will return to this point in the last section of this paper.

Note additionally that closure spaces can be also defined by other ways. In fact, it is very usual to work with *closure relations* $\vdash \wp(X) \times X$, specially when the closure spaces are obtained in the context of (Abstract) Logic. However, we will work here only with the notions given in Definition 1.

A standard order-theoretical result about closure spaces is the following

Proposition 2 *Given a CSP (X, K), the pair $(K, \subseteq)$ is a complete lattice, with*

$$\bigwedge_{i\in I}^{K} B_i = \bigcap_{i\in I} B_i \quad \text{and} \quad \bigvee_{i\in I}^{K} B_i = Cl_K(\bigcup_{i\in I} B_i).$$

Here, the greatest (lowest) element of the lattice $(K, \subseteq)$ is $B_1^K = X$ ($B_0^K = Cl_K(\emptyset)$).

All the notions above indicated will be used several times along this paper, depending on the results to be discussed. In any case, the more usual formalization of CSP to be considered here will be (X, K) based on closure systems. By the way, these c.s. will be denoted by K, $\overline{K}$, H, $\overline{H}$, and so on, with sub/superscripts if were necessary.

Proposition 3 *Given a fixed set $X \neq \emptyset$, the pair $\mathbb{CSP}(X) := (CSP(X), \subseteq)$ (considering here $CSP(X) = \{K : K$ is a CSP of $X\}$) is a complete lattice. In this case, given any family $\{K_i\}_{i\in I}$ of CSP,*

- $\bigwedge^{\mathbb{CSP}(X)}\{K_i\}_{i\in I} = \bigcap_{i\in I} K_i$,
- $\bigvee^{\mathbb{CSP}(X)}\{K_i\}_{i\in I} = \bigwedge^{\mathbb{CSP}(X)}\{K : K_i \subseteq K$ *for every* $i \in I\}$,

or, alternatively,

- $\bigvee^{\mathbb{CSP}(X)}\{K_i\}_{i\in I} = \bigcap \mathcal{F}$ *being* $\mathcal{F} := \{K : \bigcup_{i\in I} K_i \subseteq K\}$.

The previous standard results can be found on [3], for instance. They will serve as the basis of this paper.

We conclude this section recalling the notion of relative closure space (as a generalization of the relative *topological* spaces, see [8]), with the purpose of fixing some notation.

Definition 3 *Given $K \in CSP(X)$, the closure space relative to $T \subseteq X$ is $K_{\restriction T} := \{B \cap T : B \in K\}$. Note that $K_{\restriction T}$ belongs to $CSP(T)$ instead of to $CSP(X)$. In addition, we define $(K_{\restriction T})^* := K_{\restriction T} \cup \{X\}$. It is easy to see that $(K_{\restriction T})^* \in CSP(X)$.*

We remark that the definition above constitutes the underlying motivation of this notes. We will use it all along this paper.

1 $M_K(T)$-spaces and the families $RE_K(T)$

Definition 4 *Let (X, K) be a (fixed) closure space, and $T \subseteq X$. The family $M_K(T) \subseteq \wp(X)$ is defined as $M_K(T) := \{A \subseteq X : A \cap T \in K\}$. If $M_K(T)$ is closed by arbitrary intersections, then the pair $(X, M_K(T))$ will be called the meet-closure space determined by K and T. By extension, every CSP (X, H) such that there is $T \subseteq X$ with $H = M_K(T)$ will be called informally as an $M_K(T)$-space.*

It is easy to prove that $M_K(T)$ is a CSP iff $T \in K$. From this, in this paper *we will only focus on families $M_K(T)$ such that $T \in K$*. Some basic results about $M_K(T)$-spaces are summarized in the sequel.

Proposition 4 *Given a c.s. (X, K), for every T, T_1, $T_2 \in K$, it holds*
(a) *For every $B \supseteq T$, $B \in M_K(T)$.*
(b) *$T_1 \subseteq T_2$ implies $M_K(T_2) \subseteq M_K(T_1)$ (and thus, for every $B \subseteq X$, $Cl_{M_K(T_1)}(B) \subseteq Cl_{M_K(T_2)}(B)$).*
(c) *If $T = X$, then $M_K(T) = K$; $M_K(T) = [T) := \{W \subseteq X : T \subseteq W\}$ if and only if $T = B_0^K$ (being B_0^k the lowest element of K, cf. Proposition 2).*
(d) *$K \subseteq M_K(T)$ for every $T \in K$ (and thus, if $B \subseteq X$, $Cl_{M_K(T)}(B) \subseteq Cl_K(B)$).*
(e) *For every $B \subseteq X$, $Cl_{M_K(B_0^K)}(B) = B \cup B_0^K$. Therefore, $B \cup B_0^K \supseteq Cl_{M_K(T)}(B)$.*
(f) *$Cl_{M_K(T)}(B) \subseteq B \cup T$ for every $B \subseteq X$.*
(g) *For every $B \subseteq T$, it holds*
  $(g.i)$ *$B \in M_K(T)$ iff $B \in K$,*
  $(g.ii)$ *$Cl_{M_K(T)}(B) \subseteq T$,*
  $(g.iii)$ *$Cl_{M_K(T)}(B) = Cl_K(B)$.*
(h) *The lowest elements of K and of $M_K(T)$ coincide. That is, $B_0^K = B_0^{M_K(T)}$.*

In addition, the closure operator referred to any $M_K(T)$-space can be characterized as follows:

Lemma 1 *$Cl_{M_K(T)}(B) = Cl_K(B \cap T) \cup B$ for every $B \subseteq X$.*

Turning back to Proposition 2, we have that $(M_K(T), \subseteq)$ is a complete lattice for every $T \in K$. Moreover, from Lemma 1 it follows straightforwardly that given $\{B_i\}_{i \in I} \subseteq M_K(T)$,

- $\bigwedge_{i \in I}^{M_K(T)} B_i = \bigcap_{i \in I} B_i$;
- $\bigvee_{i \in I}^{M_K(T)} B_i = Cl_K\Big(\bigcup_{i \in I}(B_i \cap T)\Big) \cup \bigcup_{i \in I} B_i = \Big[\bigvee_{i \in I}^{K}(B_i \cap T)\Big] \cup \bigcup_{i \in I} B_i$.

That is, $\bigwedge_{i \in I}^{M(T)}$ $(\bigvee_{i \in I}^{M(T)})$ can be naturally expressed in terms of $\bigwedge_{i \in I}^{K}$ $(\bigvee_{i \in I}^{K})$.

A deeper result about $M_K(T)$-spaces can be proved considering Definition 3. Specifically, if we consider Proposition 4 ($g.i$), we see that this simple property generalizes the notion of relative closure spaces of the form $(T, K_{\restriction T})$. This is because these spaces verifiy that property, too. That is: for every $A \subseteq T$, $A \in K_{\restriction T}$ iff $A \in K$. A similar fact happens with the $M_K(T)$-spaces, *but in them it is considered all the set X as their support.* This comment suggests the following definition:

Definition 5 *Given a c.s. (X, K) and $T \in K$, we define the family*

$$RE_K(T) := \{H \in CSP(X) : \textit{for every } B \subseteq T, B \in K \textit{ iff } B \in H\}.$$

Proposition 5 *The poset $\mathbb{RE}_K(T) := (RE_K(T), \subseteq)$ is a bounded one, being $M_K(T)$ its greatest element and $(K_{\restriction T})^* := K_{\restriction T} \cup \{X\}$ its lowest element.*

All the previous results were proven in [9]. By the way, since obviously $K \in RE_K(T)$, we have that $(K_{\restriction T})^* \subseteq K \subseteq M_K(T)$. Proposition 5 can be improved as it follows.

Theorem 1 *$\mathbb{RE}_K(T)$ is a sublattice of $\mathbb{CSP}(X)$.*

Proof. Let $K_1, K_2 \in RE_K(T)$. Then:
(a) $K_1 \vee^{\mathbb{CSP}(X)} K_2 \in RE_K(T)$. Recall here (see Proposition 3) that $K_1 \vee^{\mathbb{CSP}(X)} K_2 = \bigcap \mathcal{F}$, being the family $\mathcal{F} := \{H \in CSP(X) : K_1 \cup K_2 \subseteq H\}$. In addition, note that $M_K(T) \in \mathcal{F}$, because it is the greatest element of $RE_K(T)$.

If $B \in K_1 \vee^{\mathbb{CSP}(X)} K_2$, then $B \in M_K(T)$, from the exposed above. Since $M_K(T) \in RE_K(T)$ and taking into account Definition 5 we have $B \in K$.

Suppose now that $B \in K$. Then obviously $B \in K_1 \subseteq K_1 \cup K_2$. Hence, for every $H \in \mathcal{F}$, $B \in H$. That is, $B \in \bigcap \mathcal{F} = K_1 \vee^{\mathbb{CSP}(\mathbb{X})} K_2$.

Therefore, $K_1 \vee^{\mathbb{CSP}(\mathbb{X})} K_2 \in RE_K(T)$, cf. Definition 5.
(b) $K_1 \wedge^{\mathbb{CSP}(\mathbb{X})} K_2 = K_1 \cap K_2 \in RE_K(T)$. In fact, for every $B \subseteq T$, K_1, $K_2 \in RE_K(T)$, one has $B \in K_1 \cap K_2$ if and only if $B \in K$. Actually, $B \in K$ if and only if $B \in K_1$ and $B \in K_2$, that is, if and only if $B \in K_1 \cap K_2$. $\square$

Note that $RE_K(T)$ is not a complete sublattice of $CSP(X)$, since $\bigwedge^{\mathbb{CSP}(\mathbb{X})} \emptyset$ $(= \wp(X))$ is not, necessarily, an element of $RE_K(T)$. However, the argument exposed for the proposition above can be adapted to arbitrary families of (specifically) elements of $RE_K(T)$. Thus, we have

Proposition 6 *$\mathbb{RE}_K(T)$ is a complete lattice itself.*

We conclude this section showing some additional results involving $RE_K(T)$, which will be useful later. Their proofs are easy (recall Definition 3 here).

Proposition 7 *For every H_1, $H_2 \in RE_K(T)$, it holds $H_{1\restriction T} = H_{2\restriction T}$, and therefore, $(H_{1\restriction T})^* = (H_{2\restriction T})^*$.*

Proposition 8 *Suppose $T \in K \in CSP(X)$. Then $K_{\restriction T} = ((K_{\restriction T})^*)_{\restriction T}$.*

2 The poset of all the $M_K(T)$-spaces

Another subposet of $\mathbb{CSP}(X)$, of special importance in this section, is the following

Definition 6 *Given a CSP (X,K), we define the poset $\mathbb{M}(K) := (M(K), \subseteq)$ being $M(K) := \{H : H = M_K(T)$ for some $T \in K\}$.*

Again, it is natural here to ask the following question: is $\mathbb{M}(K)$ a sublattice of $\mathbb{CSP}(X)$? Unfortunately, the answer is negative as the following example shows.

Example 1 Consider $X := [1,7]_{\mathbb{N}} = \{x \in \mathbb{N} : 1 \leq x \leq 7\}$. It is easy to see that the family $K_1 := \{\{1\},\{1,2\},\{1,3\},\{1,4,5\},\{1,2,6\},\{1,2,3,7\},X\}$ is a CSP with support X. Consider now $T_1 := \{1,4,5\}$ and $T_2 := \{1,2,3,7\}$. We have that $M_{K_1}(T_1)$, $M_{K_1}(T_2) \in M(K_1)$, but $M_{K_1}(T_1) \bigwedge^{\mathbb{CSP}(X)} M_{K_1}(T_2) = M_{K_1}(T_1) \cap M_{K_1}(T_2) \notin M(K_1)$. To prove our claim, let us show that for every $T \in K_1$,

$$M_{K_1}(T) \neq M_{K_1}(T_1) \cap M_{K_1}(T_2). \tag{1}$$

Note that

$$M_{K_1}(\{1\}) = M_{K_1}(\{1,2\}) = M_{K_1}(\{1,3\}) = \{W \subseteq X : 1 \in W\}.$$

Now, $M_{K_1}(\{1\}) \neq M_{K_1}(T_1) \cap M_{K_1}(T_2)$, because $\{1,4\} \in M_{K_1}(\{1\}) \setminus M_{K_1}(T_1)$. Thus, $M_{K_1}(T_1) \cap M_{K_1}(T_2) \neq M_{K_1}(\{1,2\})$ and then, $M_{K_1}(T_1) \cap M_{K_1}(T_2) \neq M_{K_1}(\{1,3\})$. In addition, $\{1,3,6\} \in M_{K_1}(T_1) \cap M_{K_1}(T_2) \setminus M_{K_1}(\{1,2,6\})$, $\{1,7\} \in M_{K_1}(T_1) \setminus M_{K_1}(T_1) \cap M_{K_1}(T_2)$, $\{1,5\} \in M_{K_1}(T_2) \setminus M_{K_1}(T_1) \cap M_{K_1}(T_2)$. Finally, $M_{K_1}(X) = K_1 \neq M_{K_1}(T_1) \cap M_{K_1}(T_2)$, obviously. From all these facts together (that can be checked in a straightforward, thorough, way), (1) is valid. Hence, $\mathbb{M}(K)$ is not a sublattice of $\mathbb{CSP}(X)$.

The previous example shows that the meet of $\mathbb{CSP}(X)$ does not belong, in a general way, to $\mathbb{M}(K)$. In a similar way, the join of $\mathbb{CSP}(X)$ is not always an element of $\mathbb{M}(K)$.

Example 2 Consider $X := [1,7]_{\mathbb{N}}$ again, and

$$K_2 := \{\{1\},\{1,2\},\{1,3\},\{1,2,6\},\{1,3,7\},\{1,2,3,5,7\},X\} \in CSP(X).$$

The families K_2 and $M(K_2)$ are ordered as it is shown in Figure 1. Suppose now that it holds

$$M_{K_2}(T_1) \vee^{\mathbb{CSP}(X)} M_{K_2}(T_2) \in M(K_2). \tag{2}$$

If we consider $T_1 := \{1,2,6\}$ and $T_2 := \{1,2,3,5,7\}$, then (supposing that (2) is valid) $M_{K_2}(T_1) \vee^{\mathbb{CSP}(X)} M_{K_2}(T_2) = M_{K_2}(T_1) \vee^{\mathbb{M}(K_2)} M_{K_2}(T_2)$

($= M_{K_2}(\{1\}) = [\{1\})$, according to Proposition 4, (c), and Figure 1). Consider now the family $\overline{K} := [\{1\}) \setminus \mathcal{G}$, with

$$\mathcal{G} := \{\{1,5,6\};\{1,3,5,6\};\{1,5,6,7\};\{1,3,5,6,7\};\{1,4,5,6\};\\ \{1,3,4,5,6\};\{1,4,5,6,7\};\{1,3,4,5,6,7\}\}\,.$$

By a straightforward checking, $\overline{K}$ is a CSP, and $M_{K_2}(T_1) \cup M_{K_2}(T_2) \subseteq \overline{K}$. These facts would imply (assuming (2))

$$M_{K_1}(T_1) \vee^{\mathbb{CSP}(X)} M_{K_2}(T_2) = [\{1\}) \subseteq \overline{K}.$$

But this is not the case. Therefore, $M_{K_2}(T_1) \vee^{\mathbb{CSP}(X)} M_{K_2}(T_2) \notin M(K_2)$.

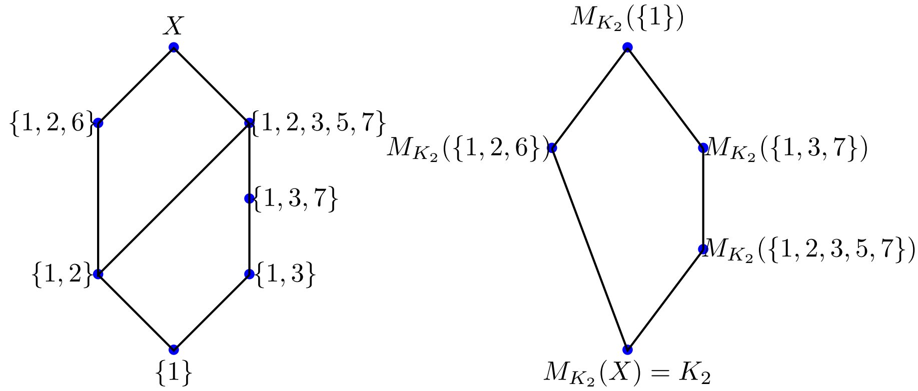

Figure 1: $K_2 \to M(K_2)$

Note here that in the previous example, $\mathbb{M}(K_2) = (M(K_2), \subseteq)$ is a lattice itself. Actually, $\mathbb{M}(K_2)$ is isomorphic to the "pentagon" M_5. In a similar way, it can be verified that $\mathbb{M}(K_1)$ of Example 1 is isomorphic to the non-distributive lattice N_5 (both denominations are taken from [4]). Actually, these cases are not isolated: we will prove in the sequel that the system $\mathbb{M}(K)$ is always a lattice, independently of the properties of the lattice $\mathbb{CSP}(X)$. For that, we need the following

Definition 7 *We define the equivalence relation $\sim \subseteq K \times K$ by: $T_1 \sim T_2$ iff $M_K(T_1) = M_K(T_2)$. In addition, for every $T \in K$, we define $T^0 := \bigcap\{A \in K : A \sim T\}$. That is, T^0 is the smallest set (according to $\subseteq$) of $[T]_\sim$, the equivalence class of T with respect to the relation $\sim$.*

Proposition 9 *Let $\overline{T} \in K$: for every $T \in K$ such that $M_K(T) \subseteq M_K(\overline{T})$, it holds $M_K(T \cap \overline{T}^0) = M_K(\overline{T}^0)$ being $\overline{T}^0$ given as in Definition 7.*

Proof. Since $T \cap \overline{T}^0 \subseteq \overline{T}^0$, we have $M_K(\overline{T}^0) \subseteq M_K(T \cap \overline{T}^0)$ by Proposition 4 (b). For the other inclusion, consider $F \in M_K(T \cap \overline{T}^0)$. Thus, $F \cap (T \cap \overline{T}^0) \in K$, and hence $(F \cap \overline{T}^0) \cap T \in K$. That is, $F \cap \overline{T}^0 \in M_K(T) \subseteq M_K(\overline{T}^0)$ by Hypothesis. In other words, $(F \cap \overline{T}^0) \cap \overline{T} = F \cap \overline{T}^0 \in K$. That is, $F \in M_K(\overline{T}^0)$. □

Corollary 1 *For every $\overline{T}$, $T \in K$ such that $M_K(T) \subseteq M_K(\overline{T})$, it holds $\overline{T}^0 \subseteq T$.*

Proof. By the previous proposition, $\overline{T}^0 \sim \overline{T}^0 \cap T$. Then $\overline{T}^0 \subseteq \overline{T}^0 \cap T \subseteq \overline{T}^0$. That is, $\overline{T}^0 \subseteq T$. □

Theorem 2 $\mathbb{M}(K) = (M(K), \subseteq)$ *is a lattice, where $\vee$ and $\wedge$ can be characterized as follows: for every pair $\{M_K(T_1), M_K(T_2)\} \subseteq M(K)$,*
(a) $M_K(T_1) \vee M_K(T_2) := M_K(T_1 \cap T_2)$,
(b) $M_K(T_2) \wedge M_K(T_2) := M_K(Cl_K(T_1^0 \cup T_2^0))$ being T_i^0 $(i = 1, 2)$ as presented in Definition 7.

Proof. Suppose T_1, $T_2 \in K$. To prove (a), note that by Proposition 4 it holds $M_K(T_i) \subseteq M_K(T_1 \cap T_2)$ $(i = 1, 2)$. Consider now $M_K(T) \in M(K)$ such that $M_K(T_i) \subseteq M_K(T)$ for $i = 1, 2$. We will prove that $M_K(T_1 \cap T_2) \subseteq M_K(T)$. For that, suppose $F \in M_K(T_1 \cap T_2)$ (that is, $F \cap (T_1 \cap T_2) \in K$). Hence, $F \cap T_1 \in M_K(T_2) \subseteq M_K(T)$ (by Hypothesis), and then $(F \cap T_1) \cap T \in K$. This implies that $F \cap T \in M_K(T_1) \subseteq M_K(T)$. Thus, $F \cap T = (F \cap T) \cap T \in K$. That is, $F \in M_K(T)$, proving our claim.

To prove (b), consider $i \in \{1, 2\}$. Obviously, $M_K(Cl_K(T_1^0 \cup T_2^0)) \subseteq M_K(T_i^0) = M_K(T_i)$. Then,

$$M_K(Cl_K(T_1^0 \cup T_2^0)) \subseteq M_K(T_i). \tag{3}$$

Consider now $T \in K$ such that $M_K(T) \subseteq M_K(T_i)$, $i = 1, 2$. By Corollary 1, $T_1^0 \subseteq T$ and $T_2^0 \subseteq T$, and then $Cl_K(T_1^0 \cup T_2^0) \subseteq T$. Hence,

$$M_K(T) \subseteq M(Cl_K(T_1^0 \cup T_2^0)). \tag{4}$$

(by Proposition 4 again). From (3) and (4), (b) is proved. □

Remark 2 The equivalence relation $\sim$ *is not compatible* w.r.t. to $\cup$. This is the reason why we must deal specifically with the closed sets of the form T^0 to obtain $\wedge$ in $\mathbb{M}(K)$. Anyway, $\sim$ is compatible with respect to $\cap$, which allows us to obtain the characterization of $\wedge$ in terms of the "T^0-sets".

Proposition 10 *Let $T_i \in K$, $i = 1, 2$, and let T_i^0 be as in Definition 7. Then $M_K(T_1^0 \cap T_2^0) = M_K(T_1 \cap T_2)$, that is, $T_1^0 \cap T_2^0 \sim T_1 \cap T_2$.*

Proof. Since $T_i^0 \subseteq T_i$, $i = 1, 2$, we have $M_K(T_1 \cap T_2) \subseteq M_K(T_1^0 \cap T_2^0)$. Suppose now $F \in M_K(T_1^0 \cap T_2^0)$, that is, $F \cap (T_1^0 \cap T_2^0) \in K$. Hence, $F \cap T_1^0 \in M_K(T_2^0) = M_K(T_2) \subseteq M_K(T_1 \cap T_2)$, and thus $F \cap T_1^0 \in M_K(T_1 \cap T_2)$. Then $(F \cap T_1^0) \cap (T_1 \cap T_2) \in K$ and thus $F \cap (T_1 \cap T_2) \in M_K(T_1^0) = M_K(T_1) \subseteq M_K(T_1 \cap T_2)$. Then, $F \cap (T_1 \cap T_2) \in M_K(T_1 \cap T_2)$, and therefore, $F \in M_K(T_1 \cap T_2)$. This concludes the proof. □

A parallel corollary derived from Proposition 10 is the following

Corollary 2 *If* $T_1 \sim T_1'$ *and* $T_2 \sim T_2'$, *then* $T_1 \cap T_2 \sim T_1' \cap T_2'$.

Turning back to the characterization of the join $\vee$ and the meet $\wedge$ in $\mathbb{M}(K) = (M(K), \subseteq)$, taking into account Theorem 2 and Proposition 10, these operations can be defined as follows:

(a) $M_K(T_1) \vee M_K(T_2) := M_K(T_1^0 \cap T_2^0)$,

(b) $M_K(T_2) \wedge M_K(T_2) := M_K(Cl_K(T_1^0 \cup T_2^0))$.

In both cases, T_i^0, $i = 1, 2$, are defined as it is indicated in Definition 7.

Note, finally, that the argumentation in Theorem 2 can be generalized to families of CSP in $CSP(X)$ with arbitrary cardinality. Thus, it is valid

Theorem 3 $(\mathbb{M}(K), \subseteq)$ *is a complete lattice, where* $\bigvee^{\mathbb{M}(K)}$ *and* $\bigwedge^{\mathbb{M}(K)}$ *for every family* $\{M_K(T_i)\}_{i \in I} \subseteq M(K)$, *can be characterized as follows:*
(a) $\bigvee^{\mathbb{M}(K)} \{M_K(T_i)\}_{i \in I} := M_K\Big(\bigcap_{i \in I} T_i^0\Big)$,
(b) $\bigwedge^{\mathbb{M}(K)} \{M_K(T_i)\}_{i \in I} := M_K\Big(Cl_K\Big(\bigcup_{i \in I} T_i^0\Big)\Big)$.

3 On recovery of closure spaces

Along this section we will work with several closure spaces at the same time. With this in mind, note that if $T \in K_1 \subseteq K_2$, then $M_{K_1}(T) \subseteq M_{K_2}(T)$. From this and from Proposition 2.2, it holds: if $T_1, T_2 \in K_1 \cap K_2$, $K_1 \subseteq K_2$ and $T_1 \subseteq T_2$, then

- $M_{K_1}(T_2) \subseteq M_{K_1}(T_1)$,
- $M_{K_1}(T_2) \subseteq M_{K_2}(T_2)$,
- $M_{K_1}(T_1) \subseteq M_{K_2}(T_1)$,
- $M_{K_2}(T_2) \subseteq M_{K_2}(T_1)$.

$M_{K_1}(T_1)$ and $M_{K_2}(T_2)$ are incomparable families, obviously.

Besides that, the use of different closure spaces at the same time allows getting a certain kind of "idempotency" in the application of $M_K(T)$-spaces in the following sense.

Proposition 11 *Let $T \in K \in CSP(X)$. Then $M_{M_K(T)}(T) = M_K(T)$.*

Proof. Obviously, $T \in M_K(T)$ and then $M_{M_K(T)}(T)$ is a CSP. In addition, $M_K(T) \subseteq M_{M_K(T)}(T)$, because $K \subseteq M_K(T)$. On the other hand, if $B \in M_{M_K(T)}(T)$, then $B \cap T \in M_K(T)$, and then $(B \cap T) \cap T \in K$. Thus, $B \cap T \in K$, which means that $B \in M_K(T)$. □

We will see that these preliminar results are related with the essential notion of this section.

Definition 8 *Let (K, T) be a pair such that $T \in K \in CSP(X)$. The recovery of (K, T) is the family $\mathcal{F}_{(K,T)} := \{H \in CSP(X) : M_H(T) = K\}$. Every closure space $H \in \mathcal{F}_{(K,T)}$ will be called a pre-(K, T) space.*

Roughly speaking, the recovery $\mathcal{F}_{(K,T)}$ of (K, T) can be intended as follows: consider $T \subseteq X$ fixed and suppose that $M_{\dots}(T)$ (with "..." to be fulfilled with certain CSP) can be understood as a map between certain classes of specific closure spaces (specifically, from the family of the CSP having T as a closed set, to itself), then $\mathcal{F}_{(K,T)}$ is the inverse image (for K) of such kind of map. This explains the name "recovery" for this family.

Some basic results about the families $\mathcal{F}_{(K,T)}$ are summarized in the sequel.

Proposition 12 *Given $T \in K \in CSP(X)$, it holds*
(a) For every $H \in \mathcal{F}_{(K,T)}$, $(H_{\restriction T})^* = (K_{\restriction T})^*$.
(b) If $H \in \mathcal{F}_{(K,T)}$, then $(H_{\restriction T})^* \in \mathcal{F}_{(K,T)}$, too.
(c) If $H \in \mathcal{F}_{(K,T)}$, then $M_H(T) \in \mathcal{F}_{(K,T)}$, too.

Proof. (a) Note that $(H_{\restriction T})^* \in RE_H(T)$ and $K = M_H(T) \in RE_H(T)$, too. Then, by Propositions 7 and 8, we have $(H_{\restriction T})^* = (((H_{\restriction T})^*)_{\restriction T})^* = (K_{\restriction T})^*$.

(b) We will prove that $M_{(H_{\restriction T})^*}(T) = K$ by double inclusion. Suppose $W \in M_{(H_{\restriction T})^*}(T)$, then $W \cap T \in (H_{\restriction T})^* \subseteq H$, obviously. Therefore, $W \in M_H(T) = K$. On the other hand, suppose $W \in K = M_H(T)$. Then, $W \cap T \in H$. Thus, $W \cap T = (W \cap T) \cap T \in (H_{\restriction T}) \subseteq (H_{\restriction T})^*$. Hence, $W \in M_{(H_{\restriction T})^*}(T)$.

(c) It is a consequence of Proposition 11. □

Items (b) and (c) establish that $\mathcal{F}_{(T,K)}$ is closed by a certain kind of "operators" for closure systems. Roughly speaking, (b) is focused on the "extended restriction" $(\dots_{\restriction T})^*$, meanwhile (c) proves preservation of belonging to $\mathcal{F}_{(K,T)}$ after "application of the operator $M_{\dots}(T)$". On the other hand, items (a) and (b) are referred to closure spaces of the form $(H_{\restriction T})^*$ which, as we said, are the first elements of the families $RE_H(T)$ (see Section 1). By the way, the relation of this kind of families with $\mathcal{F}_{(K,T)}$ is stronger yet.

Proposition 13 *Suppose $T \in K \in CSP(X)$. For every H_1, $H_2 \in \mathcal{F}_{(K,T)}$ it holds that $RE_{H_1}(T) = RE_{H_2}(T)$.*

Proof. Our hypothesis is $M_{H_1}(T) = M_{H_2}(T)$ $(= K)$. We will prove that $RE_{H_1}(T) \subseteq RE_{H_2}(T)$ (the other inclusion is similar). For that, suppose H in $RE_{H_1}(T)$. That is, for every $B \subseteq T$,

$$B \in H \text{ iff } B \in H_1. \tag{5}$$

Consider now any subset $F \subseteq T$. If $F \in H$, then $F \in H_1$ by (5). Thus, $F \in M_{H_1}(T) = M_{H_2}(T) \in RE_{H_2}(T)$ (by Proposition 5). Since $F \subseteq T$, we have

$$F \in H_2. \tag{6}$$

On the other hand, suppose $F \in H_2 \subseteq M_{H_2}(T) = M_{H_1}(T) \in RE_{H_1}(T)$ (by Proposition 5, again). Since $F \subseteq T$, we have $F \in H_1$. Hence, by (5),

$$F \in H. \tag{7}$$

Thus, by (6) and (7), for every $F \subseteq T$, it holds that $F \in H$ iff $F \in H_2$. This proves that $H \in RE_{H_2}(T)$. We have just proven that $RE_{H_1}(T) \subseteq RE_{H_2}(T)$, as it was desired. □

The previous result provides an interesting connection between the families $\mathcal{F}_{(K,T)}$ and the set $RE_K(T)$. In fact, it implies that if H_1, $H_2 \in \mathcal{F}_{(K,T)}$, then these closure systems can only differ in the sets B such that $B \not\subseteq T$.

An additional problem about $\mathcal{F}_{(K,T)}$ is the following: in which case the family $\mathcal{F}_{(K,T)}$ is a singleton? That is, when is it possible to know which is the only family H in $CSP(X)$ (if there is such only set, indeed) such that $M_H(T) = K$? To understand this motivation, let us see some examples/counterexamples.

Example 3 Consider $X = \{1,2,3,4,5\}$ and (K, T_1) with $T_1 := \{1,3\}$ and $K = \{\{1,3\}; \{1,2,3\}; \{1,3,4\}; \{1,3,5\}; \{1,2,3,4\}; \{1,2,3,5\}; \{1,3,4,5\}; X\}$.

We have that $\mathcal{F}_{(K,T_1)}$ is not a singleton because we have the following pre-(K, T_1)-spaces (among others):

$$H_1 = \{\{1,3\}; \{1,2,3\}; \{1,3,4\}; \{1,3,4,5\}; X\},$$

$$H_2 = \{\{1,3\}; X\}.$$

By a straightforward checking, $M_{H_1}(T_1) = M_{H_2}(T_1) = K$.

Some (partial) answers to the problem posed above are related to the status of T in the ordered structure of K. Indeed, the following results deal with some properties of $\mathcal{F}_{(K,T)}$ when T is the lowest (greatest) element of K.

Proposition 14 *For every* $H \in \mathcal{F}_{(K,T)}$, $B_0^H = B_0^K$.

Proof. Obvious, taking into account Proposition 4 (h). □

Lemma 2 *Let* K *be a CSP such that* $|K| \geq 3$, *and let* B_0^K *be its first element. Then* $\mathcal{F}_{(K,B_0^k)}$ *is not unitary.*

Proof. Recalling the notation $[B)$:=$\{W \subseteq X : B \subseteq W\}$, we have two cases:
(a) $K \neq [B_0^K)$. Then, $\mathcal{F}_{(K,B_0^K)} = \emptyset$. If there is some $H \in \mathcal{F}_{(K,B_0^K)}$, then (using Propositions 14 and 4 (c)), $K = M_H(B_0^K) = M_H(B_0^H) = [B_0^H) = [B_0^K)$, contradicting our assumption.
(b) $K = [B_0^K)$. Then, the following fact is valid: for every CSP $H \subseteq K$ such that $B_0^K \in H$, it holds that $H \in \mathcal{F}_{(K,B_0^K)}$. Indeed, $B_0^H = B_0^K$ again, and thus $M_H(B_0^K) = [B_0^K) = K$. In particular, K and $K' := \{B_0^K, X\} \in \mathcal{F}_{(K,B_0^K)}$ being $K \neq K'$ because of our hypothesis. Therefore, $|\mathcal{F}_{(K,B_0^K)}| \geq 2$ in this case. □

On the other hand, we have this obvious fact since $\mathcal{F}_{(K,X)} = \{K\}$.

Proposition 15 $\mathcal{F}_{(K,X)}$ *is always unitary for every* $K \in CSP(X)$.

The last result of this section shows that, even considering closed sets T "very near to X", some information about recovery sets is missed.

Proposition 16 *Let* $T \in K \in CSP(X)$ *such that* $T = X \setminus \{a\}$ *for some* $a \in X$. *Then*
(a) $K \setminus \{T\} \in CSP(X)$.
(b) *For every* $B \in K$, $X \neq B \neq T$, *we have* $M_K(B) = M_{K\setminus\{T\}}(B)$.
(c) *For every* $B \in K \setminus \{T\}$, $\mathcal{F}_{(M_K(B),B)}$ *is not unitary.*

Proof. (a) Let $\mathcal{G} = \{B_i\}_{i\in I}$ be a family included in $K \setminus \{T\}$. Then, obviously, $\bigcap \mathcal{G} \in K$. In addition, $\bigcap \mathcal{G} \neq T$. In fact, if $I = \emptyset$, then $\bigcap \mathcal{G} = X \neq T$. On the other hand, suppose that $I \neq \emptyset$. We have two possibilities here. If $\bigcap \mathcal{G} = B_{i_0} \in \mathcal{G}$, then $\bigcap \mathcal{G} \neq T$. If not, there are B_1, $B_2 \in \mathcal{G}$, $b_1, b_2 \in X$, such that $b_1 \in B_1 \setminus B_2$, $b_2 \in B_2 \setminus B_1$, $b_1 \neq b_2$. That is, there are at least two elements in X not belonging to $B_1 \cap B_2$. Thus, $\bigcap \mathcal{G} \subseteq B_1 \cap B_2 \neq T$ in this case, too. From all this, $\bigcap \mathcal{G} \in K \setminus \{T\}$.

(b) Given $B \in K \setminus \{T\}$, we only will prove that $M_K(B) \subseteq M_{K\setminus\{T\}}(B)$, because of the previous comments. That is, we will prove that for every $D \subseteq X$, if $D \cap B \in K$, then $D \cap B \neq T$. We have the following cases/subcases:
If $B \subseteq T$, then, by Hypothesis, $B \subset T$. Thus, $D \cap B \neq T$, obviously.
If $B \not\subseteq T$, then $a \in B$ (because $T = X \setminus \{a\}$). In addition, since $B \neq X$, there is $b \notin B$, and then $b \in T \setminus B$, $b \neq a$. Now, if $a \in D$, then $a \in B \cap D$, $a \notin T$. Otherwise, $b \notin D \cap B$. Hence, $T \neq D \cap B$ in both cases.

Finally, (c) is immediate from (b). □

4 Some applications to distributive logics

As we said at the beginning of this work, $M_K(T)$-spaces can be studied in the context of Abstract Logic. The last section of this paper is devoted to these topics. In particular, we will study some relations between the $M_K(T)$-spaces and the family of *distributive logics* to be defined in the sequel. For the self-contention of this paper, let us recall first the formal definition of *abstract logic* (this definition is based on [3, 10] and, considering its algebraic aspects, on [4]).

Definition 9 *An abstract algebra is a pair* $\mathbf{A} = (A, \{f_i\}_{i \in I})$ *such that for every* $i \in I$*,* f_i *is an operation* $f_i : A^{n_i} \to A$ *with* $n_i \in \omega$*, being* n_i *the arity of* f_i *(the ordered set of the arities of such maps is the similarity type of* $\mathbf{A}$*). An abstract logic is just a closure space of the form* $\mathcal{L} = (A, K)$*, being* A *the support of an abstract algebra* $\mathbf{A}$*.*[1]

In this paper, we will work specifically with the class of distributive logics, whose definition and basic development can be found in [11] and [12]. These logics are built up considering the notion of the closure operator Cl_K as fundamental one. Recall here that we can work with c.o. Cl_K (and with the underlying closure operators $Cl_{M_K(T)}$) instead of with closure systems because of Lemma 1. Then, this change is insignificant w.r.t. the results to be obtained. Besides that, some usual abbreviations in the context of closure operators applied to logic are the following: $Cl_K(B \cup \{a\})$ is denoted simply as $Cl_K(B, a)$. With the same spirit, $Cl_K(a, b)$ denotes $Cl_K(\{a, b\})$.

Definition 10 *A distributive logic is a logic* (A, K) *defined on an algebra* $\mathbf{A} = (A, \wedge, \vee)$ *of type* $(2, 2)$ *such that* Cl_K *verifies the following properties:*

(1) *PCI w.r.t.* $\wedge$*:* $Cl_K(a, b) = Cl_K(a \wedge b)$ *for every* $a, b \in A$*,*

(2) *PDI w.r.t* $\vee$*:* $Cl_K(Y, a) \cap Cl_K(Y, b) = Cl(Y, a \vee b)$ *for every* $Y \cup \{a, b\} \subseteq A$*,*

(3) *Fin:* $Cl_K(A) = \bigcup\{Cl_K(B) : B \in \wp_{fin}(A)\}$*, where* $\wp_{fin}(A) := \{B \subseteq A : B \text{ is finite}\}$*,*

(4) *nPA:* $Cl_K(\emptyset) = \bigcap\{Cl_K(Y) : \emptyset \neq Y \subseteq A\}$*.*

Property Fin is usually called *finitariness* (or compactness). In addition, a logic that verifies nPA is called *not pseudo-axiomatic*. This property is equivalent to say that $\emptyset \notin \mathcal{B}$ for some basis $\mathcal{B}$ of K, cf. [11] (recall here that $\mathcal{B}$ is a basis of K if and only if every $T \in K$ can be expressed by intersections of elements of $\mathcal{B}$.). Thus, every logic $\mathcal{L} = (A, K)$ such that $\emptyset \notin K$ is a not pseudo-axiomatic one.

[1]Following the standard literature in Universal Algebra and Abstract Logic, we do not denote the supports of the abstract algebras by X but by A.

We will begin this section analyzing if every one of the properties (1)-(4) is preserved in the $M_K(T)$-spaces. We will answer this question "item by item", by means of several results/counterexamples.

Example 4 PCI is not preserved by applications of $M_K(T)$. Let $(A, \vee, \wedge)$ be the algebra with $A = \{0, a, 1\}$ such that $\wedge$ ($\vee$) is the meet (join) defined on the chain $0 \leq a \leq 1$. Let (A, K) be the logic such that $K = \{\{1\}, A\}$, which verifies PCI, straightforwardly (see Table 1). On the other hand, considering $\{1\} \in K$, we have that $M_K(\{1\}) = \{\{1\}, \{0, 1\}, \{a, 1\}, A\}$. Then, $(A, M(\{1\}))$ does not verify PCI according to Table 2 below.

The same tables show that (A, K) verifies PDI, but this property is not verified by the closure space $M_K(\{1\})$. Indeed, $Cl_{M_K(\{1\})}(0 \vee a) = Cl_{M_K(\{1\})}(a) = \{a, 1\} \neq \{1\} = Cl_{M_K(\{1\})}(0) \cap Cl_{M_K(\{1\})}(a)$.

x	y	$x \wedge y$	$Cl_K(x, y)$	$Cl_K(x \wedge y)$
0	a	0	A	A
0	1	0	A	A
a	1	a	A	A

Table 1: $Cl_K(x, y)$ and $Cl_K(x \wedge y)$

x	y	$x \wedge y$	$Cl_{M_K(\{1\})}(x, y)$	$Cl_{M_K(\{1\})}(x \wedge y)$
0	a	0	X	$\{0, 1\}$
0	1	0	$\{0, 1\}$	$\{0, 1\}$
a	1	a	$\{a, 1\}$	$\{a, 1\}$

Table 2: $Cl_{M_K(\{1\})}(x, y)$ and $Cl_{M_K(\{1\})}(x \wedge y)$

Finitariness is always preserved by $M_K(T)$-spaces (see [9]). Finally, we have

Proposition 17 *If a distributive logic (A, K) verifies nPA, then for every $T \in K$, $(A, M_K(T))$ verifies nPA, too.*

Proof. Suppose (A, K) verifies $Cl_K(\emptyset) = \bigcap\{Cl_K(Y) : \emptyset \neq Y \subseteq A\}$. By Proposition 4 (d), $Cl_{M_K(T)}(Y) \subseteq Cl_K(Y)$ for every $Y \subseteq A$, $Y \neq \emptyset$. Thus, it holds $\bigcap\left\{Cl_{M_K(T)}(Y) : \emptyset \neq Y \subseteq A\right\} \subseteq \bigcap\left\{Cl_K(Y) : \emptyset \neq Y \subseteq A\right\} = Cl_K(\emptyset)$. Then, by Propositions 2 and 4 (h), $\bigcap\left\{Cl_{M_K(T)}(Y) : \emptyset \neq Y \subseteq A\right\} \subseteq Cl_{M_K(T)}(\emptyset)$. The other inclusion is obvious. □

Thus, not every property related to distributive logics is preserved by means of the "application of $M_K(T)$". However, in the sequel, we will give some conditions that are weaker than the previous ones and that will be preserved by the $M_K(T)$-closure operators.

Definition 11 *An abstract logic (A, K), with $\mathbf{A} = (A, \wedge, \vee)$ verifies weak-PDI property (WPDI for short) if and only if every $Y \cup \{a, b\} \subseteq A$ verifies $Cl_K(Y, a) \cap Cl_K(Y, b) \subseteq Cl_K(Y, a \vee b)$.*

In addition, we need the following definition.

Definition 12 *A logic (A, K) with $\mathbf{A} = (A, \wedge, \vee)$ of type $(2, 2)$ is called an idempotent logic iff for every $a \in A$, $a \vee a = a \wedge a = a$.*

Classical logic (and every logic where $\wedge$ and $\vee$ are intended as meets and joins in a lattice-structure defined on a certain set A) is an idempotent logic. On the other hand, some logics of the family of fuzzy logics are not idempotent. More specifically, if we consider the real interval $[0, 1]$ and $\wedge$ is defined by means of the operation $*$, being $* : [0, 1]^2 \to [0, 1]$ a t-norm (see [13]), we can see that in these logics is only warranted that $x * x \leq x$ for every $x \in [0, 1]$.

Lemma 3 *If an idempotent logic (A, K) verifies PDI, then for every $T \in K$, it holds that $(A, M_K(T))$ verifies WPDI.*

Proof. Suppose that (A, K) verifies PDI and consider $Y \cup \{a, b\} \subseteq A$, $T \in K$. By Lemma 1, it is easy to see the following: for every $x \notin T$,

$$Cl_{M_K(T)}(Y, x) = Cl_{M_K(T)}(Y) \cup \{x\}. \qquad (8)$$

With this in mind, let us consider the following possibilities:
(1) $a \vee b \notin T$. Then $a \notin T$. Indeed, if $a \in T$, then $T = Cl_K(T, a)$, and therefore, since (A,K) verifies PDI, $a \vee b \in Cl_K(T, a \vee b) = T \cap Cl_K(T, b) \subseteq T$, which is absurd. Then, in this case, by (8), we have that $Cl_{M_K(T)}(Y, a) = Cl_{M_K(T)}(Y) \cup \{a\}$, $Cl_{M_K(T)}(Y, b) = Cl_{M_K(T)}(Y) \cup \{b\}$ and $Cl_{M_K(T)}(Y, a \vee b) = Cl_{M_K(T)}(Y) \cup \{a \vee b\}$. Thus, $Cl_{M_K(T)}(Y, a) \cap Cl_{M_K(T)}(Y, b) = Cl_{M_K(T)}(Y) \cup (\{a\} \cap \{b\})$.

Further, if $a \neq b$, then $Cl_{M_K(T)}(Y, a) \cap Cl_{M_K(T)}(Y, b) = Cl_{M_K(T)}(Y) \subseteq Cl_{M_K(T)}(Y, a \vee b)$, and, if $a = b$, then $Cl_{M_K(T)}(Y, a) \cap Cl_{M_K(T)}(Y, b) = Cl_{M_K(T)}(Y) \cup \{a\} = Cl_{M_K(T)}(Y) \cup \{a \vee b\}$ (this is the reason why idempotency is necessary).
(2) $a \vee b \in T$. Then,

$$Cl_{M_K(T)}(Y, a \vee b) = Cl_K((Y \cap T), a \vee b) \cup Y \cup \{a \vee b\}. \qquad (9)$$

On the other hand, $Cl_{M_K(T)}(Y, a) \subseteq Cl_K((Y \cap T), a) \cup Y$ and $Cl_{M_K(T)}(Y, b) \subseteq\subseteq Cl_K((Y \cap T), b) \cup Y$. All this implies $Cl_{M_K(T)}(Y, a) \cap Cl_{M_K(T)}(Y, b) \subseteq\subseteq [Cl_K((Y \cap T), a) \cap Cl_K((Y \cap T), b)] \cup Y = Cl_K((Y \cap T), a \vee b) \cup Y$. Hence, $Cl_{M_K(T)}(Y, a) \cap Cl_{M_K(T)}(Y, b) \subseteq Cl_{M_K(T)}(Y, a \vee b)$ by (9). Thus, from (1) and (2), $(A, M_K(T))$ verifies WPDI. □

With respect to PCI, we have a weaker version, too.

Definition 13 *A logic* (A, K) *with* $\mathbf{A} = (A, \wedge, \vee)$ *of type* $(2, 2)$ *verifies weak-PCI property (w.r.t. to* $\wedge$*), WPCI, iff for every* $a, b \in A$, $C_K(a \wedge b) \subseteq C_K(a, b)$.

Lemma 4 *Let* (A, K) *be a logic defined on the basis of* $\mathbf{A} = (A, \wedge, \vee)$. *If* (A, K) *verifies PCI and, in addition, for every* $a, b \in A$, $a \wedge b \in \{a, b\}$, *then for every* $T \in K$, $(A, M_K(T))$ *verifies WPCI.*

Proof. Since (A, K) verifies PCI then for every $T \in K$,

$$\{a, b\} \subseteq T \text{ implies } a \wedge b \in T. \tag{10}$$

On the other hand, by Lemma 1 we have $C_{M_K(T)}(a, b) = C_K(T \cap \{a, b\}) \cup \{a, b\}$ and $C_{M_K(T)}(a \wedge b) = C_K(T \cap \{a \wedge b\}) \cup \{a \wedge b\}$. Comparing both sets (by a combinatorical analysis of all the possible cases), and using (10), we obtain the desired result. □

Definition 14 *A logic* (A, K) *is weakly distributive if and only if it verifies WPCI, WPDI, Fin and nPA.*

Corollary 3 *If* (A, K) *is a distributive idempotent logic such that* $a \wedge b \in \{a, b\}$ *for every* a, $b \in A$, *then* $(A, M_K(T))$ *is weakly distributive.*

Example 5 All the previous results together can be applied to the following example, based on the support of the standard matrices of Gödel n-valued logics (see [13]). For $n \geq 2$, consider the set $A_n = \{1/n, 2/n, \ldots, (n-1)/n, 1\}$ and the algebra $\mathbf{A_n} = (A_n, \wedge, \vee)$ defined by $a \wedge b = max(\{a, b\})$, $a \vee b = min(\{a, b\})$. If we consider $K_n = \{B_j : j \in A_n\}$, being $B_j = \{x \in A_n : j \leq x\}$, then (A_n, K_n) is a distributive logic for every $n \geq 2$. In fact, all these logics verify obviously PDI and PCI. Moreover, they are finitary and they verify nPA (because $\emptyset \notin K_n$). In addition, every (A_n, K_n) is $\wedge$-idempotent, and it verifies that $a \vee b \in \{a, b\}$. From all this we have that for every $n \geq 2$ and every $B_j \in K$, each logic $(A, M_{K_n}(B_j))$ is weakly distributive, according to Lemmas 3, 4, 1 and Corollary 3.

We conclude this section with the following remark: preservation of PCI and of PDI "in the inverse sense" is always valid. For that, let us take into account the following definitions, taken from [11] and [12] [2].

[2]These definitions are abstract ones. They are not concerned with any kind of established order on A.

Definition 15 *Let (A, K) be a logic and let $F \subseteq A$. We say that*
− F is $\wedge$-filter iff for every $a, b \in F$, $a \wedge b \in F$ if and only if $a \in F$ and $b \in F$;
− F is $\vee$-prime iff for every $a, b \in F$, $a \vee b \in F$ if and only if $a \in F$ or $b \in F$.

Proposition 18 *Let (A, K) be an abstract logic, $T \in K$. If $(A, M_K(T))$ verifies PCI (resp., PDI), then (A, K) verifies PCI (resp. PDI), too.*

Proof. From Proposition 3.2 in [11], it follows that a c.o. Cl verifies PCI iff every $T \in K$ is a $\wedge$-filter. Thus, if (A, K) does not verify PCI, then there is $F \in K$ such that F is not a $\wedge$-filter. Since $K \subseteq M_K(T)$, $M_K(T)$ does not verify PCI. The same reasoning can be applied to PDI using this fact (see [11], Proposition 3.4): Cl verifies PDI iff every $F \in K$ is $\vee$-prime. □

5 Final Conclusions

Besides its original motivation to extend relative closure spaces, $M_K(T)$-spaces can be considered as particular cases that can motivate the study of several interesting topics. Among them, the behavior of $M_K(T)$-spaces when considered as *operators between CSPs* deserves an in-depth study. With this in mind, we have studied the different order-theoretical aspects involving $M_K(T)$-spaces that were showed in this paper. In addition, we studied some applications of such spaces to Abstract Logic. It is worth noting that, even we were focused on $M_K(T)$-spaces, this kind of approach can be adapted to other "transformation of closure spaces", too.

Several notions of operators weaker than closure ones were studied in the literature (see, for example, [7, 15, 16]), but taking *generalized topologies* as a starting point. This notion is just the "interior-spaces general approach", dual to the "closure-spaces approach" given in this paper. Then, both concepts can be interdefined, and it would be very interesting to analyze in which way the $M_K(T)$-spaces (and all its order-theoretical considerations analyzed here) can be adapted to the interior operators mentioned above. Moreover, note that generalized topologies (as closure spaces) are just weakenings of the standard notion of Topological Spaces, as we previously said. Then, other weaker forms of T.S. can be analyzed in the context of the application of $M_K(T)$-spaces. We cite here the supra-topological spaces (see [14, 1]) and *infra-topological spaces* defined by Al-Odhari (see [18] for an actualized survey) among other related notions.

Another kind of application of $M_K(T)$-spaces deserves to be studied is its relation with the *fuzzy topological spaces* (F.T.S.) and, more specifically, with the *intuitionistic fuzzy topological spaces* (I.F.T.S.). The latter concept was defined by K. Atanassov (see [2]) based on the well-known definition of fuzzy set developed by L. Zadeh, and was further developed by D. Çoker. In the following, we briefly summarize the notions that are involved.

Given a "referential" set $X \neq \emptyset$, a *fuzzy set* is a pair $A = \{\langle x, \mu_A \rangle : x \in X\}$. Here, $\mu_A : X \to [0,1]$ indicates the "fuzzy degree of membership" (of every point $x \in X$) to A. When A is a "classical subset" of X, μ_A is nothing but its characteristic function. That is, $\mu_A(x) = 1$ if x belongs to A in the standard sense. If not (and only these two possibilities can exist), $\mu_A(x) = 0$. Now, if we add to this notion the conceptual framework of Intuitionistic Logic, then the "fuzzy membership" of x to A is independent of the "fuzzy non-membership" in the following sense: the Middle-Excluded Law does not need to be validated. Thus, an *intuitionistic fuzzy set* (I.F.S.) can be understood as a family $A = \{\langle x, \mu_A, \gamma_A \rangle : x \in X\}$ such that for every $x \in X$, $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ [3].

Once I.F.S. are defined, the respective "intuitionistic fuzzy" relations and operations (inclusion, union, intersection, and so on) can be redefined under this perspective (see [5] and [6]). These notions allow us to define I.F.T.S. in a natural way. At this point, it become obvious to study $M_K(T)$-spaces. This line of research (which was suggested by the referee) seems to be very interesting, because it focuses not only on the weakening of the notion of T.S., but on the "weak intersection" used here.

Finally, by inverting the problems posed in Section 4, the following question can be proposed: which kind of operators from/to abstract logics preserves all the properties that characterize, specifically, distributive logics?

All in all, we consider that the study of $M_K(T)$-spaces (and of other transformations between closure spaces related to them) deserve a deeper analysis, generalizing and/or improving the results presented in this paper.

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