Armenian Journal of Mathematics
Volume 7, Number 2, 2015, 85–96

Several Remarks on Pascal Automorphism and Infinite Ergodic Theory

A. M. Vershik[1]
St. Petersburg Department of Steklov Institute of Mathematics
Mathematical Department of. St. Petersburg State University
Moscow Institute for Information Transmission Problems

To Professor Arshag Hajian in connection with his jubilee

Abstract. We interpret the Pascal-adic transformation as a generalized induced automorphism (over odometer) and formulate the σ-finite analog of odometer which is also known as "Hajian-Kakutani transformation" (former "Ohio state example"). We shortly suggest a sketch of the theory of random walks on the groups on the base of σ-finite ergodic theory.

Key Words: infinite measure space, ergodic, measure preserving transformation, random walk, adic transformation, Pascal automorphism, odometer.
Mathematics Subject Classification 2010: 37A40, 28D05

Infinite Ergodic theory, i.e. the abstract theory of transformations with an infinite (σ-finite invariant measure) began with the well-known work of E. Hopf and then became one of the branches of Ergodic theory. Many of the concepts and facts of this theory failed to transfer on the case of actions with infinite measure, while not all of them were automatic. However, the authentic specificity of the area was revealed after the discovery by S. Kakutani and his disciple A. Hajian so-called *eww*-sets. In my opinion, the theory of these sets are still not took the worthy place in the dynamics. In this note, I do not touch this issue, and write only about the problems more close to me, which also are relevant to these sets. I pay tribute to Prof. Hajian for his long-standing efforts in this area and for his faithfulness to the selected topic.

[1]The research is supported by the Russian Science Foundation grant 14-11-00581.

1 Short history of Pascal automorphism

Pascal automorphism was used (without this name) in the paper by S. Kakutani [1] and coauthors in connection with a beautiful combinatorial problem about equidistribution of sequences of partitions, and also in connection with σ-finite ergodic theory. The main observation was in the paper by Kakutani [1] where he considered that combinatorial problem: *the formulas (8),(9) on the page 266 in [1] is just the formula of Pascal automorphism as measure preserving transformation of the unit interval with Lebesgue measure. It one-to-one coincides with my formula of the next paragraph which I suggested in 1981 [5].*

In the paper by A. Hajan, Y. Ito and S. Kakutani [2] it was proved that (Pascal) automorphism is ergodic (with respect to any Bernoulli measure on the interval). It is the key moment in the proof of equidistribution in the Kakutani Problem. But in [2] Pascal automorphism was used for needs of σ-finite ergodic theory. Namely the integral model over Pascal automorphism was later called "Ohio-State example" (OSE) and now "Hajian-Kakutani transformation" after one of the first presentations in [16], and later was developed by A. Hajian and his collaborators. This is a natural example of the big series of transformations with σ-finite invariant measure.

I did not know about papers [1, 2] before my visit to NE-University in 2011, when during my talk about Pascal automorphisms on seminar of Prof. A. Hajian the participants pointed out that Pascal automorphism they knew but from a different point of view.

I defined the Pascal automorphism in the paper [5] as a nice and simplest non-trivial case of what I had called "adic transformation". [2] Adic transformations was defined in the end of 70-th (see [4]) in order to develop the strong approximation in ergodic theory. Adic realization of the transformation is nothing more than sequence of the coherent Rokhlin towers considered in the space of paths of the graph, and what is important the height of towers can depend on the point and can change with n. [3]

Now adic transformations became very popular — we can speak about adic-type of dynamics as a theory of the special type of dynamical systems like symbolic dynamics. This is a sort of constructions of important examples and counter-examples in the theory. Shortly speaking adic dynamics is dynamics of paths of the graded graphs (Bratteli diagrams) rather action of the group. [4]

[2] "adic" means "p-adic" without "p", like Arnold's notion of "versal" deformation means "universal" without "uni-".

[3] The Dye's theorem about isomorphism of orbit partitions of any ergodic transformations is an easy corollary of adic realization (see [12]). The analog of Dye's theorem for ergodic transformation with σ-finite invariant measure ([15]) is corollary of the integral realization of automorphism (see next paragraph).

[4] The popular term "Bratteli-Vershik diagram" is not precise: the "adic transforma-

From topological point of view this is a dynamics on Cantor set or more exactly on the space of paths of a graded graph. From symbolic point of view this is a "transversal dynamics" on the Markov or more general but in general not stationary compact. For the stationary case this definition had been considered by S.Ito [6]. Measure-theoretical and Borel approach to the adic dynamics is useful for many asymptotical and probabilistic problems about combinatorics and topological structure of space of paths or Markov trajectories. As the examples we can mention limit shape theorems, approximation, entropy and so on, see my survey [14]. In this notes I will mention several questions which related to Pascal automorphism as well as to the σ-finite Ergodic theory. We do not touch here more deeper properties of automorphisms with infinite invariant measures, like *ww*-sets and others.

In the next section we made some definitions and define a new notion of generalized induced automorphisms which shows that Pascal automorphism is induced in the sense of new definition of odometer.

In the last section we gave a sketch of the σ-finite version of the theory of random walks on the groups.

2 The odometer and Pascal automorphism

Recall the definitions of odometer and Pascal automorphisms. Consider $X = \mathbf{Z}_2$, the compact additive dyadic group of dyadic integers with Haar measure μ and let

$$T : Tx = x + 1$$

be the addition of unity, or odometer-ergodic transformation with dyadic spectra. The p-adic odometer can be define in the same way. Moreover we can consider an arbitrary product-space of the finite sets

$$\prod_k Z_{p_k},\ p_k \in (\mathbb{N} \setminus 1),\ k = 1, 2 \ldots,$$

equipped with group structure as profinite group, and then define $\{p_k\}$-odometer. We will consider here only 2- or p-odometer.

Denote as P *the Pascal automorphism of the space* (Z_2, μ) an automorphism which is defined by the following formula in terms of dyadic decomposition:

$$x \mapsto Px; \quad P(0^m 1^k \mathbf{10} **), = 1^k 0^m \mathbf{01} ** \quad m, k = 0, 1 \ldots.$$

tion" was defined by myself as a model of approximation in the uniform topology of automorphisms, and were a self-consistent collections of Rokhlin towers for given automorphisms. The interpretation with Bratteli diagram was very convenient if we define the lexicographic order on the paths. The corresponding automorphisms and their - names for the given graphs (Pascal, Young etc.) I had used in the beginning of 80-th.

For example:

$$P(10{*}{*}) = 01{*}{*}, P(0110{*}{*}) = 1001, P(00110{*}{*}) = 10001, P(1110{*}{*}) = 1101{*}{*},$$

etc. More exactly, the rule is the following: to find the first appearance of pair **10**, change it to **01** and put all 1-s to the beginning of string, and all zeros a next. The number of zeros and ones does not change under P. So the orbit of P belongs to the orbit of action of infinite symmetric group. Moreover it is clear that the orbit partition of P coincides with the orbit partition of the action of the group $S_{\mathbb{N}}$. [5]

We have the formulas:

$$Px = x + n(x) = T^{n}(x)x,$$

Remark that $Px > x$ for all x in the sense of order in integers. So $Px - x$ is a natural (nonnegative) number $n(x)$. The arithmetic properties of the (ceiling) function $n : \mathbf{Z}_2 \to \mathbb{N}$ are very interesting.

The implicit formula for the function $n(\cdot)$ is the following:

$$n(0^m 1^k \mathbf{10} * *) = 1^k 0^m \mathbf{01} - 0^m 1^k \mathbf{10} = 2^m + 2^k - 1$$

Easy calculation shows:

$$\int_{\mathbf{Z}_2} n(x) d\mu = +\infty.$$

Consequently, the Pascal automorphism *is not the induced automorphism by odometer T and vice versa odometer is not the integral automorphism over Pascal automorphism.* By this construction Pascal automorphism is induced by the automorphism $\bar{T}$ with infinite invariant measure (see next section with the general definitions). Pascal automorphism is a partial case of the general definition of *adic* transformation, see [5], [13]].

The Pascal automorphism as a transformation on $\mathbf{Z}_2$ has the continuum mutually nonequivalent invariant ergodic measures (Bernoulli $(p, q); p + q = 1; p, q > 0$ measures; this is so called exit boundary of the space of paths of Pascal graph). In the same time the odometer has only one ergodic invariant measure (Haar measure). It is possible to extent all Bernoulli measures up to invariant measures under the automorphism $\bar{T}$ (see [2]). There is the sufficiently large literature on Pascal automorphism after [5](see f.e.[7, 13], and references therein). The exceptionally interesting papers are [8, 9] on the loosely Bernoulli property and Takage curve, see recent continuation [10]. There are many open problems about it, e.g. the weak mixing (my

[5]The initial definition of P starts from the fact that the space of paths of infinite Pascal triangle is exactly the set of all infinite sequences of 0-1 on $\mathbf{Z}_2$. In the paper [13] we define Pascal automorphism as the inverse automorphism P^{-1} as it was defined above.

conjecture in 80-th) is still (11.2015) not proved, however there are several plans to attack this problem, see [13]. [6].

I will give one example of the problem. Important characteristic of Pascal automorphism and more general — adic transformations) is the following. Consider a point $x = \{x_n, n \in \mathbb{N}\} \in \mathbb{Z}_2$ and for each n fix the n-fragment of x of length n: $\{x_k, k < n\}$. Suppose that this fragment as a sequence of $0, 1$ has m zeros and $n - m$ ones, and so belongs to the linear ordered set of all sequences of length n and m zeros; suppose $t_n(x)$ is its ordinal number in this linear ordered set.

Conjecture 1 *(Non-linear limit theorem) There exist such a sequence a_n of natural numbers and a sequence of positive numbers b_n such that for almost all $x \in \mathbb{Z}_2$ with respect to Haar measure there exists the limit*

$$\lim_n m\{x : \frac{t_n(x) - a_n}{b_n} \leq \alpha\} = \Psi(\alpha),$$

where m is Haar measure on $\mathbb{Z}_2$ and $\Psi(.)$ is a non-degenerated distribution on $\mathbb{R}$.

It is intriguing question if this problem has any connection with Takagi curve in the sense of the paper [9].

3 Generale model

3.1 The notion of generalized induced automorphisms

We want to generalize the notion of induced automorphism due to S.Kakutani.

Theorem 2 *Let T be a m.p. automorphism of the space (X, μ). The automorphism P of the same space is called the generalized induced automorphism over T if one of the following equivalent conditions takes place:*

1. $Px = T^{n(x)}x$, where $n(x) \geq 0$ for almost all x.

2. The orbit partition $\tau(P)$ of automorphism P is a subpartition of the orbit partition of the automorphism T:

$$\tau(P) \succ \tau(T)$$

and the order on the orbits of P is induced by the order on the orbits of T.

[6] The title of that article is not a claim but used to express the old certitude of author that the spectra of Pascal is pure continuous, and the assurance that the proof will be done soon

The equivalence of the conditions is evident.

The notion of induced automorphism T_A where A is a measurable set of positive measure evidently agrees with this generalization. Usually define $T_A x = x$ if $x \notin A$. In this case $T_A x = T^{n(x)}x$ where $n(x)$ is the moment of the first return to A if $x \in A$ and 0 if $x \notin A$; it is clear that in this case $\int_X n(x)d\mu < \infty$.

The class of generalized induced automorphisms is a proper subclass of all automorphisms which has the form $Px = T^{n(x)}x$ with arbitrary function $n(\cdot)$ (which defines an automorphism).

Problem. *Describe all automorphisms which are generalized induced over the odometer (dyadic or p-adic).*

It is easy to answer this question in terms of adic transformation:

Proposition 1 *Suppose the directed graph Γ has the property: the number of edges which started from the vertices of the level n to the level $n+1$ does not exceed p. Then any adic automorphism on the graph is generalized induced over the p-odometer.*

But this description does not answer on the question about the properties of this class of automorphisms like spectra, rank and so on,

Evidently any automorphism can be represented as generalized induced over an odometers of the spaces $(\prod_n Z_{k_n}, \prod_n m_{k_n})$ with sufficiently large growth of the sequence k_n — this is a corollary of the universality of adic realization ([5]).

In the previous examples it is easy to express $\int_X n(x)d\mu(x)$ via "binomial coefficients"— number of paths. Usually it is equal to infinity, so the typical generalized induced automorphism is not ordinary induced.

3.2 HK-automorphism as quasi-odometer

Pascal automorphism is the simplest generalized induced automorphism over dyadic odometer. But because of infinity of the integral $\int_X n(x)dm(x)$ it *is not the induced automorphism by odometer T and consequently odometer is not the integral automorphism over Pascal automorphism.*

In the same time we can define the integral automorphism over Pascal automorphism, using the same function $n(x)$ — that will be the measure preserving automorphism of the space with **infinite measure**, we call it now Haijan-Kakutani transformation.

We have the scheme:

$$T \to P = T^{n(\cdot)} \to \bar{T} = P^{n(\cdot)}$$

Here T is 2-odometer, P is Pascal automorphism and $\bar{T}$ is HK or quasi-odometer which is integral over P and is measure preserving automorphism with σ-finite measure $\bar{\mu}$.

More exactly: define the space with σ-finite measure and measure preserving transformation on it:

$$(\bar{X}, \bar{\mu}, \bar{T}).$$

Here $\bar{X}$ is the intersection of the lattice Z^2 with the subgraph of function $n \geq 0$ (from the formula $Px = T^{n(x)}x$); the measure $\bar{\mu}$ is a σ finite measure which coincides with μ on the base X, and whose conditional measure on the finite sets $((x,0),(x,1),\dots(x,n(x))$ is the counting measure; finally $\bar{T}$ is defined as the automorphism which has the automorphism T as induced on the subset $(X,0)$ and acts as $(x,i) \mapsto (x,i+1)$ if $i = 0,\dots n(x)-1$.

This construction is universal: one can use any automorphism instead of odometer T and any generalized induced over it. This gives the class of realizatiosn of automorphisms with σ-finite invariant measure.

Suppose T and P are m.p.(=measure preserving) automorphisms of the space (X,μ) and $\tau(T),\tau(P)$ are their orbit partitions; assume that $\tau(P) \succ \tau(T)$ which means that each orbit of P belongs to an orbit of T. In this case the formula takes place:

$$Px = T^{n(x)}x,$$

where $n(x)$ is a measurable integer-valued function on x. Assume also that for almost all x the T-orbit of x consists of infinitely many orbits of P. As was proved in [11] this means that the function $n(\cdot)$ is not integrable. Remark that this means that the function $n(\cdot)$ is far from to be arbitrary integer values function on X — in contrast to the case of construction of the integral automorphisms where the *ceiling function* is arbitrary. Thus we have a special class of the integer-valued functions $n(\cdot)$ for which we want to give a model of infinite measure preserving transformations.

Theorem 3 *Each ergodic automorphism P with finite invariant measure can be realized in the form*

$$Px = T^{n(x)}$$

where T is 2-odometer and the orbit partitions $\tau(\cdot)$ of T and P have property $\tau(P) \succ \tau(T)$ and almost each orbit of T contains countably many orbits of P.

The proof is similar to the proof of Dye's theorem and is based on the fact that each ergodic P has *universal* adic realization in the space of paths of the distinguish graph of unordered pairs (UP) — see [14] — this is the strengthen of the theorem on adic realization [5]. But the space of paths of UP can be identified with the dyadic group $\mathbf{Z}_2$ preserving the tail filtrations.

Of course the dyadic odometer can be changed on any space of type $X = \prod_{k=1}^{\infty} \mathbf{p}_k$ where $p_k > 1$ is any sequence of naturals, $p_k > 1$. Sometimes this is more convenient than $\mathbf{p}_n \equiv 2$.

It is convenient to make the following form for $\bar{T}$. Consider the space $\mathbf{Q}_2$ of rational dyadic numbers (as a space but not as an additive group):

$$\mathbf{Q}_2 = \left(\sum_{n<0} \mathbb{Z}_2\}\right) \times \prod_{n\geq 0} \mathbb{Z}_2 \equiv \mathbf{N} \times \mathbf{Z}_2;$$

it is useful to consider the first summand as natural numbers with addition as operation and second summand as dyadic integers with natural 2-adic operation. We equip $\mathbf{Q}_2$ with (infinite) Haar measure m with normalization $m(\mathbf{Z}_2) = 1$.

Consider the automorphism P which satisfies our assumptions

$$Px = T^{n(x)}x = x + n(x), x \in \mathbf{Z}_2,$$

where T is 2-odometer and n is a suitable function. *Define the automorphism $\bar{T}$ of $\boldsymbol{Q}_2$ with invariant Haar measure by the following formula:*

$$\bar{T}(y,x) = (y', x'); y, y' \in \sum_{n<0} \mathbf{Z}_2, x, x' \in \prod_{n\geq 0} \mathbf{Z}_2,$$

where

$x' = Px = T^{n(x)}x, \quad \text{if} \quad y = n(x);$

$x' = x, \quad \text{if} \quad y \neq n(x);$

$y' = y + 1, \quad \text{if} \quad y \neq n(x);$

$y' = 0, \quad \text{if} \quad y = n(x).$

We emphasize that the function $n(x)$ plays two roles - as ceiling function and as time change for P. It is natural to call $\bar{T}$ the "pseudo-odometer", or "reflection of odometer" with respect to the automorphism P.

It is possible to give another transparent description of this construction for $\bar{T}$:

The measure space can be realized in the space $\mathbf{Q}_2$ in the form which used the previous notations: $x \in \mathbb{Z}_2; y \in \mathbb{N}, y \leq n(x)$:

$$\bar{T}(x,y) = (T^1_y(x), T^2_x(y)),$$

where

$$T^1_x y = y + 1, \quad \text{if} \quad y < n(x), \quad \text{and} \quad T^1_x y = 0 \in \mathbb{N} \quad \text{if} \quad y = n(x).$$

For T^2 we have:

$$T^2_y x = x, \quad \text{if} \quad y < n(x), T^2_{n(x)} x = Px.$$

This is simply "horizontal" expression of the fact $\bar{T}$ is generalized the induced automorphism. But this form is very convenient for operation and for graph interpretation.

In general $\{T_x\}$ is a measurable family of permutaions of $\mathbb{N}$ and $\{T_y\}$ is a countable family of $\mathbb{Z}_2$ (This construction could be called "bi-skew-product"). But our realization is a little bit special.

Problem For what ergodic automorphism $\bar{T}$ with infinite invariant measure there exists such representation (integral representation) where T is odometer and with the some function $n(\cdot)$?

Recall also that the simple generalization of well-known Dye's theorem (see f.e.[15]) claimed that the orbit partition of any ergodic automorphism S is isomorphic to the partition of the space $\mathbf{Q}_2$ into cosets with respect to the subgroup $\mathbb{Q}$ (rational numbers). The model above gives more or less simple way to represent automorphism (like "interval exchange"). It is possible to give a graph interpretation of a representation our model which is similar to the adic model of the automorphism with finite invariant measure.

For this we need to consider $\mathbb{Z}$-graded (not $\mathbb{N}$=graded) graphs, but of the special type on "minus infinity": the space of double infinite paths must have stabilization on minus infinity, which means that there is a path t such that each path coincides with it at $-\infty$. We hope that this σ-finite adic model will be useful for infinite ergodic theory.

Problem To define the analog of Bratteli-Vershik model for σ-finite measure preserving transformation.

Actuality of this problem lies in necessity of the simple models of ergodic automorphisms with σ-finite measure.

4 σ-finite theory of the random walk on the groups.

It was mentioned in the paper [3] (see p. 458, item 5.) the following important point: the natural theory of the random walks on the groups must be developed in the framework of the ergodic theory with infinite invariant measure. The reason is the possibility to consider a two-sided process and the shift of the trajectories as measure preserving transformation. Unfortunately, we are still far from the serious achievements in this direction. The analysis of the new situation must take into account the fact that the global measure is infinite and the space of trajectory of a random walk is not compact. The last facts demand a serious revision of the many ordinary notions like ergodicity, regularity, notion of boundary, entropy, new look on the role of functional spaces (l^2, L^∞), etc.

Of course the situation seems to be very clear: σ-finite measure in the space of trajectories of random walk is a direct product of Bernoulli measure with σ-finite (Haar) measure on the group, and the automorphism is simply skew-product with Bernoulli automorphism as the base and translations on the group as the fibers. But this does not mean that all analysis is reduced to the ordinary ergodic theory.

We give here only evident example of the preference of the "σ-finite" point of view, more exactly — preference of the consideration of two-sided processes (the time is $\mathbb{Z}$ but not $\mathbb{Z}_+$) with necessarily σ finite measures.

How to define exit and entrance (Dynkin) as well as Poisson-Furstenberg boundaries of the random walk using the shift in the space of trajectories with σ-finite measure?

Let G be a countable (infinite) finitely generated group and S be the set of generators. We assume that the group G is the semigroup generated by S). Denote as $\bar{\nu}$ the measure on S^{-1}: $(\bar{\nu})(g) = \nu(g^{-1})$.

Consider the random walk on the countable group G with a probability measure μ on S. The set of trajectories of generalized Markov chain is the subspace $\mathcal{M}$ of $G^{\mathbb{Z}}$:

$$\mathcal{M} = \{\{y_n\}_{n\in\mathbb{Z}} : {y_n}^{-1} \cdot y_{n+1} \in S, n \in \mathbb{Z}\}.$$

We can represent the space $\mathcal{M}$ as the direct product

$$\mathcal{M} = S^{\mathbb{Z}} \times G : \{y_n\} = (\{s_n\}_{n\in\mathbb{Z}}, y_0),$$

where $s_n = {y_n}^{-1} \cdot y_{n+1}, n \in \mathbb{Z}$. We will not define a Markov measure on $\mathcal{M}$, but define only transition and co-transition probabilities:

$$P\{y_n \equiv y_{n-1}s|y_{n-1}\} = \nu(s) \quad P\{y_n \equiv y_{n+1}s|y_{n+1}\} = \bar{\nu}(s) = \nu(s^{-1}), n \in \mathbb{Z}.$$

The dynamics on the space $\mathcal{M}$ is defined by the shift T which is linear *automorphism of the space* $\mathcal{M}$: $\{T(\{y_n\})\}_n = y_{n-1}$.

We the stationary (shift-invariant) Markov measures on $\mathcal{M}$[7].

Definition 2 *Exit boundary of a random walk corresponding to the pair (G, ν) where G is countable group, and ν is measure with the finite support S which generate G as a semigroup is the set of all ergodic shift-invariant two-sided σ-finite Markov measures μ on the space $\mathcal{M}$ with given co-transition probabilities defined above;*

Entrance boundary of random walk corresponding to the pair (G, μ) is the set of all ergodic shift-invariant two sided σ-finite Markov measures μ on the space $\mathcal{M}$ with transition probabilities defined above.

Here we call the Markov measure on the space of double infinite sequences ergodic if it is regular (terms of Kolmogorov) or Kolmogorovian (modern term). This means that filtration of the past and filtration of the future have trivial intersections. In other words -no nontrivial events on plus and

[7]Definition of Markov property of the σ-finite measures is the same as usual:independence of past and future for fixed value of the present.

minus infinity. Remark that the initial distribution does not mentioned in the definition. It is clear that both boundaries are topological spaces (with weak topology on the space of measures). It is very important to understand the topological properties of these spaces.

Of course, the exit (correspondingly, entrance) boundary in this definition is the same as exit (entrance) boundary for one-sided system of co-transition (correspondingly, transition) probabilities in the sense of old papers of Dynkin or in the new paper [14]. But two-sided invariance gives more possibilities to study these objects.

In the case of symmetric measure $\nu = \bar{\nu}$ both boundaries are tautologically coincided.

Measure ν (correspondingly $\bar{\nu}$) defines a Laplace operators in $l^2(G)$ (these two Laplace operators are mutually conjugate) in the space l^2.

The Laplace operators $L_\nu, L_{\bar{\nu}}$ are convex combinations of the isometries of the shifts $U_s, s \in S$, or $s \in S^{-1}$ in the space $l^2(G)$ (or in the spaces $L^\infty_\mu(\mathcal{M})$).

The PF-boundary of random walk in this case is a measure space with harmonic measure which is the "part" of the exit boundary which is a topological space. Remark that the exit boundary could be nontrivial even if the PF-boundary is trivial. Both boundaries could be identifying with the spectrum of Laplace operator in the space L^∞, and more exactly — with the space of minimal non-negative harmonic functions with so called harmonic measure.

One of the most intriguing and (as I know) open questions on σ-finite theory of random walks (even for a simple walk on $\mathbb{Z}$), is how to describe weakly wandering sets for them.

References

[1] S. Kakutani, *A problem of equidistribution on the unit interval* $[0, 1]$. Lect.Note in Math. Springer. No. 541 (1976) 369-375.

[2] A. Hajan, Y. Ito, S. Kakutani *Invariant Measures and orbits of Dissipative Transformations.* Advances in Math., **9** (1972) 52-65.

[3] V. Kaimanovich, A. Vershik, *Random walks on discrete groups:Boundary and entropy.* Ann. Prob. **11**, no.3 (1983) 457-490.

[4] A. Vershik, *Uniform algebraic approximations of shift and multiplication operators.* Sov. Math. Dokl., **24** (1981) 97-100.

[5] A. Vershik, *A theorem on periodical Markov approximation in ergodic theory.* Zapiski Nauchn. Sem. LOMI, **115** (1982) 72-82. English translation: J. Sov. Math. 28, 667-674 (1985).

[6] S. Ito, A construction of transversal flows for maximal Markov automorphisms. Tokyo J. Math. **1**, no. 2 (1978) 305-324.

[7] X. Mela, K. Petersen, *Dynamical properties of the Pascal adic transformation.* ETDS, **25**, no.1 (2005), 227-256.

[8] E. Janvresse, T. de la Rue, *The Pascal adic transformation is loosely Bernoulli.* Ann.Inst.H.Poincare. Prob. Statist. **40** no.2 (2004) 133-139.

[9] E. Janvresse, T. de la Rue, Y. Velenik, *Self-similar corrections to the ergodic theorem for Pascal-adic thransformation.* Stoc.Dyn. **5**, no.1 (2005) 1-25.

[10] A. Lodkin, A. Minabutdinov, *Limit curve for Pascal automorphisms.* Zapiski Nauchn. Sem. POMI, **437**, ser. *Teoria predstavlenii, Dynamicheskie sistemy, Kombinatorika.* XXVI, (2015)145-183. .

[11] R. Beilnskaya, *Generalized power of automorphisms.* Sibirsk. Math. Journ., **11**, no.4 (1970) 739-749.

[12] A. Vershik, *A theorem on the lacunary isomorphism of monotonic sequence of partitions.* Funct. Anal. Appl., **2** no.3 (1968) 17-21.

[13] A. Vershik, *The Pascal automorphism has a continuous spectrum.* Funct. Anal. Appl., **45**, no. 3 (2011) 173-186.

[14] A. Vershik *Asymptotic theory of space paths of graded graphs and its applications.* Submitted to Japan. Journ of Math.

[15] W. Krieger, *On Non-singular Transformations of a Measure Space* I., II. Z.Wahrscheinlichkeitstheorie. Geb., **11** (1969) 83-97,98-119.

[16] A. Hajian, S. Kakutani, *Example of an ergodic measure preserving transformation on an infinite measure space.* In: Contributions to Ergodic Theory and Probab. (Proc. Conf., Ohio State Univ., Columbus, Ohio, 1970), Springer, Berlin (1970) 45-52.

Anatoly Vershik
St. Petersburg Department of Stekloff Mathematical Institute,
Russian Academy of Sciences
27, Fontanka, 191023 St.Petersburg, Russia
avershik@gmail.com

Please, cite to this paper as published in
Armen. J. Math., V. **7**, N. 2(2015), pp. 85–96