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Operator-Valued Fourier Multipliers on Vector-Valued Orlicz Spaces and their Applications

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Abstract. In this paper, we extend the operator-valued Fourier multiplier theorem on Lebesgue spaces to vector-valued Orlicz spaces. Then we characterize the growth bound of a C_0-semigroup via Fourier multipliers in vector-valued Orlicz spaces and establish the relationship between exponential stability and Fourier multipliers in vector-valued Orlicz spaces.

Key Words: Fourier Multiplier, Orlicz Space, Operator-Valued, Semigroup
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Introduction

Let X and Y be Banach spaces. Let $\mathcal{L}(X,Y)$ be the space of bounded linear operators from X to Y. When $X = Y$, we denote $\mathcal{L}(X,Y)$ by $\mathcal{L}(X)$ for simplicity. We denote by $S(\mathbb{R}^n; X)$ the space of rapidly decreasing functions from $\mathbb{R}^n$ to X. The Fourier transform $\mathcal{F} : S(\mathbb{R}^n; X) \to S(\mathbb{R}^n; X)$ is defined by

$$(\mathcal{F}f)(t) \equiv \hat{f}(t) := \int_{\mathbb{R}^n} e^{-it\cdot s} f(s)\mathrm{d}s,$$

which is a bijection and whose inverse is given by

$$(\mathcal{F}^{-1}f)(t) \equiv \check{f}(t) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{it\cdot s} f(s)\mathrm{d}s,$$

where $f \in S(\mathbb{R}^n; X)$ and $t \in \mathbb{R}^n$.

We say that a bounded and strongly measurable function $m : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(X,Y)$ is a Fourier multiplier on $L^p(\mathbb{R}^n; X)$ if the expressions

$$T_m f = (m(\cdot)[\hat{f}(\cdot)])^\vee, \; f \in S(\mathbb{R}^n; X),$$

are well defined, and T_m extends to a bounded operator $T_m : L^p(\mathbb{R}^n; X) \to L^p(\mathbb{R}^n; Y)$. Fourier multiplier theorems play an important role in analysis. In particular, they can be used to prove maximal regularity of certain parabolic evolution equations [3] or stability theory [5, 7, 13].

For $X, Y = \mathbb{C}$, we have the scalar-valued case. When $p \in \{1, 2, \infty\}$, Weiss and Stein [18] gave sufficient conditions for m to be a Fourier multiplier. In $L^p(\mathbb{R}^d)$ for $p \notin \{1, 2, \infty\}$, a sufficient condition is

$$\sup_{\alpha \in \{0,1\}^d, x \in \mathbb{R}^d} |x^\alpha||D^\alpha m(x)| < \infty,$$

which was obtained by Mihlin in [12]. In $L^p(\mathbb{T})$, Marcinkiewicz [11] gave the bounded variation condition

$$\sup_{k \in \mathbb{Z}} |m_k| < \infty, \ \sup_{n \in \mathbb{N}} \sum_{2^n \le |k| < 2^{n+1}} |m_{k+1} - m_k| < \infty$$

for m to be a Fourier multiplier.

The study of operator-valued Fourier multipliers began in 1962 by Benedek, Calderón and Panzone [4], who gave a sufficient condition for a function m to be a L^q Fourier multiplier with $1 < q < \infty$. Amann [1] proved that for a multiplier $M : \mathbb{R}^n \to \mathcal{L}(X, Y)$ satisfying the modified Mihlin condition

$$\|D^\alpha M(\xi)\|_{\mathcal{L}(X,Y)} \le c_\alpha (1 + |\xi|)^{m-\alpha}, \quad |\alpha| \le n + 1,$$

the corresponding operator maps the vector-valued Besov space $B^{s+m}_{p,q}(\mathbb{R}^n; X)$ continuously into $B^s_{p,q}(\mathbb{R}^n; Y)$ for all values of $s \in \mathbb{R}^n$ and $p, q \in [1, \infty]$. In [2, 3], Arendt and Bu described an operator-valued multiplier theorem for Hilbert spaces and showed that the operator-valued Marcinkiewicz and Mihlin Fourier multiplier theorems are valid if and only if the underlying Banach space is isomorphic to a Hilbert space.

Fourier multipliers also have important applications in the study of L^p-maximal regularity, the asymptotic properties of solutions of evolution equations, and the semigroup of operators. However, asymptotic behavior can be deduced from the associated resolvent operators $R(\lambda, A) = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. A uniform bound for the resolvent is not sufficient to ensure exponential stability on general Banach spaces, but it was shown in [7] that exponential stability can be characterized in terms of L^p Fourier multiplier properties of the resolvent. Van Neerven [14] proved that a C_0-semigroup on Banach space X is uniformly exponentially stable if and only if it acts boundedly on $L^p(\mathbb{R}_+; X)$ by convolution. Wark [20] gave a necessary and sufficient condition for the boundedness of Fourier-Haar multiplier operators from $L^1([0, 1], X)$ to $L^1([0, 1], Y)$, where X is an arbitrary finite dimensional Banach space and Y is an arbitrary Banach space. Recently, Rozendaal [17] gave an overview of some recent results on operator-valued

(L^p, L^q) Fourier multipliers and stability theory for evolution equations and indicated how operator-valued (L^p, L^q) Fourier multipliers can be applied to functional calculus theory. Motivated by the Riesz transform, Vodák [19] proved that singular integrals satisfying Calderón-Zygmund conditions are well-defined on Orlicz spaces (for details on Orlicz spaces, which are natural extensions of Lebesgue spaces, we refer to [6, 9, 15]).

Motivated by [14, 17, 19], in this paper, we extend the relationship between Fourier multipliers and evolution equations to vector-valued Orlicz spaces.

The paper is organized as follows. In Section 1, we recall some notions and results on vector-valued Orlicz spaces to be used in the sequel. In Section 2, we prove the operator-valued Fourier multiplier theorem on vector-valued Orlicz spaces. In Section 3, we give a characterization of the growth bound of a C_0-semigroup via Fourier multipliers in Orlicz spaces and the relationship between exponential stability and Fourier multipliers in vector-valued Orlicz spaces.

1 Preliminaries

First, we recall the definition of Young function.

Definition 1 *A function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if it satisfies the following conditions:*

(i) Φ *is an increasing, convex, left continuous function;*

(ii) $\Phi(0) = 0$, $\lim_{s\to 0^+} \Phi(s) = 0$ *and* $\lim_{s\to\infty} \Phi(s) = \infty$*;*

(iii) $\lim_{s\to 0^+} \Phi(s)/s = 0$ *and* $\lim_{s\to\infty} \Phi(s)/s = \infty$.

Let Φ be a Young function. Then

$$k\Phi(t) \le \Phi(kt), \quad k \ge 1,\ t \ge 0;$$

and

$$\Phi(kt) \le k\Phi(t), \quad 0 \le k \le 1,\ t \ge 0.$$

Note also that Φ^{-1} is defined for $0 \le y$ by

$$\Phi^{-1}(y) = \inf\{x \ge 0 : \Phi(x) \ge y\}.$$

Let X be a Banach space. We denote by $L^0(\mathbb{R}^n; X)$ the space of all X-valued strongly measurable functions on $\mathbb{R}^n$.

Definition 2 *If Φ is a Young function, by a modular we mean a functional ρ_Φ defined on the set of strongly measurable functions f by the following formula*

$$\rho_\Phi(f) := \int_{\mathbb{R}^n} \Phi(\|f(x)\|)\mathrm{d}x.$$

We set

$$L^{\Phi}(\mathbb{R}^n;X) := \{f \in L^0(\mathbb{R}^n;X) : \rho_{\Phi}(\lambda f) < \infty \text{ for some } \lambda > 0\},$$

and

$$E^{\Phi}(\mathbb{R}^n;X) := \{f \in L^0(\mathbb{R}^n;X) : \rho_{\Phi}(\lambda f) < \infty \text{ for any } \lambda > 0\}.$$

Directly from above it follows that

$$E^{\Phi}(\mathbb{R}^n;X) \subset L^{\Phi}(\mathbb{R}^n;X).$$

The space $L^{\Phi}(\mathbb{R}^n;X)$ equipped with the Luxemburg norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^n;X)} := \inf\left\{\lambda > 0 : \rho_{\Phi}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

is a Banach spaces.

Definition 3 *Given a Young function Φ, we define its complemented function $\Phi^* : [0,\infty) \to [0,\infty)$ by the Legendre transform,*

$$\Phi^*(t) := \sup_{s\geq 0}\big\{st - \Phi(s)\big\} \textit{ for } t \geq 0.$$

Note that Φ^* is a Young function as well. Moreover, one can check that the complemented function of $\Phi^*(\cdot)$ equals $\Phi(\cdot)$, i.e., $\Phi^{**} = \Phi$ (see [6]).

Remark 1 *The pair (Φ, Φ^*) is called a complementary pair of Young functions. Its elements satisfy*

$$x \leq \Phi^{-1}(x)\Phi^{*-1}(x) \leq 2x, \quad x \geq 0,$$

and Young's inequality

$$xy \leq \Phi(x) + \Phi^*(y), \quad x, y \geq 0.$$

Thus, for $x \in X$ and $x^* \in X^*$, we get

$$|\langle x^*, x\rangle| \leq \Phi(\|x\|) + \Phi^*(\|x^*\|).$$

The following Orlicz norm will also be useful in the sequel:

$$\|f\|_{X,\Phi} := \sup\left\{\int_{\mathbb{R}^n}\big\langle f(x), g(x)\big\rangle \mathrm{d}x : \int_{\mathbb{R}^n}\Phi^*(\|g(x)\|)\mathrm{d}x \leq 1\right\}.$$

The equivalence between Orlicz norm and Luxemburg norm is well-known in the Orlicz spaces for scale-valued setting (see [10]); for the Banach space valued functions, we also have

$$\|f\|_{L^{\Phi}(\mathbb{R}^n;X)} \leq \|f\|_{X,\Phi} \leq 2\|f\|_{L^{\Phi}(\mathbb{R}^n;X)}. \tag{1}$$

Similar to the scalar-valued functions, we have the following result.

Lemma 1 *Let Φ be an Young function and Φ^* be its complemented function. Suppose $f \in L^{\Phi}(\mathbb{R}^n; X)$ and $g \in L^{\Phi^*}(\mathbb{R}^n; X^*)$. Then*

$$\int_{\mathbb{R}^n} \left|\left\langle f(x), g(x)\right\rangle\right| \mathrm{d}x \leq 2\|f\|_{L^{\Phi}(\mathbb{R}^n;X)}\|g\|_{L^{\Phi^*}(\mathbb{R}^n;X^*)}.$$

Definition 4 *A Young function $\Phi : [0, \infty) \to [0, \infty)$ is said to be in Δ_2 (denoted $\Phi \in \Delta_2$) if there exists a constant $c_{\Delta_2} > 0$ such that*

$$\Phi(2s) \leq \ c_{\Delta_2}\Phi(s), \quad s \geq 0.$$

Similar to scalar-valued case, one can prove the following result.

Lemma 2 *Let Φ be an Young function. If $\Phi \in \Delta_2$, then*

$$E^{\Phi}(\mathbb{R}^n; X) = L^{\Phi}(\mathbb{R}^n; X).$$

Lemma 3 *[9] Let Φ be a Young function satisfying the Δ_2-condition. Then there exist $p > 1$ and $b > 1$ such that*

$$\frac{\Phi(t_2)}{t_2^p} \leq \frac{b\Phi(t_1)}{t_1^p}, \quad 0 < t_1 < t_2.$$

Definition 5 *A function Φ is said to be quasiconvex if there exist a convex function ω and a constant $c > 0$ such that*

$$\omega(t) \leq \Phi(t) \leq \omega(ct), \quad t \in [0, \infty).$$

Given a Banach space X, we denote by $L^{p,X} = L^p(\mathbb{R}^n; X),\ p < \infty$, the space consisting of all X-valued measurable functions f defined in $\mathbb{R}^n$ such that

$$\|f\|_{L^{p,X}} = \left(\int_{\mathbb{R}^n} \|f(x)\|_X^p \mathrm{d}x\right)^{1/p} < \infty.$$

Similarly, the space $WL^{p,X} =$ weak-$L^{p,X}$ is formed by all X-valued functions f such that

$$\|f\|_{WL^{p,X}} = \sup_{t>0} t\left|\{x \in \mathbb{R}^n : \|f(x)\|_X > t\}\right|^{1/p} < \infty.$$

Here and in what follows, $|E|$ denotes the measure of measurable set E in $\mathbb{R}^n$.

Definition 6 *An operator T is said to be of weak type (p, q) if*

$$\lambda(\alpha, Tf) \leq \left(\frac{C\|f\|_p}{\alpha}\right)^q, \quad f \in L^p(\mathbb{R}^n; X),\ \alpha > 0,$$

with C independent of f, where and in what follows $\lambda(\alpha, Tf) := |\{x \in \mathbb{R}^n : \|Tf(x)\| > \alpha\}|$. An operator T is said to be of weak type (Φ, Φ) if

$$\Phi(\alpha)\lambda(\alpha, Tf) \leq C \int_{\mathbb{R}^n} \Phi(\|f(x)\|)\mathrm{d}x,\ f \in L^{\Phi}(\mathbb{R}^n; X),\ \alpha > 0,$$

with C independent of f.

2 Fourier multipliers

Given a Young function Φ, the space of all functions $m \in L^\infty(\mathbb{R}^n, \mathcal{L}(X,Y))$ for which T_m has a bounded extension from $L^\Phi(\mathbb{R}^n;X)$ to $L^\Phi(\mathbb{R}^n;Y)$ will be denoted by $\mathcal{M}_\Phi(\mathbb{R}^n;X,Y)$.

Direct calculation of T_m may encounter some problems that can not be solved at present. For this reason, we consider acting on them in $C_0^\infty(\mathbb{R}^n;X)$, the set of infinitely differentiable functions compactly supported in $\mathbb{R}^n$ and taking values in X. We will prove the density of $C_0^\infty(\mathbb{R}^n;X)$ in $L^\Phi(\mathbb{R}^n;X)$ with respect to the Luxemburg norm by the method developed in [21].

First, we give some notations and definitions. By $B_c(\mathbb{R}^n;X)$ we denote the set of bounded measurable functions compactly supported in $\mathbb{R}^n$ and taking values in X. By Definition 1 for any constant number $c>0$ and for every compact set $K \subset \mathbb{R}^n$,

$$\int_K \Phi(c)\mathrm{d}x < \infty. \tag{2}$$

Indeed, since $\Phi : [0,\infty) \to [0,\infty)$, we have $\Phi(c) < \infty$ for any $c \in [0,\infty)$. Then for every compact set $K \subset \mathbb{R}^n$, we have $\int_K \Phi(c)\mathrm{d}x < \infty$, because we have $|K| < \infty$ for any compact set in $\mathbb{R}^n$. In the sequel, we shall use C to denote a constant which may differ from line to line.

For $h \in \mathbb{R}^n$, let $\tau_h u$ stand for the translation operator defined by

$$\tau_h u(x) = \begin{cases} u(x+h), & \text{for } x \in \mathbb{R}^n \text{ and } x+h \in \mathbb{R}^n, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1 *Let Φ be a Young function. Then for any $u \in B_c(\mathbb{R}^n;X)$ and every $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that for $h \in \mathbb{R}^n$ with $|h| < \eta$, we have*

$$\|\tau_h u - u\|_{L^\Phi(\mathbb{R}^n;X)} < \varepsilon.$$

Proof. For $u \in B_c(\mathbb{R}^n;X)$, let supp $u := U \subset B_R$, where by B_R we denote a ball with radius $R > 0$. Let $h \in \mathbb{R}^n$ with $|h| < 1$. We have supp $\tau_h u \subset B_{R+1}$. Let $\overline{B}_{R+1}$ stand for the closed ball with radius $R+1$. Thanks to (2), for any constant $C > 0$ and any compact subset $K \subset \mathbb{R}^n$, we have $\Phi(C) \in L^1(K)$. Therefore, for arbitrary $\varepsilon \in [0,1)$, there is $\nu > 0$ such that for every measurable subset $G \subset K$,

$$\int_G \Phi(C)\mathrm{d}x < \frac{\varepsilon}{2} \text{ whenever } |G| < \nu. \tag{3}$$

For this ν, there exists $\varrho \in (0,1)$ such that $|H_\varrho| < 4^{-1}\nu$, where

$$H_\varrho = \{x \in B_{R+1} : \operatorname{dist}(x, \partial B_{R+1}) \le \varrho\}.$$

Define $U_\varrho = B_{R+1} \setminus H_\varrho$. Since u is measurable on U_ϱ, Luzin's theorem ensures that for $\nu > 0$, there exists a closed set $F_{1,\nu} \subset U_\varrho$ such that the restriction of u to $F_{1,\nu}$ is continuous and $|U_\varrho \setminus F_{1,\nu}| < 4^{-1}\nu$. Then we have

$$|B_{R+1} \setminus F_{1,\nu}| = |((B_{R+1} \setminus H_\varrho) \setminus F_{1,\nu}) \cup H_\varrho| = |(U_\varrho \setminus F_{1,\nu}) \cup H_\varrho| < 2^{-1}\nu.$$

The function u is uniformly continuous on the compact set $F_{1,\nu}$. It follows that for a fixed ε, there exists an $\eta \in (0, \varrho)$ such that for all $x, x+h \in F_{1,\nu}$ and $|h| < \eta$, we have

$$\|u(x+h) - u(x)\| < \frac{\varepsilon}{2(\int_U \Phi(1)\mathrm{d}x + 1)}. \tag{4}$$

Define two sets

$$F_{2,\nu} = \{x \in U, x+h \in F_{1,\nu}\} \text{ and } F_\nu = F_{1,\nu} \cap F_{2,\nu}.$$

The set F_ν is a closed subset of $\mathbb{R}^n$. In addition, we have $|B_{R+1} \setminus F_\nu| < \nu$. Indeed, since the Lebesgue measure is translation invariant we get

$$|B_{R+1} \setminus F_{1,\nu}| = |B_{R+1} \setminus F_{2,\nu}|.$$

Therefore,

$$\begin{aligned}|B_{R+1} \setminus F_\nu| &= |B_{R+1} \setminus (F_{1,\nu} \cap F_{2,\nu})| \\ &= |(B_{R+1} \setminus F_{1,\nu}) \cup (B_{R+1} \setminus F_{2,\nu})| \\ &\le |B_{R+1} \setminus F_{1,\nu}| + |B_{R+1} \setminus F_{2,\nu}| < \nu.\end{aligned}$$

If $x \notin B_{R+1}$, then for $|h| < \eta$ we have $x+h \notin B_R$, since otherwise we would get $x \in B_{R+1}$. Hence, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} \Phi(\|\tau_h u(x) - u(x)\|)\mathrm{d}x = \int_{B_{R+1}} \Phi(\|\tau_h u(x) - u(x)\|)\mathrm{d}x \\ &= \int_{B_{R+1}\cap F_\nu} \Phi(\|\tau_h u(x) - u(x)\|)\mathrm{d}x + \int_{B_{R+1}\setminus F_\nu} \Phi(\|\tau_h u(x) - u(x)\|)\mathrm{d}x \\ &=: I_1 + I_2.\end{aligned}$$

By (4) and the convexity of Φ, for $\varepsilon/2 < 1$, I_1 can be estimated as

$$\begin{aligned}I_1 &\le \int_{B_{R+1}\cap F_\nu} \Phi\left(\frac{\varepsilon}{2(\int_U \Phi(1)\mathrm{d}x + 1)}\right)\mathrm{d}x \le \frac{\varepsilon}{2}\int_{B_{R+1}\cap F_\nu} \Phi\left(\frac{1}{\int_U \Phi(1)\mathrm{d}x + 1}\right)\mathrm{d}x \\ &\le \frac{\varepsilon}{2}\int_{B_{R+1}\cap F_\nu} \Phi\left(\frac{1}{\int_U \Phi(1)\mathrm{d}x}\right)\mathrm{d}x < \frac{\varepsilon}{2}.\end{aligned}$$

As regards I_2, we use the fact that $u \in B_c(\mathbb{R}^n; X)$ is bounded by a constant $c > 0$ and then (3) to obtain

$$I_2 \le \int_{B_{R+1}\setminus F_\nu} \Phi(2c)\mathrm{d}x \le \frac{\varepsilon}{2}.$$

Hence, for any $\epsilon > 0$, there exists $\eta > 0$ such that

$$\int_{\mathbb{R}^n} \Phi(\|\tau_h u(x) - u(x)\|)\mathrm{d}x < \varepsilon \text{ as soon as } |h| < \eta.$$

Therefore, for $u\varepsilon^{-1} \in B_c(\mathbb{R}^n; X)$, we get

$$\int_{\mathbb{R}^n} \Phi(\|\tau_h u(x) - u(x)\|\varepsilon^{-1})\mathrm{d}x \leq 1,$$

which gives

$$\|\tau_h u - u\|_{L^\Phi(\mathbb{R}^n;X)} \leq \varepsilon \text{ whenever } |h| < \eta.$$

This finishes the proof.□

Let J stand for the Friedrichs mollifier kernel defined on $\mathbb{R}^n$ by

$$J(x) = \begin{cases} ke^{-1/(1-|x|^2)}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

where $k > 0$ is such that $\int_{\mathbb{R}^n} J(x)\mathrm{d}x = 1$. For $\varepsilon > 0$, we define $J_\varepsilon(x) = \varepsilon^{-n} J(x\varepsilon^{-1})$ and $u_\varepsilon = J_\varepsilon * u$ by

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} J_\varepsilon(x - y)u(y)\mathrm{d}y = \int_{B(0,1)} u(x - \varepsilon y)J(y)\mathrm{d}y. \tag{5}$$

A direct consequence of Theorem 1 is the following approximation result.

Corollary 1 *Let Φ be a Young function and let $u \in B_c(\mathbb{R}^n; X)$. For any $\varepsilon > 0$, we have $u_\varepsilon \in C_0^\infty(\mathbb{R}^n; X)$. Furthermore,*

$$\|u_\varepsilon - u\|_{L^\Phi(\mathbb{R}^n;X)} \to 0, \quad \text{as } \varepsilon \to 0^+.$$

Proof. Let $u \in B_c(\mathbb{R}^n; X)$. The function u_ε defined in (5) belongs to $C_0^\infty(\mathbb{R}^n; X)$ whenever $\varepsilon > 0$. Let Φ^* stand for the complementary Young function of Φ and let $v \in L^{\Phi^*}(\mathbb{R}^n; X^*)$. By Fubini's theorem and Hölder's inequality we can write

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\langle v(x), u_\varepsilon(x) - u(x)\rangle|\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left|\left\langle v(x), \int_{B(0,1)} u(x - \varepsilon y)J(y)\mathrm{d}y - u(x)\right\rangle\right|\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left|\left\langle v(x), \int_{B(0,1)} u(x - \varepsilon y)J(y)\mathrm{d}y - \int_{\mathbb{R}^n} u(x)J(y)\mathrm{d}y\right\rangle\right|\mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} \left|\left\langle v(x), \int_{\mathbb{R}^n} (u(x - \varepsilon y) - u(x))J(y)\mathrm{d}y\right\rangle\right|\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\langle v(x), u(x-\varepsilon y) - u(x)\rangle| J(y)\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\langle v(x), u(x-\varepsilon y) - u(x)\rangle| J(y)\mathrm{d}x\mathrm{d}y \\
&\leq 2\|v\|_{L^{\Phi^*}(\mathbb{R}^n;X^*)} \int_{|y|\leq 1} \|\tau_{-\varepsilon y}u - u\|_{L^{\Phi}(\mathbb{R}^n;X)} J(y)\mathrm{d}y.
\end{aligned}$$

Hence, by the definition of the Orlicz norm and the inequality (1), we obtain

$$\|u_\varepsilon - u\|_{L^{\Phi}(\mathbb{R}^n;X)} \leq 2\int_{|y|\leq 1} \|\tau_{-\varepsilon y}u - u\|_{L^{\Phi}(\mathbb{R}^n;X)} J(y)\mathrm{d}y.$$

We can now use Theorem 1. Given $\mu > 0$, there exists $\eta > 0$ such that for $\varepsilon < \eta$, we get

$$\|\tau_{-\varepsilon y}u - u\|_{L^{\Phi}(\mathbb{R}^n;X)} \leq \mu$$

for every y with $|y| \leq 1$. Then we conclude that

$$\|u_\varepsilon - u\|_{L^{\Phi}(\mathbb{R}^n;X)} \leq 2\mu \int_{|y|\leq 1} J(y)\mathrm{d}y = 2\mu,$$

which gives the result. □

Lemma 4 *Let Φ be a Young function. Then $B_c(\mathbb{R}^n;X)$ is dense in $E^{\Phi}(\mathbb{R}^n;X)$ with respect to the strong topology in $L^{\Phi}(\mathbb{R}^n;X)$.*

Proof. If $u \in E^{\Phi}(\mathbb{R}^n;X)$, then for all $\lambda > 0$, we have $\Phi(||u||\lambda^{-1}) \in L^1(\mathbb{R}^n;X)$. Denote by T_j, $j > 0$, the truncation function at levels $\pm j$ defined on $\mathbb{R}$ by $T_j(s) = \max\{-j, \min\{j,s\}\}$. We define the sequence $\{u_j\}$ by

$$u_j = T_j u \chi_{K_j},$$

where χ_{K_j} stands for the characteristic function of the set

$$K_j = \{x \in \mathbb{R}^n : |x| \leq j\}.$$

Hence, the functional sequence $\{u_j\}$ belongs to $B_c(\mathbb{R}^n;X)$ and converges almost everywhere to u in $\mathbb{R}^n$. Thus, $\Phi(\lambda^{-1}\|u_j(x) - u(x)\|) \to 0$ a.e. in $\mathbb{R}^n$, and

$$\Phi((2\lambda)^{-1}\|u_j(x) - u(x)\|) \leq \Phi(\lambda^{-1}\|u(x)\|) \in L^1(\mathbb{R}^n;X).$$

Therefore, by the Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^n} \Phi((2\lambda)^{-1}\|u_j(x) - u(x)\|)\mathrm{d}x \leq 1 \text{ for } j \text{ large enough},$$

which yields $\lim_{j\to+\infty} \|u_j - u\|_{L^{\Phi}(\mathbb{R}^n;X)} \leq 2\lambda$. Since λ is an arbitrary positive number, we get

$$\lim_{j\to+\infty} \|u_j - u\|_{L^{\Phi}(\mathbb{R}^n;X)} = 0.$$

This yields the result. □

Combining Corollary 1 and Lemma 4, we obtain

Theorem 2 *Let* Φ *be a Young function. Then* $C_0^\infty(\mathbb{R}^n;X)$ *is dense in* $E^\Phi(\mathbb{R}^n;X)$ *with respect to the strong topology in* $E^\Phi(\mathbb{R}^n;X)$.

Combining Theorem 2 and Lemma 2, we obtain

Theorem 3 *Let* Φ *be a Young function and* $\Phi \in \Delta_2$. *Then* $C_0^\infty(\mathbb{R}^n;X)$ *is dense in* $L^\Phi(\mathbb{R}^n;X)$ *with respect to the strong topology in* $L^\Phi(\mathbb{R}^n;X)$.

Now we present our main results. The proof of the following one is similar to the scalar-valued case (see Theorem 3.2 in [22]) and therefore will be omitted

Theorem 4 *Let* $\Phi \in \Delta_2$. *Suppose that* T_m *is an operator of weak type* $(1,1)$ *and weak type* (p,p), $p>1$. *Then there exists* $c>0$ *such that*

$$\Phi(\alpha)\lambda(\alpha,T_mf) \leq c\int_{\mathbb{R}^n}\Phi(\|f(x)\|)\mathrm{d}x$$

for $f \in L^\Phi(\mathbb{R}^n;X)$ *and for all* $\alpha \in [0,\infty)$.

Theorem 5 *Let* $\Phi \in \Delta_2$ *and let* Φ^γ *be quasiconvex for some* $\gamma \in (0,1)$. *Suppose that* T_m *is an operator of weak type* $(1,1)$ *and weak type* (p,p), $p>1$. *Then there exists* $C>0$ *such that*

$$\int_{\mathbb{R}^n}\Phi(\|T_mf(x)\|)\mathrm{d}x \leq C\int_{\mathbb{R}^n}\Phi(\|f(x)\|)\mathrm{d}x \tag{6}$$

and

$$\|T_mf\|_{L^\Phi(\mathbb{R}^n;Y)} \leq C\|f\|_{L^\Phi(\mathbb{R}^n;X)} \tag{7}$$

for all $f \in L^\Phi(\mathbb{R}^n;X)$.

Proof. The proof of (6) is similar to the scalar-valued case, see Theorem 3.3 in [19].

To prove (7), we suppose $\|f\|_{L^\Phi(\mathbb{R}^n;X)} = 1$. Then

$$\int_{\mathbb{R}^n}\Phi(\|f(x)\|)\mathrm{d}x \leq 1,$$

and therefore,

$$\int_{\mathbb{R}^n}\Phi(\|T_mf(x)\|)\mathrm{d}x \leq C.$$

If $C \leq 1$, we have $\|T_mf\|_{L^\Phi(\mathbb{R}^n;Y)} \leq \|f\|_{L^\Phi(\mathbb{R}^n;X)}$. If $C>1$, then $1/C<1$, and we find that

$$\frac{1}{C}\int_{\mathbb{R}^n}\Phi(\|T_mf(x)\|)\mathrm{d}x \leq 1.$$

Since Φ is a convex function, we have

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{1}{C}\|T_m f(x)\|\Big)\mathrm{d}x \leq 1.$$

Hence,

$$\|T_m f\|_{L^\Phi(\mathbb{R}^n;Y)} \leq C,$$

therefore,

$$\|T_m f\|_{L^\Phi(\mathbb{R}^n;Y)} \leq \max\{1, C\}.$$

By the scale argument, we obtain

$$\|T_m f\|_{L^\Phi(\mathbb{R}^n;Y)} \leq C\|f\|_{L^\Phi(\mathbb{R}^n;X)},$$

which finishes the proof. □

In the following we apply Theorem 5 to Fourier multipliers which satisfy the Hörmander condition.

Definition 7 *Let X, Y be Banach spaces. We say that $k : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(X, Y)$ satisfies the Hörmander condition if k satisfies the size condition*

$$\sup_{R>0} \int_{R\leq|x|\leq 2R} \|k(x)\|\mathrm{d}x = A_1 < \infty,$$

and smoothness condition

$$\sup_{y\neq 0} \int_{|x|>2|y|} \|k(x-y) - k(x)\|\mathrm{d}x = A_2 < \infty.$$

Lemma 5 *(Theorem 11.2.5 in [8]) Let X and Y be Banach spaces and $p_0 \in [1, \infty]$. Let*

$$T \in \mathcal{L}(L^{p_0}(\mathbb{R}^n; X), L^{p_0,\infty}(\mathbb{R}^n; Y))$$

with norm $A_0 := \|T\|_{\mathcal{L}(L^{p_0}(\mathbb{R}^n;X),L^{p_0,\infty}(\mathbb{R}^n;Y))}$. If T has a Hörmander kernel K, then

(i) *T extends uniquely to $T \in \mathcal{L}(L^p(\mathbb{R}^n; X), L^p(\mathbb{R}^n; Y))$ for all $p \in (1, p_0)$, and*

$$\|T\|_{\mathcal{L}(L^p(\mathbb{R}^n;X),L^p(\mathbb{R}^n;Y))} \leq c_d\Big(\frac{p_0 - 1}{(p_0 - p)(p - 1)}\Big)^{1/p}(A_0 + \|K\|);$$

(ii) *T extends uniquely to $T \in \mathcal{L}(L^1(\mathbb{R}^n; X), L^{1,\infty}(\mathbb{R}^n; Y))$ and*

$$\|T\|_{\mathcal{L}(L^1(\mathbb{R}^n;X),L^{1,\infty}(\mathbb{R}^n;Y))} \leq c_d(A_0 + \|K\|).$$

where $\|K\|$ is the smallest constant that makes the Hörmander condition hold.

Combining Theorem 5 and Lemma 5, we obtain the following corollary.

Corollary 2 *Let k be as in the assumptions of Definition 7. Let $\Phi \in \Delta_2$, Φ^γ be quasiconvex for some $\gamma \in (0,1)$. Suppose that the operator T given by convolution with k maps $L^r(\mathbb{R}^n;X)$ to $L^r(\mathbb{R}^n;Y)$ for some $1 < r \leq \infty$. Then there exists a positive constant $C > 0$ such that*

$$\|Tf\|_{L^\Phi(\mathbb{R}^n;Y)} \leq C\|f\|_{L^\Phi(\mathbb{R}^n;X)},$$

where $Tf(x) = \lim_{\epsilon\to 0^+}\int_{|x-y|>\epsilon} k(x-y)f(y)\mathrm{d}y$.

3 Applications to evolution equations

The purpose of this section is to generalize the results in [17] to Orlicz spaces, namely, to find the relationship between exponential stability and Fourier multipliers in vector-valued Orlicz spaces. Due to the relationship between the generator and semigroups, we can examine the spectral properties of generator A instead of semigroups $(T(t))_{t\geq 0}$. This approach to the asymptotic behavior of solutions to evolution equations is useful in practice, because the resolvent is typically more accessible than the semigroup itself (see, for example, [7, 14, 16]).

To state our results, we need notations and lemmas. The identity operator on X is denoted by I_X, and we typically write λ for λI_X when $\lambda \in \mathbb{C}$. The domain of a closed operator A on X is $D(A)$, a Banach space with the norm

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_X, \quad x \in D(A).$$

The resolvent set is $\rho(A) = \mathbb{C}\backslash\sigma(A)$. We write $R(\lambda, A) = (\lambda - A)^{-1}$ for the resolvent operator of A at $\lambda \in \rho(A)$.

Let T be a C_0-semigroup with generator A. By

$$s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$$

we denote the spectral bound of A, and the growth bound

$$\omega_0(T) := \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0}\|e^{-\omega t}T(t)\| < \infty\right\}.$$

Then $s(A) \leq \omega_0(T)$ but strict inequality may occur. For examples illustrating this fact, we refer to [13]. For this reason, it is interesting to characterize the $\omega_0(T)$ of T by properties of the resolvent $R(\lambda, A)$ of A.

Lemma 6 *[5] Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space X, and take constants $\omega' \in \mathbb{R},\ M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega' t}$$

for $t \geq 0$. For the generator A of $(T(t))_{t\geq 0}$, the following properties hold:

(i) *if* $\lambda \in \mathbb{C}$ *is such that* $R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)x\mathrm{d}s$ *exists for all* $x \in X$, *then* $\lambda \in \rho(A)$ *and* $R(\lambda, A) = R(\lambda)$;

(ii) *if* $\mathrm{Re}\lambda > \omega'$, *then* $\lambda \in \rho(A)$, *and the resolvent is given by the integral expression in* (i);

(iii) $\|R(\lambda, A)\| \leq M/(\mathrm{Re}\lambda - \omega')$ *for all* $\mathrm{Re}\lambda > \omega'$.

Without loss of generality, suppose that $\omega' = 0$ in Lemma 6. Then the resolvent $R(\omega + i\xi, A)$ exists for all $\omega > 0$ and $\xi \in \mathbb{R}$, and the integral expression for resolvent is given by

$$R(\omega + i\xi, A)x = \int_0^\infty e^{-t(\omega+i\xi)}T(t)x\mathrm{d}t$$

for all $x \in X$. We can invert this Laplace transform:

$$e^{-\omega t}T(t)x = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi}R(\omega + i\xi, A)x\mathrm{d}\xi \tag{8}$$

for $t \geq 0$, where the integral converges absolutely for x in suitable dense subspaces of X, such as the fractional domain $D((-A)^\alpha)$ for $\alpha > 1$ (see [16]). For a general $x \in X$, we can use information about $R(\omega + i\xi, A)x$ to bound the integral in (8) and obtain $C' \geq 0$ such that

$$\|T(t)x\|_X \leq C'e^{\omega t}, \quad t \geq 0. \tag{9}$$

If $\omega < 0$, we also can obtain decay estimates for the orbit $t \to T(t)x$. Suppose that $i\mathbb{R} \subseteq \rho(A)$ and that $\sup_{\xi\in\mathbb{R}} \|R(i\xi, A)\|_{\mathcal{L}(X)} < \infty$. Then a Neumann series argument yields that for any $\omega < 0$,

$$\omega + i\mathbb{R} \subseteq \rho(A)$$

and

$$\sup_{\xi\in\mathbb{R}} \|R(\omega + i\xi, A)\|_{\mathcal{L}(X)} < \infty.$$

Using the above equation, we obtain $\int_{\mathbb{R}} \|R(\omega + i\xi, A)x\|_X\mathrm{d}\xi < \infty$ for all $x \in D((-A)^\alpha)$ and $\alpha > 1$. And then (8) indeed implies (9) for such x.

The resolvent also appears as a Fourier multiplier in the asymptotic theory of evolution equations (see [16]). Suppose A generates a uniformly bounded C_0-semigroup $(T(t))_{t\geq 0}$ and $\sup_{\xi\in\mathbb{R}} \|R(i\xi, A)\|_{\mathcal{L}(X)} < \infty$. Then we may fix $\omega < 0$ such that $\sup_{\xi\in\mathbb{R}} \|R(\omega + i\xi, A)\|_{\mathcal{L}(X)} < \infty$.

Lemma 7 *[16] Let* $n \in \mathbb{N}_0$, $x \in X$ *and* $\xi \in \mathbb{R}$, *and let* A *be the generator of a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X. *Suppose that* $i\xi \in \rho(A)$ *and* $[t \mapsto t^nT(t)x] \in L^1([0, \infty), X)$. *Then*

$$\mathcal{F}[t \mapsto t^nT(t)x](\xi) = n!(i\xi - A)^{-n-1}x$$

and

$$\mathcal{F}\bigg(\int_0^\infty t^nT(t)g(\cdot-t)x\mathrm{d}t\bigg)(\xi)=\hat{g}(\xi)n!(i\xi-A)^{-n-1}x,\ g\in L^1(\mathbb{R}).$$

Definition 8 *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and let $n\in\mathbb{N}_0$. A Banach space Y which is continuously embedded in X is (A,n)-admissible if the following conditions hold:*

(i) *there exists a constant $C_T\in[0,\infty)$ such that $T(t)Y\subseteq Y$ and*

$$\|T(t)\|_{\mathcal{L}(Y)}\leq C_T\|T(t)\|_{\mathcal{L}(X)}\ \textit{for}\ t\in[0,\infty);$$

(i) *there exists a dense subspace $Y_0\subseteq Y$ such that $[t\mapsto t^nT(t)y]\in L^1([0,\infty);X)$ for all $y\in Y_0$.*

Next result establishes polynomial stability of the semigroup $(T(t))_{t\geq 0}$.

Theorem 6 *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, and assume that $(T(t))_{t\geq 0}$ is uniformly bounded. Let $n\in\mathbb{N}_0$ and let Y be an (A,n)-admissible space, $\Phi\in\Delta_2$. Suppose $R(i\cdot,A)^k\in\mathcal{M}_\Phi(\mathbb{R};Y,X)$ for $k\in\{n,n+1\}\cap\mathbb{N}_0$. Then*

$$\sup_{t\geq 0}\|t^nT(t)\|_{\mathcal{L}(Y,X)}<\infty.$$

Proof. First, let $M,\omega\geq 1$ be such that $\|T(t)\|_{\mathcal{L}(X)}\leq Me^{(\omega-1)t}$, for all $t\geq 0$, and set

$$\begin{aligned}m(\xi):&=n!R(i\xi,A)^n(I_X+\omega R(i\xi,A))\\&=n!R(i\xi,A)^n+n!\omega R(i\xi,A)^{n+1}\in\mathcal{L}(Y,X),\quad \xi\in\mathbb{R}\setminus\{0\}.\end{aligned}$$

It follows from the assumptions that for $f\in L^\Phi(\mathbb{R};Y)$,

$$T_m:L^\Phi(\mathbb{R},Y)\to L^\Phi(\mathbb{R};X)$$

is bounded:

$$\|T_m\|\leq n!(C_n+\omega C_{n+1}). \tag{10}$$

Here C_k is a suitable constant of $\|R(i\xi,A)^k\|_{\mathcal{M}_\Phi(\mathbb{R};Y,X)}$ for $k\in\mathbb{N}$, and $C_0:=\|I_Y\|_{\mathcal{L}(Y,X)}$. Now let $Y_0\subseteq Y$ be as in Definition 8 and fix $x\in Y_0$. Lemma 7 yields

$$\mathcal{F}[t\mapsto t^nT(t)x](\xi)=n!(i\xi-A)^{-n-1}x=n!R(i\xi,A)^{n+1}x. \tag{11}$$

Set

$$t\mapsto f(t):=\begin{cases}e^{-\omega t}T(t)x, & t\geq 0,\\ 0, & t<0.\end{cases}$$

According to Definition 8, we have

$$\|f(t)\|_Y \le \|e^{-\omega t}T(t)\|_{\mathcal{L}(Y)}\|x\|_Y \le C_T\|e^{-\omega t}T(t)\|_{\mathcal{L}(X)}\|x\|_Y \le C_T M e^{-t}\|x\|_Y. \tag{12}$$

Hence, $f \in L^1(\mathbb{R}; Y) \cap L^\infty(\mathbb{R}; Y)$. Indeed, as t tends to infinity, the value of e^{-t} tends to 0. Since $\Phi \in \Delta_2$, from Lemma 3 we find that

$$\int_0^\infty \Phi\left(\frac{\|f(t)\|_Y}{C_T M\|x\|_Y}\right)\mathrm{d}t \le \int_0^\infty \Phi(e^{-t})\mathrm{d}t \le C\int_0^\infty e^{-tp}\Phi(1)\mathrm{d}t \le Cp^{-1}\Phi(1).$$

If $Cp^{-1}\Phi(1) \le 1$, we get $\|f\|_{L^\Phi(\mathbb{R};Y)} \le C_T M\|x\|_Y$.
If $Cp^{-1}\Phi(1) > 1$, then $(Cp^{-1}\Phi(1))^{-1} < 1$, and we have

$$\begin{aligned}&\int_0^\infty \Phi\left(\frac{\|f(t)\|_Y}{C_T M\|x\|_Y Cp^{-1}\Phi(1)}\right)\mathrm{d}t\\ &\le (Cp^{-1}\Phi(1))^{-1}\int_0^\infty \Phi\left(\frac{\|f(t)\|_Y}{C_T M\|x\|_Y}\right)\mathrm{d}t = 1.\end{aligned}$$

Hence,

$$\|f\|_{L^\Phi(\mathbb{R};Y)} \le C_T M\|x\|_Y Cp^{-1}\Phi(1).$$

Therefore,

$$\|f\|_{L^\Phi(\mathbb{R};Y)} \le C\|x\|_Y,$$

where C is independent of x.

Moreover, $\hat{f}(\cdot) = R(\omega + i\cdot, A)x$. Therefore, by the resolvent identity,

$$m(\xi)\hat{f}(\xi) = n!R(i\xi, A)^{n+1}x.$$

Thus,

$$\mathcal{F}[t \mapsto t^n T(t)x](\xi) = n!R(i\xi, A)^{n+1}x = m(\xi)\hat{f}(\xi),$$

i.e.,

$$[t \mapsto t^n T(t)x] = \mathcal{F}^{-1}(m(\cdot)\hat{f}(\cdot)) = T_m f.$$

Combining (11) and (12) with (10) yields

$$\sup_{t\ge 0}\|t^n T(t)x\|_X \le \|T_m f\|_{L^\Phi(\mathbb{R};X)} \le \|T_m\| \cdot \|f\|_{L^\Phi(\mathbb{R};Y)} \le C\|x\|_Y$$

where C is independent of x. The required result now follows since $Y_0 \subseteq Y$ is dense. □

Further, we establish general exponential stability.

Theorem 7 *Let Y be a Banach space that is continuously embedded in X and is dense in X. Let $(T(t))_{t\ge 0}$ be a C_0-semigroup with generator A on a*

Banach space X, $\Phi \in \Delta_2$. Let $\omega' \in \mathbb{R}$ be such that $\lambda \in \rho(A)$ for all $\lambda \in \mathbb{C}$ with $Re(\lambda) \geq \omega'$ and

$$\sup_{\mathrm{Re}(\lambda)\geq\omega'} \|R(\lambda, A)\|_{\mathcal{L}(X)} < \infty.$$

For each $\omega > \omega'$, suppose $R(\omega + i\cdot, A) \in \mathcal{M}_\Phi(\mathbb{R}; Y, X)$. Then there exists $C \geq 0$ such that $\|T(t)x\|_X \leq Ce^{\omega t}\|x\|_Y$ for all $x \in Y$ and $t \geq 0$.

Proof. First, we choose $\widetilde{\omega} > 0$ such that $(e^{-\widetilde{\omega}t}T(t))_{t\geq 0}$ is exponentially stable, i.e., there exist $M > 0, \mu > 1$ such that $\|e^{-\widetilde{\omega}t}T(t)x\|_X \leq Me^{-\mu t}\|x\|_Y$ for all $t \geq 0,\ x \in Y$, and set

$$m(\xi) := I_X + (\widetilde{\omega} - \omega)R(\omega + i\xi, A) \in \mathcal{L}(Y, X).$$

It follows from the assumptions that for $f \in L^\Phi(\mathbb{R}; Y)$,

$$T_m : L^\Phi(\mathbb{R}; Y) \to L^\Phi(\mathbb{R}; X)$$

is bounded by

$$\|T_m\| \leq C_0 + (\widetilde{\omega} - \omega)C_1.$$

Here, C_1 is a suitable constant of $\|R(\omega+i\xi, A)\|_{\mathcal{M}_\Phi(\mathbb{R};Y,X)}$ and $C_0 := \|I_Y\|_{\mathcal{L}(Y,X)}$. Set $f(t) := e^{-\widetilde{\omega}t}T(t)x$ for $t \geq 0$, and $f \equiv 0$ on $(-\infty, 0)$. Then

$$\|f(t)\|_Y = \|e^{-\widetilde{\omega}t}T(t)x\|_Y \leq Me^{-\mu t}\|x\|_Y.$$

Similar to the proof of Theorem 6, we find that

$$\|f\|_{L^\Phi(\mathbb{R};Y)} \leq C\|x\|_Y,$$

where C is independent of x.

Moreover, $\hat{f}(\cdot) = R(\widetilde{\omega}+i\cdot, A)x$ and by the resolvent identity we can write

$$\begin{aligned} m(\xi)\mathcal{F}f(\xi) &= (I_X + (\widetilde{\omega} - \omega)R(\omega + i\xi, A))R(\widetilde{\omega} + i\xi, A)x \\ &= R(\widetilde{\omega} + i\xi, A)x + (\widetilde{\omega} - \omega)R(\omega + i\xi, A)R(\widetilde{\omega} + i\xi, A)x \\ &= R(\widetilde{\omega} + i\xi, A)x + R(\omega + i\xi, A)x - R(\widetilde{\omega} + i\xi, A)x \\ &= R(\omega + i\xi, A)x. \end{aligned}$$

Hence, for $x \in Y$, we can use $\int_{\mathbb{R}} \|R(\omega+i\xi, A)x\|_X \mathrm{d}\xi < \infty$ to take the inverse Fourier transform in (8) and obtain

$$e^{-\omega t}T(t)x = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi}R(\omega + i\xi, A)x\mathrm{d}\xi = \mathcal{F}^{-1}(m \cdot \mathcal{F}f)(t) = T_m(f)(t)$$

for $t \geq 0$, i.e., $[t \mapsto e^{-\omega t}T(t)x] = T_m(f)$. Since the resolvent is absolutely integrable for $x \in Y$,

$$\sup_{t\geq 0} \|e^{-\omega t}T(t)x\|_X \leq \|T_m f\|_{L^\Phi(\mathbb{R};X)} \leq \|T_m\| \cdot \|f\|_{L^\Phi(\mathbb{R};Y)} \leq C\|x\|_Y.$$

Thus,

$$\|T(t)x\|_X \leq Ce^{\omega t}\|x\|_Y, \quad t \geq 0,$$

where C is independent of x. □

Directly from Theorem 7, it follows

Corollary 3 *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A. Let $\Phi \in \Delta_2$. Then*

$$\omega_0(T) = \inf\left\{\mu > s(A) : \sup_{\alpha\geq\mu} \|R(\alpha + i\cdot, A)\|_{\mathcal{M}_\Phi(\mathbb{R};X)} < \infty\right\}.$$

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