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Asymptotic Estimates for Quasi-Periodic Interpolations

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Contents

Introduction

The problem of trigonometric interpolation has a long history of decades ([1]). Investigations in this area have started over 100 years ago by works of Burkhardt [2], Vallée-Poussin [3], Jackson [4–6], Fejér [7], Feldheim [8] and are urgent till now. Numerous authors consider this problem and rather short list includes [9–20] with references therein.

Let $\{x_k\}_{k=-N}^{N}$, $N \geq 1$ be some grid on $[-1, 1)$. The classical trigonometric interpolation $I_N(f, x)$ is completely identified by the following two conditions:

a) interpolation condition

$$I_N(f, x_k) = f(x_k),\ |k| \leq N,\ f \in C[-1, 1], \tag{0.1}$$

b) exactness condition

$$I_N\left(e^{i\pi nx}, x\right) \equiv e^{i\pi nx},\ |n| \leq N,\ x \in [-1, 1]. \tag{0.2}$$

Let

$$I_N(f, x) = \sum_{k=-N}^{N} a_k(x) f(x_k),\ x \in [-1, 1], \tag{0.3}$$

where a_k are some unknown functions to be determined. The exactness condition *b)* leads to the following system of linear equations with the Vandermone matrix $e^{i\pi n x_k}$

$$e^{i\pi nx} \equiv \sum_{k=-N}^{N} a_k(x) e^{i\pi n x_k},\ |n| \leq N,\ x \in [-1, 1]. \tag{0.4}$$

When $x_k \neq x_j$, $k \neq j$, system (0.4) has unique solution and the classical interpolation has the following representation ([1])

$$I_N(f, x) = \sum_{k=-N}^{N} f(x_k) \prod_{\substack{j=-N \\ j \neq k}}^{N} \frac{\sin \frac{\pi}{2}(x - x_j)}{\sin \frac{\pi}{2}(x_k - x_j)} \tag{0.5}$$

which also holds the interpolation condition *a)*.

In general, the geometric structure of the set x_k is of great importance for convergence properties of I_N to f as $N \to \infty$. Mostly investigated and important in applications is the case of equidistant nodal points when $e^{i\pi x_k}$, $|k| \le N$ are equally spaced over the circumference of the unit circle. Further, in this work, we assume that the classical trigonometric interpolation is realized for the equidistant set of points

$$x_k = \frac{2k}{2N+1}, \ k = -N, \dots, N. \tag{0.6}$$

In this case, system (0.4) can be solved also by application of discrete Fourier transform ([21])

$$a_k(x) = \frac{1}{2N+1} \sum_{n=-N}^{N} e^{i\pi n(x-x_k)} \tag{0.7}$$

and consequently

$$I_N(f,x) = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) D_N(x-x_k), \tag{0.8}$$

where D_N is the Dirichlet kernel

$$D_N(x) = \frac{\sin\left(N+\frac{1}{2}\right)\pi x}{\sin\frac{\pi x}{2}}. \tag{0.9}$$

Representation (0.7) allows to rewrite the classical interpolation in the form which has far-going importance for applications

$$I_N(f,x) = \sum_{n=-N}^{N} \check{f}_n e^{i\pi n x}, \tag{0.10}$$

where $\{\check{f}_n\}$ is the discrete Fourier transform of $\{f(x_k)\}$

$$\check{f}_n = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) e^{-i\pi n x_k}. \tag{0.11}$$

The classical Cooley-Tukey algorithm ([22, 23]) computes the discrete Fourier transform for n given complex coefficients in $n \log n$ operations instead of n^2 operations and is known as the fast Fourier transform (FFT) algorithm.

Let

$$R_N(f,x) = f(x) - I_N(f,x). \tag{0.12}$$

It is well known ([1, 15]) that convergence $R_N \to 0$ highly depends on the smoothness of 2-periodic extension of f onto the real line. Kress ([9]) showed that for 2-periodic analytic function the order of convergence is $O(e^{-cN})$, $N \to \infty$ with some $c > 0$ constant. When 2-periodic extension of f is discontinuous then regardless of the smoothness of f on $[-1,1]$ (it means that $f(1) \neq f(-1)$) the classical interpolation has slow L_2 and pointwise convergence on $(-1,1)$ and the Gibbs phenomenon at the points $x = \pm 1$. Next theorems describe this in more exact terms.

Let

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1). \tag{0.13}$$

The following theorems provide estimates for pointwise convergence.

Theorem 0.1 (A. Poghosyan, [24]) *Let $q \geq 0$ be even, $f^{(q+1)} \in AC[-1,1]$ and*

$$A_k(f) = 0, \ k = 0, \dots, q-1. \tag{0.14}$$

Then, $R_N(f,x) = O(N^{-q-1})$ for $|x| < 1$ as $N \to \infty$.

Theorem 0.2 (A. Poghosyan, [24]) *Let $q \geq 1$ be odd, $f^{(q+2)} \in AC[-1,1]$ and*

$$A_k(f) = 0, \ k = 0, \dots, q-1. \tag{0.15}$$

Then, $R_N(f,x) = O(N^{-q-2})$ for $|x| < 1$ as $N \to \infty$.

Now, we provide estimate in the L_2-norm. Let

$$\|f\|_{L_2[-1,1]} = \left(\int_{-1}^{1} |f(x)|^2 dx \right)^{\frac{1}{2}}. \tag{0.16}$$

Theorem 0.3 (A. Nersessian and N. Hovhannisyan, [25]) *Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$ and $A_k(f) = 0, \ k = 0, \dots, q-1$. Then, the following estimate holds*

$$\lim_{N \to \infty} N^{q+\frac{1}{2}} \|R_N(f,x)\|_{L_2[-1,1]} = |A_q(f)| c_q, \tag{0.17}$$

where

$$c_q = \frac{1}{\pi^{q+1}} \left(\frac{1}{2q+1} + \frac{1}{2} \int_{-1}^{1} \left| \sum_{s \neq 0} \frac{(-1)^s}{(x+2s)^{q+1}} \right|^2 dx \right)^{1/2}. \tag{0.18}$$

Estimate (0.17) is valid also for $q = 0$ if $f' \in L_2[-1,1]$.

Next theorem characterizes behavior of the classical interpolation at the points $x = \pm 1$ in terms of the limit functions. Similar technique was utilized by series of authors for investigation of convergence of the classical interpolation ([3, 11, 12]).

Theorem 0.4 (A. Nersessian and N. Hovhannisyan, [26]) *Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$ and $A_k(f) = 0,\ k = 0,\dots,q-1$. Then, the following estimate holds*

$$\lim_{N\to\infty} N^q R_N\left(f, \pm\left(1 - \frac{h}{N}\right)\right) = A_q(f)\ell_{x\to\pm1,q}(h),\ h \geq 0, \tag{0.19}$$

where

$$\ell_{x\to\pm1,q}(h) = \frac{1}{2(i\pi)^{q+1}}\left(\int_{-1}^{1} e^{\mp i\pi ht}\sum_{s\neq0}\frac{(-1)^s}{(t+2s)^{q+1}}dt - \int_{|t|>1}\frac{e^{\mp i\pi ht}}{t^{q+1}}dt\right). \tag{0.20}$$

Estimate of (0.19) is valid also for $q = 0$ if $f' \in L_2[-1,1]$ and $h > 0$. Direct calculations show that $\ell_{x\to\pm1,0}(0) = \pm\frac{1}{2}$.

Taking into account that $|\ell_{x\to1,q}(h)| = |\ell_{x\to-1,q}(h)|$, we put

$$\ell_q^* = \max_{h\geq0}|\ell_{x\to1,q}(h)|, \tag{0.21}$$

which characterizes the asymptotic ($N \to \infty$) uniform error of the classical interpolation.

Presented results show that when f is smooth on $[-1,1]$, but its 2-periodic extension is discontinuous ($f(1) \neq f(-1)$), the classical interpolation has convergence rate $O(N^{-\frac{1}{2}})$ in the L_2-norm, $O(N^{-1})$ pointwise convergence and $O(1)$ uniform error.

Different methods for convergence acceleration of trigonometric interpolation are known in the literature. We are interested in two approaches: application of rational and polynomial corrections to the errors of interpolations.

The classical idea of interpolation by rational functions is in interpolation of f by the following quotient

$$S_{L,M}(x) = \frac{P_L(x)}{Q_M(x)}, \tag{0.22}$$

where P_L and Q_M are trigonometric polynomials of degrees at most L and M, respectively. The coefficients of nominator and denominator must be determined from the interpolation condition

$$f(x_j) = S_{L,M}(x_j) \tag{0.23}$$

for some grid x_k, where the number of nodes equals the number of unknown coefficients. Rational interpolation can encounter standard problems: the problem is not always solvable as in some cases, the interpolation condition (0.23) cannot be satisfied and the interpolating function

$S_{L,M}(x)$ can have poles in the interval of interpolation, which make the interpolation useless unless f is singular at the same points, too. Different authors studied rational interpolations trying construct well-conditioned, reliable and stable interpolations. Rather short list includes [27–38] with references therein.

Application of polynomial corrections is an efficient approach for convergence acceleration of the classical trigonometric interpolations and approximations (see [24, 39–53] with references therein).

In this work, we study trigonometric interpolations and approximations of smooth on $[-1, 1]$ functions which have discontinuous or low-smooth 2-periodic extensions on the real line. When the basis functions have periods equal to the length of interval, we call such interpolations and approximations as classical. We achieve better accuracy by considering trigonometric interpolations with basis functions which periods are greater than the length of the interval but tending to it as the number of nodes tends to infinity. Such interpolations are known as Quasi-Periodic (QP) interpolations ([54]).

First, we derive an explicit formula for realization of the QP interpolation and then study its convergence in different frameworks: pointwise, L_2 and uniform convergence. The essence of our approach is derivation of the exact constants of asymptotic errors and their comparison with the classical analogues. When theoretical conclusions are impossible, we use the methods of numerical analysis.

Second, we apply rational and polynomial corrections for convergence acceleration of the QP interpolation and investigate the convergence of the resultant QP Polynomial interpolation and QP Rational, QP Rational Polynomial approximations. We consider rational functions as corrections of the error of corresponding interpolations. In case of the classical interpolation this approach is considered in [35, 36, 55] (see also references therein) which leads to the classical rational trigonometric (RT) interpolation. There, the pointwise and L_2 convergence were investigated. Here, we explore the behavior of the classical RT interpolation in terms of the limit functions. Then, we apply this approach for convergence acceleration of the QP interpolation and derive QP Rational (QPR) approximation. Actually, in this case, the rational corrections distort the condition of interpolation although the latest is also written via discrete

Fourier coefficients of f.

The main results of this work are published in [56–60] and presented in conferences and workshops [61–64].

Let us consider trigonometric interpolation of f with basis functions

$$\left\{e^{i\pi n\sigma x}\right\}_{n=-N}^{N}, \ 0 < \sigma < 1, \tag{0.24}$$

which have period $2/\sigma > 2$. Similar idea considered in [65] named as sub-periodic trigonometric interpolation. Similar to (0.4) the exactness condition leads to the following system of linear equations with Vandermonde matrix for determination of the unknowns $a_k(x)$

$$e^{i\pi n\sigma x} \equiv \sum_{k=-N}^{N} a_k(x) e^{i\pi n\sigma x_k}, \ |n| \leq N, \ x \in [-1, 1]. \tag{0.25}$$

If parameter σ and grid x_k are chosen such that the system has unique solution then the sub-periodic interpolation can be realized.

Following the idea introduced in [54], we assume that parameter σ depends on N and consider interpolation which is exact for the following system of quasi-periodic exponents

$$\left\{e^{i\pi n\sigma x}\right\}_{n=-N}^{N}, \ \sigma = \frac{2N}{2N+m+1}, \ x \in [-1, 1], \ m \in Z, \ m \geq 0 \tag{0.26}$$

with the periods $2/\sigma \to 2$ as $N \to \infty$. Such interpolations are known as QP interpolations. We denote it by $I_{N,m}(f, x)$. In papers [25, 26, 54] such interpolations were investigated only by the methods of numerical analysis.

Let $\{x_k\}_{k=-N}^{N}$, $N \geq 1$ be some grid on $[-1, 1]$. The QP interpolation can be completely characterized by the following two conditions:

a) interpolation condition

$$I_{N,m}(f, x_k) = f(x_k), \ |k| \leq N, \ f \in C[-1, 1], \tag{0.27}$$

b) exactness condition

$$I_{N,m}\left(e^{i\pi n\sigma x}, x\right) \equiv e^{i\pi n\sigma x}, \ |n| \leq N, \ x \in [-1, 1]. \tag{0.28}$$

Throughout this work, we assume the following grid for the QP interpolation

$$x_k = \frac{k}{N}, \ k = -N, \ldots, N, \tag{0.29}$$

which includes also the endpoints $x = \pm 1$ of the interval. Such interpolations are known as the "full-interpolations". It is easy to verify that system (0.25) has unique solution for such choices of σ and x_k.

Let us establish the connection between the classical trigonometric and the QP interpolations. Consider a new function $f^*(t)$ defined on $[-\sigma, \sigma]$ by the following change of variable

$$f^*(t) = f\left(\frac{t}{\sigma}\right) = f(x),\ x \in [-1,1],\ t \in [-\sigma, \sigma],\ t = \sigma x. \tag{0.30}$$

This implies $e^{i\pi n \sigma x} = e^{i\pi n t}$ and transformation of the grid

$$t_k = \sigma x_k = \frac{2k}{2N+m+1},\ |k| \leq N. \tag{0.31}$$

Hence, we derived the classical trigonometric interpolation (which is exact for $e^{i\pi n t}$) of $f^*(t)$ on grid t_k. When $m = 0$, the grid t_k is equidistant on $[-1, 1]$ and we readily derive the explicit formula for $I_{N,0}(f, x)$

$$I_{N,0}(f,x) = I_N(f^*, t) = \sum_{n=-N}^{N} \check{f}_n^* e^{i\pi n t} = \sum_{n=-N}^{N} \check{f}_n^* e^{i\pi n \sigma x}, \tag{0.32}$$

where

$$\begin{aligned} \check{f}_n^* &= \frac{1}{2N+1} \sum_{k=-N}^{N} f^*\left(\frac{2k}{2N+1}\right) e^{-i\pi n t_k} \\ &= \frac{1}{2N+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-i\pi n \frac{2k}{2N+1}}. \end{aligned} \tag{0.33}$$

In the case of $m > 0$, the grid t_k is non-equidistant as

$$t_k - t_{k-1} = \frac{2}{2N+m+1},\ k = -N+1, \ldots,\ N, \tag{0.34}$$

while

$$1 - t_N + t_{-N} - (-1) = (m+1)\frac{2}{2N+m+1}. \tag{0.35}$$

It is also worth noting that f^* depends on N, and although $f^* \to f$ as $N \to \infty$, but this dependence essentially changes interpolation properties.

Chapter 1 studies the convergence of the QP interpolation in different frameworks. Actually it establishes the convergence theory of the QP interpolation. Comparison with similar results of the classical interpolation confirms observations derived in [54] by the methods of numerical analysis: better accuracy of the QP interpolation compared to the classical one in the L_2 and uniform norms and higher pointwise convergence rate.

Section 1.1 presents some results for the classical trigonometric interpolation.

Section 1.2 presents the process of solution of system (0.25) with grid (0.29) and σ defined by (0.26). As we mentioned above, grid (0.31) is non-equidistant on $[-1,1]$ when $|k| \leq N$, but it becomes equidistant when index k changes from $k=-N$ to $k=N+m$. Hence, by adding some additional terms in the sum on the right-hand side of (0.31) with some additional unknowns and also by adding some new equations, we enlarge the matrix of system (0.25) into unitary matrix of the discrete Fourier transform

$$e^{i\pi n\sigma x_k} = e^{i\pi n\frac{2k}{2N+m+1}}, \; k,n=-N,\ldots,N+m. \tag{0.36}$$

Additional added unknowns can be found via inversion of some Vandermonde matrix of size $m \times m$. Finally, the QP interpolation has the following representation

$$I_{N,m}(f,x) = \sum_{n=-N}^{N} F_{n,m} e^{i\pi n\sigma x}, \tag{0.37}$$

where

$$F_{n,m} = \check{f}_{n,m} - \sum_{\ell=1}^{m} \theta_{n,\ell} \check{f}_{\ell+N,m}, \tag{0.38}$$

$$\check{f}_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-\frac{2i\pi nk}{2N+m+1}} \tag{0.39}$$

and

$$\theta_{n,\ell} = e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi n(s-N-m-1)}{2N+m+1}}. \tag{0.40}$$

By $v_{\ell,s}^{-1}$, we denoted the elements of the matrix which is the inverse of the following Vandermonde matrix

$$v_{s,\ell} = \alpha_\ell^{s-1}, \; \alpha_\ell = e^{\frac{2i\pi(\ell+N)}{2N+m+1}}, \; s,\ell = 1,\ldots,m. \tag{0.41}$$

When $m=0$, (0.37) coincides with (0.32).

Section 1.3 studies pointwise convergence of the QP interpolation and derives exact constant of the main term of asymptotic error. The main results of this section are the following theorems.

Let

$$A_{ks}(f) = f^{(s)}(1) - (-1)^{k+s} f^{(s)}(-1) \tag{0.42}$$

and

$$\Phi_{k,m}(e^{i\pi x}) = e^{\frac{i\pi}{2}(m-1)x} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}}. \tag{0.43}$$

Theorem 0.5 *[60] Let $f^{(q+2m)} \in AC[-1,1]$ for some $q \geq 0$, $m \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \ldots, q-1. \tag{0.44}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$\begin{aligned} R_{N,m}(f,x) = \frac{(-1)^N}{N^{q+m+1}} & \left[\sin\left(\pi(N+1)\sigma x\right) \sum_{k=0}^{\left[\frac{m}{2}\right]} \binom{m-k}{k} \frac{(-1)^k}{2^{2k+1}\cos^{2k+2}\frac{\pi x}{2}} \right. \\ & \left. - \sin\left(\pi N \sigma x\right) \sum_{k=0}^{\left[\frac{m}{2}\right]-1} \binom{m-k-2}{k} \frac{(-1)^k}{2^{2k+3}\cos^{2k+4}\frac{\pi x}{2}} \right] \\ & \times \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k+1} i^{k-1} \pi^{k-m+1}(q-k)!} \Phi_{k,m}^{(m)}(-1) + o(N^{-q-m-1}). \end{aligned} \tag{0.45}$$

Theorem 0.6 *[60] Let $f^{(q+1)} \in AC[-1,1]$ for some $q \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, \ldots, q-1. \tag{0.46}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$R_{N,0}(f,x) = \frac{(-1)^N}{2^{q+1}N^{q+1}} \frac{\sin \pi N x}{\cos \frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(f) 2^k}{(q-k)! i^k \pi^{k+1}} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r+1)^{k+1}} + o(N^{-q-1}). \tag{0.47}$$

Assume that f is infinitely differentiable on $[-1,1]$, but $f(1) \neq f(-1)$. Hence, estimates of Theorems 0.1, 0.5 and 0.6 are valid with $q = 0$. The QP and the classical interpolations have the same convergence rate $O(N^{-1})$ for $m = 0$. When $m > 0$, the QP interpolation has convergence rate $O(N^{-m-1})$ against rate $O(N^{-1})$ of the classical interpolation. Improvement is by factor $O(N^{-m})$.

Section 1.4 studies the QP interpolation in the framework of the L_2-norm. The main results are:

Theorem 0.7 *[56, 57] Let* $f^{(q+m)} \in AC[-1,1]$ *for some* $q, m \geq 0$, $q+m \neq 0$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, ..., q-1. \tag{0.48}$$

Then, the following estimate holds

$$\lim_{N\to\infty} N^{q+\frac{1}{2}} ||R_{N,m}(f,x)||_{L_2(-1,1)} = c_{q,m}(f), \tag{0.49}$$

where

$$\begin{aligned} c_{q,m}^2(f) = &-\frac{m+1}{8}\int_{-1}^{1}\left|\int_{-1}^{1}\nu_{q,m}(f,h)e^{i\pi\frac{m+1}{2}xh}dh - \int_{|h|>1}\mu_{q,m}(f,h)e^{i\pi\frac{m+1}{2}xh}dh\right|^2 dx \\ &+\frac{1}{2}\int_{-1}^{1}\left|\nu_{q,m}(f,x)\right|^2 dx + \frac{1}{2}\int_{|x|>1}\left|\mu_{q,m}(f,x)\right|^2 dx, \end{aligned} \tag{0.50}$$

and

$$\begin{aligned} \mu_{q,m}(f,x) &= \sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(q-k)!(i\pi x)^{k+1}}, \\ \nu_{q,m}(f,x) &= \sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(i\pi)^{k+1}(q-k)!} \\ &\times\left(\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}} - e^{-i\pi\frac{m-1}{2}x}\sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi_{k,m}^{(\tau)}(-1)\left(e^{i\pi x}+1\right)^{\tau}\right). \end{aligned} \tag{0.51}$$

Estimate (0.49) is valid also for $q = m = 0$ if $f' \in L_2[-1,1]$.

Comparison with Theorem 0.3 shows the same convergence rate if other conditions are satisfied. More detailed comparison of the classical and the QP interpolations is possible to perform via analysis of ratio $|A_q(f)|c_q/c_{q,m}(f)$. In general, this ratio depends on f and comparison of both interpolations is possible only for a specified function. However, in **Section 1.6**, we show that in important special case $q = 0$, the ratio is independent of f and comparison of both interpolations can be performed independently of interpolated function. The values of the ratio for different values of m show that the QP interpolation is much more precise in the L_2-norm than the classical interpolation when $q = 0$. For example, when $m = 7$, the QP interpolation is more than 3000 times more accurate than the classical interpolation.

Section 1.5 explores the behavior of the QP interpolation at the endpoints of the interval in terms of limit functions. The main result is:

Theorem 0.8 *[57] Let* $f^{(q+m)} \in AC[-1,1]$ *for some* $q, m \geq 0$, $q+m \neq 0$ *and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, ..., q-1. \tag{0.52}$$

Then, the following estimates hold

$$\lim_{N\to\infty} N^q R_{N,m}\left(f, \pm\left(1-\frac{h}{N}\right)\right) = \ell_{x\to\pm1,q,m}(f,h), \;\; h>0, \tag{0.53}$$

where

$$\ell_{x\to\pm1,q,m}(f,h) = \frac{1}{2}\int_{-1}^{1} \nu_{q,m}(f,t)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)t}dt - \frac{1}{2}\int_{|t|>1} \mu_{q,m}(f,t)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)t}dt \tag{0.54}$$

and functions $\mu_{q,m}$ *and* $\nu_{q,m}$ *are defined in Theorem 0.7.*

Estimate (0.53) is valid also for $q = m = 0$ if $f' \in L_2[-1,1]$. Note that $R_{N,m}(f,\pm1) = 0$ as $x = \pm1$ $(h = 0)$ are the nodes of interpolation.

We put

$$\ell_{q,m}(f) = \max\{\max_{h>0}|\ell_{x\to1,q,m}(f,h)|\,,\; \max_{h>0}|\ell_{x\to-1,q,m}(f,h)|\} \tag{0.55}$$

which characterizes the uniform error of the QP interpolation. Theorems 0.4 and 0.8 show the same convergence rate for both interpolations and more detailed comparison of accuracies is possible via analysis of ratio $|A_q(f)|\ell_q^*/\ell_{q,m}(f)$. In general, such analysis is possible to perform only by specifying f. However, in important special case $q = 0$, we perform such analysis independently of f (see **Section 1.6**). The values of the ratio for different m show that the QP interpolation is more precise than the classical interpolation and as bigger is m as more bigger is difference in accuracies. For example, when $m = 7$, the QP interpolation is more than 4500 times more accurate than the classical interpolation in the uniform norm.

Chapter 2 considers rational trigonometric interpolations and approximations.

Section 2.1 considers convergence acceleration of the classical trigonometric interpolation by rational corrections and investigate the convergence of the resultant interpolation.

The classical RT interpolation $I_N^p(f,x)$ and its error $R_N^p(f,x)$ are defined by (2.3) and (2.4), respectively. It is represented as a sum of the classical trigonometric interpolation $I_N(f,x)$ and rational functions (in terms of $e^{i\pi x}$) as corrections of the error $R_N(f,x)$. Rational corrections contain some parameters λ_k (in general, complex numbers) which determination is a crucial problem. Different approaches are known for determination of these parameters. An approach, which is applicable for cases when interpolated function is smooth on $[-1,1]$ is

$$\lambda_k = \lambda_{-k} = 1 - \frac{\tau_k}{N}, \;\; k = 1,\ldots,p, \tag{0.56}$$

where new parameters τ_k are independent of N and f ([35]). Below, we present some convergence theorems that outline behavior of interpolations with such choice of parameters.

Another approach is determination of λ_k from the following system of equations (see (2.1))

$$\delta_n^p(\lambda, \{\check{f}_s\}) = 0, \ n = N - p + 1, \cdots, N + p. \tag{0.57}$$

Such interpolations are known as the Fourier-Pade (FP) interpolation ([36]). **Section 2.5** considers some numerical results that outline behavior of such interpolations.

Next theorems which we need for further comparisons provide estimates for the pointwise convergence. In these theorems parameters τ_k are undefined.

Theorem 0.9 (A. Poghosyan, [35]) *Let $q \geq 0$ be even and $f^{(q+2p+1)} \in AC[-1,1]$ for some $p \geq 1$. Let $A_k(f) = 0$, $k = 0, \ldots, q-1$ and parameters λ_k be chosen as in* (0.56). *Then, $R_N^p(f,x) = O(N^{-2p-q-1})$ for $|x| < 1$ as $N \to \infty$.*

Theorem 0.10 (A. Poghosyan, [35]) *Let $q \geq 1$ be odd and $f^{(q+2p+2)} \in AC[-1,1]$ for some $p \geq 1$. Let $A_k(f) = 0$, $k = 0, \ldots, q-1$ and parameters λ_k be chosen as in* (2.36). *Then, $R_N^p(f,x) = O(N^{-2p-q-2})$ for $|x| < 1$ as $N \to \infty$.*

As parameters τ_k are undefined, we have additional freedom for getting better accuracy by vanishing the main terms of the asymptotic errors. Paper [35] presents these optimal values for different p and q. As a result, we get more accurate interpolations with

$$R_N^p(f,x) = o(N^{-2p-q-1}), \ R_N^p(f,x) = o(N^{-2p-q-2}) \tag{0.58}$$

for even and odd values of q, respectively.

Comparison with Theorems 0.1 and 0.2 shows that for enough smooth functions the classical RT interpolations are asymptotically more precise than the classical interpolation and improvement is by the factor $O(N^{-2p})$ as $N \to \infty$.

The main result of this section is the next theorem (not published) which characterizes the behavior of the RT interpolation at the points $x = \pm 1$ in terms of the limit functions.

Theorem 0.11 *Let $f^{(q+2p)} \in AC[-1,1]$ for some $p \geq 1$, $q \geq 1$ and $A_k(f) = 0$, $k = 0, \dots, q-1$ Let λ_k be chosen as in* (0.56)*. Then, the following estimate holds*

$$\lim_{N\to\infty} N^q R_N^p \left(f, \pm\left(1 - \frac{h}{N}\right)\right) = A_q(f)\ell^p_{x\to\pm1,q}(h), \ h \geq 0, \tag{0.59}$$

where

$$\ell^p_{x\to\pm1,q}(h) = \frac{(-1)^{p+1}}{2(i\pi)^{q+1}q!} \frac{1}{\prod_{s=1}^p(\tau_s^2 + \pi^2 h^2)} \sum_{s=0}^p (-1)^s \gamma_s(\tau) \sum_{k=0}^p \gamma_k(\tau)(2p-k-s+q)! \\ \times \left(\int_{|t|>1} \frac{e^{\mp i\pi ht}}{t^{2p-s-k+q+1}} dt - \int_{-1}^1 e^{\mp i\pi ht} \sum_{r\neq 0} \frac{(-1)^r}{(2r+t)^{2p-k-s+q+1}} dt \right). \tag{0.60}$$

Estimate (0.59) is valid also for $q = 0$ and $h > 0$. Case $h = 0$ must be considered separately. It can be verified that

$$\lim_{N\to\infty} R_N^p (f, \pm 1) = A_0(f)\ell^p_{x\to\pm1,0}(0), \tag{0.61}$$

where

$$\ell^p_{x\to\pm1,0}(0) = \pm\frac{1}{2}. \tag{0.62}$$

Taking into account that

$$|\ell^p_{x\to1,q}(h)| = |\ell^p_{x\to-1,q}(h)|, \tag{0.63}$$

we put

$$\ell_q^{p,*} = \max_{h\geq 0} |\ell^p_{x\to1,q}(h)| \tag{0.64}$$

which characterizes the asymptotic ($N \to \infty$) uniform error of the classical RT interpolation. Estimate (0.59) outlines another approach for determination of parameters τ_k by minimization of constant $\ell_q^{p,*}$. In **Section 2.1**, we show the corresponding values of τ_k for $p = 1$ and $p = 2$. Analysis of ratio $\ell_q^*/\ell_q^{p,*}$ shows that, when $q > 0$, the classical RT interpolation is asymptotically more accurate with optimal parameters than the classical interpolation in the uniform norm. However, when $q = 0$, both interpolations provide the same asymptotic accuracy in the uniform norm, independently of the choice of parameters τ_k.

Section 2.2 considers convergence acceleration of the QP interpolation by rational (in terms of $e^{i\pi\sigma x}$) correction functions. The resultant QP Rational (QPR) approximation $I^p_{N,m}(f,x)$ and its error $R^p_{N,m}(f,x)$ are defined by (2.34) and (2.35), respectively. The QPR approximation is represented as a sum of the QP interpolation and correction functions. Rational corrections, as in case of the classical RT, contain some parameters λ_k. One approach for their determination is (0.56). Another approach is determination of parameters λ_k similar to (0.57) with some modification as follows

$$\delta_n^p(\lambda, \{F_{s,m}\}) = 0, \ n = N - p + \left[\frac{m}{2}\right] + 1, \cdots, N + p + \left[\frac{m}{2}\right] \tag{0.65}$$

which we call as QP FP approximation. **Section 2.5** presents some numerical results for these approximations and compares the results with the classical FP interpolations.

Section 2.3 studies the pointwise convergence of the QPR approximation. Let

$$\prod_{s=1}^{p} (1 + \tau_s x) = \sum_{k=0}^{p} \gamma_k(\tau) x^k, \tag{0.66}$$

where $\tau = \{\tau_1, \ldots, \tau_p\}$. The main results are:

Theorem 0.12 *[59] Let $f^{(q+2p+m)} \in AC[-1,1]$ for some $q \geq 0$, $p, m \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \ldots, q-1. \tag{0.67}$$

Let parameters λ_k be chosen as in (0.56)*. Then, the following estimate holds for $|x| < 1$*

$$R_{N,m}^p(f, x) = \frac{D_{N,m}^p(f, x)}{N^{q+2p+1}} + o(N^{-q-2p-1}), \ N \to \infty. \tag{0.68}$$

where

$$\begin{aligned} D_{N,m}^p(f,x) &= \frac{(-1)^N}{2^{q+2p+1}} \frac{\sin\left(\pi\sigma(N+\frac{1}{2})x - \frac{\pi m}{2}\right)}{\cos^{2p+1}\frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k}2^k}{(q-k)! i^{k-m}\pi^{k+1}} \\ &\times \left(\frac{(-1)^p}{k!}\psi_{p,m,k}^{+}(\tau) - \sum_{t=0}^{m-1} \frac{\Phi_{k,m}^{(t)}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu - m + 1, \tau)\right), \end{aligned} \tag{0.69}$$

$$\psi_{p,m,j}^{\pm}(\tau) = \sum_{t=0}^{p} (-1)^t \gamma_t(\tau) \sum_{k=0}^{p} \gamma_k(\tau)(2p - k - t + j)! \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{(2r \pm 1)^{2p-k-t+j+1}}, \tag{0.70}$$

$$h_p(\beta, \tau) = \left(\frac{\pi\beta}{2}\right)^{2p} \sum_{t=0}^{p} \gamma_t(\tau) \sum_{k=0}^{p} (-1)^k \gamma_k(\tau) \left(\frac{2}{i\pi\beta}\right)^{k+t}. \tag{0.71}$$

Estimate (0.68) is valid also for $m = 0$ if $f^{(q+2p+1)} \in AC[-1,1]$. Comparison with Theorems 0.5 and 0.6 shows that if $2p > m$ then the QPR approximation has better convergence rate than the QP interpolation and improvement is by factor $O(N^{2p-m})$. Note that estimate (0.68) is valid independently of parameters τ_k.

In estimate (0.68), function $D^p_{N,m}(f,x)$ has an interesting property. When m is odd and $q=0$ then $D^p_{N,m}(f,x)=0$ for all $p\ge 1$ independently of f and parameters τ_k. We summarize this property in the next theorem (not yet published):

Theorem 0.13 *Let $f^{(2p+m)}\in AC[-1,1]$ for some $p\ge 1$, odd m and*

$$f(-1)\neq f(1). \tag{0.72}$$

Let parameters λ_k be chosen as in (0.56). Then, the following estimate holds for $|x|<1$

$$R^p_{N,m}(f,x)=o(N^{-2p-1}),\ \ N\to\infty. \tag{0.73}$$

If f has additional smoothness then estimate (0.73) can be improved to $R^p_{N,m}(f,x)=O(N^{-2p-2})$. Again, estimate (0.73) is valid without determination of parameters τ_k.

Section 2.4 performs analysis of the limit functions of the QPR approximation. Main result is (not yet published):

Theorem 0.14 *Let $f^{(q+2p+m)}\in AC[-1,1]$ for some $q,m\ge 0$, $p\ge 1$ and*

$$f^{(k)}(-1)=f^{(k)}(1)=0,\ \ k=0,\dots,q-1. \tag{0.74}$$

Let parameters λ_k be chosen as in (0.56). Then, the following estimate holds

$$\lim_{N\to\infty} N^q R^p_{N,m}\left(f,\pm\left(1-\frac{h}{N}\right)\right)=\ell^p_{x\to\pm1,q,m}(f,h),\ \ h\ge 0, \tag{0.75}$$

where

$$\begin{aligned}\ell^p_{x\to\pm1,q,m}(f,h) &= \frac{1}{\prod_{s=1}^p(\tau_s^2+\pi^2(\frac{m+1}{2}+h)^2)}\\ &\times\left(\int_{-1}^1 \nu^p_{q,m}(f,x)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)x}dx-\int_{|x|>1}\mu^p_{q,m}(f,x)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)x}dx\right)\end{aligned} \tag{0.76}$$

and

$$\begin{aligned}\mu^p_{q,m}(f,x) &= \sum_{\ell=0}^q \frac{A_{\ell q}(f)(m+1)^{q-\ell}}{2^{q+1-\ell}(i\pi)^{\ell+1}(q-\ell)!}\sum_{t=0}^p(-1)^{p+t}\gamma_t(\tau)\sum_{k=0}^p\gamma_k(\tau)\\ &\times\frac{(\ell+2p-k-t)!}{\ell!}\frac{1}{x^{2p-k-t+\ell+1}},\end{aligned} \tag{0.77}$$

$$\nu_{q,m}^{p}(f,x) = \sum_{\ell=0}^{q} \frac{A_{\ell q}(f)(m+1)^{q-\ell}}{2^{q+1-l}(i\pi)^{\ell+1}(q-\ell)!}$$
$$\times \left(\sum_{t=0}^{p} (-1)^{p+t}\gamma_t(\tau) \sum_{k=0}^{p} \gamma_k(\tau) \frac{(2p-k-t+\ell)!}{\ell!} \sum_{r\neq 0} \frac{(-1)^{r(m+1)}}{(2r+x)^{2p-k-t+\ell+1}} \right.$$
$$\left. - \sum_{t=0}^{m-1} \frac{\Phi_{\ell,m}^{(t)}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} e^{\frac{i\pi(2\mu-m+1)x}{2}} h_p(2\mu-m+1,\tau) \right). \tag{0.78}$$

In this section, we perform analysis of the QPR approximation with the classical and the QP interpolations and discuss approaches for determination of parameters τ_k based on the asymptotic estimates presented above. The main recommendation, for the case $q=0$, is realization of the QPR approximation for $m=1$ and utilization of parameters τ_k that minimize the uniform error.

Numerical results of **Section 2.5** show comparison of the corresponding interpolations and approximations for such choice of parameters.

Chapter 3 considers convergence acceleration of the QP interpolation and the QPR approximation by polynomial corrections.

Section 3.1 introduces polynomial corrections. The idea is to construct two families of polynomials $\xi_{k,q}(x)$ and $\eta_{k,q}(x)$, $k=0,\ldots,q-1$ with the following properties

$$\xi_{k,q}^{(s)}(1) - \xi_{k,q}^{(s)}(-1) = \delta_{k,s},\ \xi_{k,q}^{(s)}(1) + \xi_{k,q}^{(s)}(-1) = 0,\ k,s=0,\ldots,q-1 \tag{0.79}$$

and

$$\eta_{k,q}^{(s)}(1) + \eta_{k,q}^{(s)}(-1) = \delta_{k,s},\ \eta_{k,q}^{(s)}(1) - \eta_{k,q}^{(s)}(-1) = 0,\ k,s=0,\ldots,q-1 \tag{0.80}$$

and write the following representation of f

$$f(x) = G(x) + \sum_{k=0}^{q-1} A_k^{-}(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^{+}(f)\eta_{k,q}(x), \tag{0.81}$$

where

$$A_k^{-}(f) = f^{(k)}(1) - f^{(k)}(-1),\ A_k^{+}(f) = f^{(k)}(1) + f^{(k)}(-1). \tag{0.82}$$

Function G has the same smoothness as f, but the following important property for our approaches

$$G^{(k)}(1) = G^{(k)}(-1) = 0,\ k=0,\ldots,q-1. \tag{0.83}$$

Section 3.2 defines the QP Polynomial (QPP) interpolation and the QPR Polynomial (QPRP) approxiamtion based on representation (0.81). Application of the QP interpolation and the QPR approximation to G leads to the following QPP interpolation and QPRP approximation

$$I_{N,m,q}(f,x) = I_{N,m}(G,x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x), \tag{0.84}$$

$$I^p_{N,m,q}(f,x) = I^p_{N,m}(G,x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x), \tag{0.85}$$

respectively. We assumed that the exact values of $A_k^-(f)$ and $A_k^+(f)$ are known. Discrete Fourier coefficients of G can be easily calculated from (0.81).

All theorems concerning the convergence of the QP interpolation and the QPR approximation can be reformulated for the QPP interpolation and the QPRP approximation without condition $f^{(k)}(1) = f^{(k)}(-1) = 0$, $k = 0, \ldots, q-1$. For example, the theorem concerning the pointwise convergence of the QPP interpolation can be reformulated as follows:

Theorem 0.15 *[58] Let $f^{(q+2m)} \in AC[-1,1]$ for some $q, m \geq 1$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$f(x) - I_{N,m,q}(f,x) = \frac{D_{N,m,q}(G,x)}{N^{q+m+1}} + o(N^{-q-m-1}), \tag{0.86}$$

where

$$\begin{aligned} D^p_{N,m,q}(G,x) &= \frac{(-1)^N}{2^{q+2p+1}} \frac{\sin\left(\pi\sigma(N+\frac{1}{2})x - \frac{\pi m}{2}\right)}{\cos^{2p+1}\frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(G)(m+1)^{q-k}2^k}{(q-k)!\, i^{k-m}\pi^{k+1}} \\ &\times \left[\frac{(-1)^p}{k!}\psi^+_{p,m,k}(\tau) - \sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu-m+1,\tau)\right]. \end{aligned} \tag{0.87}$$

Section 3.3 shows the procedure of approximation of $A_k^-(f)$ and $A_k^+(f)$, $k = 1, \ldots, q-1$ by the discrete Fourier coefficients of f. Procedure that we applied was introduced by Eckhoff before in a series of papers [66–68]. We repeat it with some modifications and derive systems of linear equations for determining approximate values $A_k^-(f,N)$ and $A_k^+(f,N)$. Then, we define the corresponding interpolations and approximations:

$$\begin{aligned} \widetilde{I}_{N,m,q}(f,x) &= I_{N,m}(\widetilde{G},x) + \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x) \\ &+ A_0^-(f)\xi_{0,q}(x) + A_0^+(f)\eta_{0,q}(x) \end{aligned} \tag{0.88}$$

and

$$\widetilde{I}^p_{N,m,q}(f,x) = I^p_{N,m}(\widetilde{G},x) + \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x)$$
$$+ A_0^-(f)\xi_{0,q}(x) + A_0^+(f)\eta_{0,q}(x), \tag{0.89}$$

where

$$\widetilde{G}(x) = f(x) - \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) - \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x)$$
$$- A_0^-(f)\xi_{0,q}(x) - A_0^+(f)\eta_{0,q}(x). \tag{0.90}$$

We are not considering the convergence of the latest approaches theoretically and present only some results of numerical experiments in **Section 3.4**.

CHAPTER 1

Quasi-periodic interpolation

In this chapter, we study convergence of the Quasi-Periodic (QP) interpolation in different frameworks: pointwise convergence, L_2-convergence and in terms of the limit functions.

Section 1.1 recounts some well-known results concerning the convergence of the classical trigonometric interpolation which are needed for further comparisons. Section 1.2 presents the statement of the QP interpolation and derives explicit formula for it. Sections 1.3, 1.4 and 1.5 study the QP interpolation in terms of pointwise convergence, L_2-convergence and limit functions, respectively. Section 1.6 discusses some results of numerical experiments.

We recap details from [56, 57, 60]. Topics discussed here are presented also in [61, 62].

1.1 The Classical Trigonometric Interpolation

Let $I_N(f, x)$ be the classical trigonometric interpolation on equidistant grid

$$x_k = \frac{2k}{2N+1}, \ |k| \le N. \tag{1.1}$$

As we mentioned above (see Introduction), the classical interpolation with equidistant nodes can be rewritten via discrete Fourier coefficients

$$I_N(f, x) = \sum_{n=-N}^{N} \check{f}_n e^{i\pi n x}, \tag{1.2}$$

where

$$\check{f}_n = \frac{1}{2N+1} \sum_{k=-N}^{N} f(x_k) e^{-i\pi n x_k}. \tag{1.3}$$

Let

$$R_N(f, x) = f(x) - I_N(f, x). \tag{1.4}$$

Next theorems address the convergence of the classical trigonometric interpolation.

Let $f \in C^{q-1}[-1,1]$ and

$$A_k(f) = f^{(k)}(1) - f^{(k)}(-1), \ k = 0, \dots, q-1. \tag{1.5}$$

The following theorems study the pointwise convergence.

Theorem 1.1 *[24] Let $q \geq 0$ be even, $f^{(q+1)} \in AC[-1,1]$ and*

$$f^{(k)}(1) = f^{(k)}(-1), \ k = 0, \dots, q-1. \tag{1.6}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$R_N(f,x) = A_q(f) \frac{(-1)^{N+\frac{q}{2}}}{2\pi^{q+1}N^{q+1}} \frac{\sin \frac{\pi x}{2}(2N+1)}{\cos \frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+1}} + o(N^{-q-1}). \tag{1.7}$$

Theorem 1.2 *[24] Let $q \geq 1$ be odd, $f^{(q+2)} \in AC[-1,1]$ and*

$$f^{(k)}(1) = f^{(k)}(-1), \ k = 0, \dots, q-1. \tag{1.8}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$\begin{aligned} R_N(f,x) = {} & A_q(f) \frac{(-1)^{N+\frac{q+1}{2}+1}(q+1)}{4\pi^{q+1}N^{q+2}} \frac{\sin \frac{\pi x}{2} \sin \frac{\pi x}{2}(2N+1)}{\cos^2 \frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+2}} \\ & + A_{q+1}(f) \frac{(-1)^{N+\frac{q+1}{2}}}{2\pi^{q+2}N^{q+2}} \frac{\sin \frac{\pi x}{2}(2N+1)}{\cos \frac{\pi x}{2}} \sum_{s=-\infty}^{\infty} \frac{(-1)^s}{(2s+1)^{q+2}} + o(N^{-q-2}). \end{aligned} \tag{1.9}$$

Remark 1.1 *Theorem 1.1 shows that for $q = 0$, the error is $O(N^{-1})$ for fixed $x \neq \pm 1$.*

Next theorem shows the exact constant of the asymptotic error in the L_2-norm (see (0.16)).

Theorem 1.3 *[25] Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$ and*

$$f^{(k)}(1) = f^{(k)}(-1), \ k = 0, \dots, q-1. \tag{1.10}$$

Then, the following estimate holds

$$\lim_{N \to \infty} N^{q+\frac{1}{2}} \| R_N(f,x) \|_{L_2[-1,1]} = |A_q(f)| c_q, \tag{1.11}$$

where

$$c_q = \frac{1}{\pi^{q+1}} \left(\frac{1}{2q+1} + \frac{1}{2} \int_{-1}^{1} \left| \sum_{s \neq 0} \frac{(-1)^s}{(x+2s)^{q+1}} \right|^2 dx \right)^{1/2}. \tag{1.12}$$

Remark 1.2 *Estimate of Theorem 1.3 is valid also for $q = 0$ when $f(1) \neq f(-1)$. In this case condition $f \in AC[-1,1]$ should be $f' \in L_2[-1,1]$. The classical interpolation has slow convergence of order $O(N^{-\frac{1}{2}})$ for $q = 0$ even if f is infinitely differentiable on $[-1,1]$.*

Table 1.1 shows numerical values of c_q for some q.

q	0	1	2	3	4	5	6	7
c_q	0.33	0.084	0.019	0.0055	0.0015	$4.4 \cdot 10^{-4}$	$1.3 \cdot 10^{-4}$	$4.0 \cdot 10^{-5}$

Table 1.1: Numerical values of c_q.

The behavior of the classical interpolation at the points $x = \pm 1$ can be characterized in terms of the limit function ([3]). Helmberg utilized this technique ([11, 12]) for investigation of the Gibbs phenomenon for the classical interpolation with equidistant nodes.

Theorem 1.4 *[26] Let $f^{(q)} \in AC[-1,1]$ for some $q \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1), \; k = 0, \ldots, q-1. \tag{1.13}$$

Then, the following estimate holds

$$\lim_{N\to\infty} N^q R_N\left(f, \pm\left(1 - \frac{h}{N}\right)\right) = A_q(f)\ell_{x\to\pm1,q}(h), \; h \geq 0, \tag{1.14}$$

where

$$\ell_{x\to\pm1,q}(h) = \frac{1}{2(i\pi)^{q+1}}\left(\int_{-1}^{1} e^{\mp i\pi ht}\sum_{s\neq0}\frac{(-1)^s}{(t+2s)^{q+1}}dt - \int_{|t|>1}\frac{e^{\mp i\pi ht}}{t^{q+1}}dt\right). \tag{1.15}$$

Remark 1.3 *Estimate of Theorem 1.4 is valid also for $q = 0$ when $f(1) \neq f(-1)$ and $h > 0$. In this case condition $f \in AC[-1,1]$ should be $f' \in L_2[-1,1]$. The case $h = 0$ must be considered separately. It is easy to verify that*

$$\lim_{N\to\infty} R_N\left(f, \pm1\right) = A_0(f)\ell_{x\to\pm1,0}(0), \; \ell_{x\to\pm1,0}(0) = \pm\frac{1}{2}. \tag{1.16}$$

Taking into account that

$$|\ell_{x\to1,q}(h)| = |\ell_{x\to-1,q}(h)|, \tag{1.17}$$

we put

$$\ell_q^* = \max_{h\geq 0} |\ell_{x\to 1,q}(h)| \tag{1.18}$$

which characterizes the asymptotic ($N \to \infty$) uniform error of the classical interpolation.

Table 1.2 shows numerical values of ℓ_q^* for different q.

q	0	1	2	3	4	5	6	7
ℓ_q^*	0.5	0.16	0.011	0.0066	$7.2 \cdot 10^{-4}$	$4.1 \cdot 10^{-4}$	$5.1 \cdot 10^{-5}$	$3.0 \cdot 10^{-5}$

Table 1.2: Numerical values of ℓ_q^*.

From Table 1.2 and Remark 1.3, we see that for $q = 0$

$$\max_{x\in[-1,1]} |R_N(f,x)| \nrightarrow 0, \;\; N \to \infty. \tag{1.19}$$

1.2 QP Interpolation

The previous section showed that the classical trigonometric interpolation had poor convergence when 2-periodic extension of the interpolated function was discontinuous or had low smoothness on the real line.

Improved convergence, compared to the classical interpolation, has the Quasi-Periodic interpolation considered in [54]. There, and in [25, 26], the interpolation was studied only by numerical experiments. In the current work, we establish (see [56–60]) theoretical background of such interpolations.

Let $f \in C[-1,1]$ and $\{x_k\}_{k=-N}^{N} \subset [-1,1]$, $N \geq 1$ be some grid on $[-1,1]$. We consider interpolation of f on grid x_k which is exact for the following quasi-periodic exponents

$$\left\{e^{i\pi n\sigma x}\right\}_{n=-N}^{N}, \;\sigma = \frac{2N}{2N+m+1}, \; x \in [-1,1], \; m \in Z, \; m \geq 0 \tag{1.20}$$

with the periods $2/\sigma \to 2$ as $N \to \infty$. Such interpolations are known as Quasi-Periodic (QP) interpolations which we denote by $I_{N,m}(f,x)$.

Throughout the paper, we assume that

$$x_k = \frac{k}{N}, \; k = -N, \ldots, N. \tag{1.21}$$

We seek the QP interpolation in the form

$$I_{N,m}(f,x) = \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) a_k(x), \quad x \in [-1,1] \tag{1.22}$$

with unknowns a_k. Condition of exactness for quasi-periodic exponents implies the following system of linear equations for determination of the unknowns

$$e^{i\pi\ell\sigma x} = \sum_{k=-N}^{N} e^{\frac{2i\pi\ell k}{2N+m+1}} a_k(x), \quad |\ell| \le N. \tag{1.23}$$

In order to apply the discrete Fourier transform, we add some new unknowns and equations getting the following enlarged system of equations

$$e^{i\pi\ell\sigma x} = \sum_{k=-N}^{N+m} e^{\frac{2i\pi\ell k}{2N+m+1}} a_k^*(x) + \varepsilon_\ell(x), \quad \ell = -N, ..., N+m, \tag{1.24}$$

where

$$\begin{aligned} & a_k^*(x) \equiv a_k(x), \ |k| \le N, \\ & a_k^*(x) \equiv 0, \ k = N+1, ..., N+m, \\ & \varepsilon_\ell(x) \equiv 0, \ |\ell| \le N. \end{aligned} \tag{1.25}$$

We multiply the both sides of equation (1.24) by $e^{-\frac{2i\pi\ell s}{2N+m+1}}$ and sum over ℓ

$$\begin{aligned} \sum_{\ell=-N}^{N+m} e^{\frac{2i\pi\ell(Nx-s)}{2N+m+1}} &= \sum_{\ell=-N}^{N+m}\sum_{k=-N}^{N+m} e^{\frac{2i\pi\ell(k-s)}{2N+m+1}} a_k^*(x) + \sum_{\ell=N+1}^{N+m} e^{-\frac{2i\pi\ell s}{2N+m+1}} \varepsilon_\ell(x), \\ s &= -N, \dots, N+m. \end{aligned} \tag{1.26}$$

Taking into account that

$$\sum_{\ell=-N}^{N+m} e^{\frac{2i\pi\ell(k-s)}{2N+m+1}} = (2N+m+1)\delta_{k,s}, \ \ k,s = -N, \dots, N+m, \tag{1.27}$$

we get

$$a_s^*(x) = \frac{1}{2N+m+1}\left(\sum_{\ell=-N}^{N+m} e^{\frac{2i\pi\ell(Nx-s)}{2N+m+1}} - \sum_{\ell=N+1}^{N+m} e^{-\frac{2i\pi\ell s}{2N+m+1}} \varepsilon_\ell(x)\right). \tag{1.28}$$

Conditions (1.25) deduce to the following system with Vandermonde matrix for determination of $\varepsilon_\ell(x)$

$$\sum_{\ell=N+1}^{N+m} e^{-\frac{2i\pi\ell s}{2N+m+1}} \left(\varepsilon_\ell(x) - e^{\frac{2i\pi\ell Nx}{2N+m+1}}\right) = \sum_{t=-N}^{N} e^{\frac{2i\pi t(Nx-s)}{2N+m+1}}, \ \ s = N+1, ..., N+m. \tag{1.29}$$

After some transformations, we obtain

$$\sum_{\ell=1}^{m} v_{s,\ell}\,\tilde{\varepsilon}_\ell(x) = \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m-1)}{2N+m+1}}, \quad s = 1, ..., m, \tag{1.30}$$

where

$$\tilde{\varepsilon}_\ell(x) = e^{-\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \left(\varepsilon_{\ell+N}(x) - e^{\frac{2i\pi(\ell+N)Nx}{2N+m+1}} \right) \tag{1.31}$$

and

$$v_{s,\ell} = \alpha_\ell^{s-1}, \;\; \alpha_\ell = e^{\frac{2i\pi(\ell+N)}{2N+m+1}}. \tag{1.32}$$

Following [69] (see also [51, 70–72]), where the explicit form of the inverse of Vandermonde matrix was constructed, we write

$$v_{\ell,s}^{-1} = -\frac{1}{\alpha_\ell^s \prod_{i=1, i\neq\ell}^{m} (\alpha_\ell - \alpha_i)} \sum_{j=0}^{s-1} \beta_j \alpha_\ell^j, \;\; \ell, s = 1, ..., m, \tag{1.33}$$

where β_j are the coefficients of the polynomial

$$\prod_{i=1}^{m} (x - \alpha_i) = \sum_{j=0}^{m} \beta_j x^j. \tag{1.34}$$

Now, the solution of (1.29) can be written explicitly in the form

$$\begin{aligned} \varepsilon_\ell(x) &= e^{\frac{2i\pi \ell Nx}{2N+m+1}} + e^{\frac{2i\pi\ell(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell-N,s}^{-1} \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m-1)}{2N+m+1}}, \\ &\ell = N+1, ..., N+m. \end{aligned} \tag{1.35}$$

Substituting $\varepsilon_\ell(x)$ into (1.28), we get the unique solution of (1.23)

$$\begin{aligned} a_k(x) = \frac{1}{2N+m+1} \Bigg(&\sum_{\ell=-N}^{N} e^{\frac{2i\pi\ell Nx}{2N+m+1}} e^{-\frac{2i\pi\ell k}{2N+m+1}} - \sum_{\ell=N+1}^{N+m} e^{\frac{2i\pi\ell(N+m)}{2N+m+1}} e^{-\frac{2i\pi\ell k}{2N+m+1}} \\ &\times \sum_{s=1}^{m} v_{\ell-N,s}^{-1} \sum_{t=-N}^{N} e^{\frac{2i\pi tNx}{2N+m+1}} e^{\frac{2i\pi t(s-N-m-1)}{2N+m+1}} \Bigg), \;\; |k| \leq N. \end{aligned} \tag{1.36}$$

Substitution of (1.36) into (1.22) leads to the following explicit formula for the QP interpolation (see [56, 57])

$$I_{N,m}(f, x) = \sum_{n=-N}^{N} F_{n,m} e^{i\pi n\sigma x}, \tag{1.37}$$

where

$$F_{n,m} = \check{f}_{n,m} - \sum_{\ell=1}^{m} \theta_{n,\ell} \check{f}_{\ell+N,m}, \tag{1.38}$$

$$\check{f}_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-\frac{2i\pi nk}{2N+m+1}} \tag{1.39}$$

and

$$\theta_{n,\ell} = e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi n(s-N-m-1)}{2N+m+1}}. \tag{1.40}$$

When $m = 0$

$$I_{N,0}(f,x) = \sum_{n=-N}^{N} f_{n,0} e^{i\pi n\sigma x},\ \sigma = \frac{2N}{2N+1}, \tag{1.41}$$

where

$$\check{f}_{n,0} = \frac{1}{2N+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) e^{-\frac{2i\pi nk}{2N+1}}. \tag{1.42}$$

We see that (1.41), (1.42) coincide with (0.32), (0.33).

Let us check that $I_{N,m}(f,x)$ interpolates f on grid $x_k = k/N$, $|k| \le N$.

Theorem 1.5 *[56, 57] Let $f \in C[-1,1]$. Then $I_{N,m}(f,x)$ interpolates f on grid* (1.21).

Proof. We will proof straightforwardly. Let $x = k/N$

$$I_{N,m}\left(f, \frac{k}{N}\right) = \sum_{n=-N}^{N} \left(\check{f}_{n,m} - \sum_{\ell=1}^{m} \theta_{n,\ell} \check{f}_{\ell+N,m}\right) e^{\frac{2i\pi nk}{2N+m+1}}. \tag{1.43}$$

From (1.40), we have

$$\begin{aligned} I_{N,m}\left(f, \frac{k}{N}\right) &= \sum_{n=-N}^{N} \left(\check{f}_{n,m} - \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi n(s-N-m-1)}{2N+m+1}} \check{f}_{\ell+N,m}\right) e^{\frac{2i\pi nk}{2N+m+1}} \\ &= \sum_{n=-N}^{N} \check{f}_{n,m} e^{\frac{2i\pi nk}{2N+m+1}} \\ &- \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \check{f}_{\ell+N,m} \sum_{s=1}^{m} v_{\ell,s}^{-1} \sum_{n=-N}^{N} e^{\frac{2i\pi n(s-N-m+k-1)}{2N+m+1}}. \end{aligned} \tag{1.44}$$

Taking into account that (see (1.32))

$$\begin{aligned} \sum_{s=1}^{m} v_{\ell,s}^{-1} \sum_{n=-N}^{N} e^{\frac{2i\pi n(s-N-m+k-1)}{2N+m+1}} &= -\sum_{s=1}^{m} v_{\ell,s}^{-1} \sum_{t=1}^{m} e^{\frac{2i\pi(t+N)(s-N-m+k-1)}{2N+m+1}} \\ &= -\sum_{s=1}^{m} v_{\ell,s}^{-1} \sum_{t=1}^{m} e^{\frac{2i\pi(t+N)(k+s+N)}{2N+m+1}} \\ &= -\sum_{t=1}^{m} e^{\frac{2i\pi(t+N)(k+N+1)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi(t+N)(s-1)}{2N+m+1}} \\ &= -e^{\frac{2i\pi(\ell+N)(k+N+1)}{2N+m+1}}, \end{aligned} \tag{1.45}$$

we obtain the required result

$$
\begin{aligned}
I_{N,m}\left(f,\frac{k}{N}\right) &= \sum_{n=-N}^{N} \check{f}_{n,m} e^{\frac{2i\pi nk}{2N+m+1}} + \sum_{\ell=1}^{m} \check{f}_{\ell+N,m} e^{\frac{2i\pi(\ell+N)k}{2N+m+1}} \\
&= \frac{1}{2N+m+1} \sum_{t=-N}^{N} f\left(\frac{t}{N}\right)\left(\sum_{n=-N}^{N} e^{\frac{2i\pi n(k-t)}{2N+m+1}} + \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(k-t)}{2N+m+1}}\right) \\
&= \frac{1}{2N+m+1} \sum_{t=-N}^{N} f\left(\frac{t}{N}\right) \sum_{n=-N}^{N+m} e^{\frac{2i\pi n(k-t)}{2N+m+1}} \qquad (1.46)\\
&= \sum_{t=-N}^{N} f\left(\frac{t}{N}\right) \delta_{k,t} \\
&= f\left(\frac{k}{N}\right).
\end{aligned}
$$

□

We frequently use the following estimates. First, (1.32) implies

$$\alpha_s - \alpha_i = O(1/N). \qquad (1.47)$$

Second, (1.33) and (1.47) provide

$$v_{s,\ell}^{-1} = O\left(N^{m-1}\right),\ N \to \infty. \qquad (1.48)$$

Third, from (1.40) and (1.48), we obtain

$$\theta_{n,\ell} = O\left(N^{m-1}\right),\ N \to \infty. \qquad (1.49)$$

Then, from (1.32) and (1.40), it follows that

$$
\begin{aligned}
\theta_{N+k,\ell} &= e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi(N+k)(s-N-m-1)}{2N+m+1}} \\
&= e^{\frac{2i\pi(\ell-k)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v_{\ell,s}^{-1} e^{\frac{2i\pi(N+k)(s-1)}{2N+m+1}} \qquad (1.50)\\
&= \delta_{k,\ell},\ \ell, k = 1, ..., m.
\end{aligned}
$$

Finally, taking into account (1.50) and the periodicity of $F_{n,m}$ from (1.38), it follows that

$$F_{N+k,m} = 0,\ F_{-N-k,m} = 0,\ k = 1, ..., m. \qquad (1.51)$$

1.3 Pointwise Convergence

In this section, we investigate the pointwise convergence of the QP interpolations and derive exact constants for the main terms of asymptotic errors ([60]). Comparison with the corresponding results for the classical trigonometric interpolation shows extra accuracy of the QP interpolation for enough smooth functions.

1.3.1 Preliminaries

Here, we collect some basic lemmas and formulae that we need for further investigations. Consider a sequence of complex numbers $\{y_s\}_{s=-\infty}^{\infty}$. Let us define some sequences of finite differences

$$\begin{aligned}\delta_n^0\left(\{y_s\}_{s=-\infty}^{\infty}\right) &= \delta_n^0\left(\{y_s\}\right) = y_n,\\ \delta_n^p\left(\{y_s\}_{s=-\infty}^{\infty}\right) &= \delta_n^p\left(\{y_s\}\right)\\ &= \delta_{n+1}^{p-1}\left(\{y_s\}\right) + 2\delta_n^{p-1}\left(\{y_s\}\right) + \delta_{n-1}^{p-1}\left(\{y_s\}\right),\ p \geq 1\end{aligned} \tag{1.52}$$

and

$$\begin{aligned}\Delta_n^0\left(\{y_s\}_{s=-\infty}^{\infty}\right) &= \Delta_n^0\left(\{y_s\}\right) = y_n,\\ \Delta_n^p\left(\{y_s\}_{s=-\infty}^{\infty}\right) &= \Delta_n^p\left(\{y_s\}\right)\\ &= \Delta_n^{p-1}\left(\{y_s\}\right) + \Delta_{n-1}^{p-1}\left(\{y_s\}\right),\ p \geq 1.\end{aligned} \tag{1.53}$$

It is easy to verify that

$$\delta_n^p\left(\{y_s\}\right) = \Delta_{n+p}^{2p}\left(\{y_s\}\right), \tag{1.54}$$

$$\Delta_n^p\left(\{y_s\}\right) = \sum_{k=0}^{p} \binom{p}{k} y_{n-k} \tag{1.55}$$

and

$$\delta_n^p\left(\{y_s\}\right) = \sum_{k=0}^{2p} \binom{2p}{k} y_{n+p-k}. \tag{1.56}$$

Let

$$\begin{aligned}B_n(j) &= \frac{(-1)^{n+1}}{2(i\pi n)^{j+1}},\ n \neq 0,\\ B_0(j) &= 0.\end{aligned} \tag{1.57}$$

Lemma 1.1 *[60] The following estimate holds for $p, m \geq 0$ and $n \in \mathbb{Z}$ as $N \to \infty$*

$$\delta_n^p \left(\left\{ (-1)^s e^{\frac{i\pi\beta s}{2N+m+1}} \right\}_{s=-\infty}^{\infty} \right) = \frac{(-1)^n (\pi\beta)^{2p}}{(2N+m+1)^{2p}} e^{\frac{i\pi\beta n}{2N+m+1}} + O\left(N^{-2p-1}\right), \tag{1.58}$$

where $\beta \in \mathbb{R}$ is a constant.

Proof. According to (1.54)

$$\delta_n^p \left(\left\{ (-1)^s e^{\frac{i\pi\beta s}{2N+m+1}} \right\} \right) = \Delta_{n+p}^{2p} \left(\left\{ (-1)^s e^{\frac{i\pi\beta s}{2N+m+1}} \right\} \right). \tag{1.59}$$

Utilization of (1.55) implies

$$\begin{aligned}
\Delta_n^p \left(\left\{ (-1)^s e^{\frac{i\pi\beta s}{2N+m+1}} \right\} \right) &= (-1)^n e^{\frac{i\pi\beta n}{2N+m+1}} \sum_{k=0}^{p} \binom{p}{k} (-1)^k e^{-\frac{i\pi\beta k}{2N+m+1}} \\
&= (-1)^n e^{\frac{i\pi\beta n}{2N+m+1}} \sum_{k=0}^{p} \binom{p}{k} (-1)^k \sum_{t=0}^{\infty} \frac{(i\pi\beta)^t (-1)^t k^t}{t!(2N+m+1)^t} \\
&= (-1)^n e^{\frac{i\pi\beta n}{2N+m+1}} \sum_{t=0}^{\infty} \frac{(-1)^t (i\pi\beta)^t}{t!(2N+m+1)^t} \omega_{p,t},
\end{aligned} \tag{1.60}$$

where (see [73])

$$\begin{aligned}
\omega_{p,t} &= \sum_{k=0}^{p} \binom{p}{k} (-1)^k k^t \sim p^t, \ t \to \infty, \\
\omega_{p,t} &= 0, \ 0 \leq t < p, \\
\omega_{p,p} &= (-1)^p \ p!.
\end{aligned} \tag{1.61}$$

Hence,

$$\Delta_n^p \left(\left\{ (-1)^s e^{\frac{i\pi\beta s}{2N+m+1}} \right\} \right) = \frac{(-1)^n (i\pi\beta)^p}{(2N+m+1)^p} e^{\frac{i\pi\beta n}{2N+m+1}} + O\left(N^{-p-1}\right), \ N \to \infty \tag{1.62}$$

which completes the proof in view of (1.59). □

Lemma 1.2 *The following estimate holds for $p, j, m \geq 0$ as $N \to \infty$*

$$\begin{aligned}
\delta_{\pm N}^p \left(\left\{ \sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(j) \right\}_{n=-\infty}^{\infty} \right) &= \frac{(-1)^{N+p+1}}{2(i\pi N)^{j+1} N^{2p}} \frac{(j+2p)!}{j!} \\
&\times \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{(2r \pm 1)^{2p+j+1}} + O\left(N^{-j-2p-2}\right).
\end{aligned} \tag{1.63}$$

Proof. According to (1.54)

$$\delta_{\pm N}^p \left(\left\{ \sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(j) \right\} \right) = \Delta_{\pm N+p}^{2p} \left(\left\{ \sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(j) \right\} \right). \tag{1.64}$$

In view of (1.55)

$$\Delta^p_{\pm N+\ell}\left(\left\{\sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(j)\right\}\right)=\sum_{k=0}^{p}\binom{p}{k}\sum_{r=-\infty}^{\infty} B_{\pm N+\ell-k+r(2N+m+1)}(j). \tag{1.65}$$

Using definition of $B_n(j)$ (see (1.57)), we get

$$\begin{aligned}\sum_{r=-\infty}^{\infty} B_{\pm N+\ell+r(2N+m+1)}(j) &= \frac{(-1)^{N+\ell+1}2^j}{(i\pi)^{j+1}(2N+m+1)^{j+1}}\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{\left(2r+\frac{2(\pm N+\ell)}{2N+m+1}\right)^{j+1}}\\ &= \frac{(-1)^{N+\ell+1}2^j}{(i\pi)^{j+1}(2N+m+1)^{j+1}}\sum_{t=0}^{\infty}\binom{t+j}{j}\frac{(-1)^t\left(2\ell\mp(m+1)\right)^t}{(2N+m+1)^t}\\ &\times\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r\pm 1)^{t+j+1}}.\end{aligned} \tag{1.66}$$

Substituting the last one into (1.65) and taking into account (1.61), we derive

$$\begin{aligned}&\Delta^p_{\pm N+\ell}\left(\left\{\sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(j)\right\}\right)=\frac{(-1)^{N+\ell+1}2^j}{(i\pi)^{j+1}(2N+m+1)^{j+1}}\\ &\times\sum_{t=0}^{\infty}\binom{t+j}{j}\frac{(-1)^t}{(2N+m+1)^t}\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r\pm 1)^{t+j+1}}\\ &\times\sum_{\tau=0}^{t}(-1)^{\tau}2^{\tau}\omega_{p,\tau}\left(2\ell\mp(m+1)\right)^{t-\tau}\\ &=\frac{(-1)^{N+p+\ell+1}}{2(i\pi N)^{j+1}N^p}\frac{(j+p)!}{j!}\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r\pm 1)^{p+j+1}}+O\left(N^{-j-p-2}\right),\ N\to\infty.\end{aligned} \tag{1.67}$$

This concludes the proof in view of (1.64). □

Lemma 1.3 *[60] The following estimate holds for $p,j,m\ge 0$ and $|n|\le N$*

$$\delta^p_n\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(j)\right\}_{s=-\infty}^{\infty}\right)=O\left(N^{-j-2p-1}\right),\ N\to\infty. \tag{1.68}$$

Proof. According to (1.54)

$$\delta^p_n\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(j)\right\}\right)=\Delta^{2p}_{n+p}\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(j)\right\}\right). \tag{1.69}$$

Equations (1.55) and (1.57) deduce

$$\begin{aligned}\Delta^p_{n+\ell}\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(j)\right\}\right)&=\sum_{k=0}^{p}\binom{p}{k}\sum_{r\neq 0} B_{n+\ell-k+r(2N+m+1)}(j)\\ &=\frac{(-1)^{n+\ell+1}}{2(i\pi)^{j+1}}\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k}{(2N+m+1)^{j+1}}\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{\left(r+\frac{n+\ell-k}{2N+m+1}\right)^{j+1}}.\end{aligned} \tag{1.70}$$

Taking into account (1.61), we can write

$$\begin{aligned}\Delta_{n+\ell}^{p}\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(j)\right\}\right) &= \frac{(-1)^{n+\ell+1}}{2(i\pi)^{j+1}}\sum_{r\neq 0}\sum_{t=0}^{\infty}\binom{t+j}{j} \\ &\times \frac{(-1)^{r(m+1)}\omega_{p,t}}{(2N+m+1)^{t+j+1}\left(r+\frac{n+\ell}{2N+m+1}\right)^{t+j+1}} \\ &= \frac{(-1)^{n+\ell+p+1}}{2(i\pi)^{j+1}(2N+m+1)^{p+j+1}}\frac{(p+j)!}{j!} \\ &\times \sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{\left(r+\frac{n+\ell}{2N+m+1}\right)^{p+j+1}} + O\left(N^{-j-p-2}\right),\ N\to\infty.\end{aligned} \tag{1.71}$$

This completes the proof together with (1.69). □

Lemma 1.4 *[24] The following estimate holds for $p, j \geq 0$*

$$\delta_n^p\left(\{B_s(j)\}_{s=-\infty}^{\infty}\right) = O\left(n^{-j-2p-1}\right),\ n\to\infty. \tag{1.72}$$

Proof. According to (1.54)

$$\delta_n^p\left(\{B_s(j)\}\right) = \Delta_{n+p}^{2p}\left(\{B_s(j)\}\right). \tag{1.73}$$

Taking into account (1.55), (1.57) and (1.61), we get

$$\begin{aligned}\Delta_n^p\left(\{B_s(j)\}\right) &= \frac{(-1)^{n+1}}{2(i\pi n)^{j+1}}\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k}{\left(1-\frac{k}{n}\right)^{j+1}} \\ &= \frac{(-1)^{n+1}}{2(i\pi n)^{j+1}}\sum_{k=0}^{p}\binom{p}{k}\sum_{t=0}^{\infty}\binom{t+j}{j}\frac{(-1)^k k^t}{n^t} \\ &= \frac{(-1)^{n+1}}{2(i\pi n)^{j+1}}\sum_{t=0}^{\infty}\binom{t+j}{j}\frac{\omega_{p,t}}{n^t} \\ &= \frac{(-1)^{n+p+1}}{2(i\pi n)^{j+1}n^p}\frac{(p+j)!}{j!} + O\left(n^{-p-j-2}\right)\end{aligned} \tag{1.74}$$

which completes the proof. □

Below, we analyze separately the cases $m \geq 1$ and $m = 0$.

1.3.2 Analysis of the case $m \geq 1$

Let $f \in C^{\alpha}[-1,1]$, $\alpha \geq 0$. We put

$$f^*(x) = \begin{cases} f_{left}(x), & x\in[-1,-\sigma), \\ f\left(\frac{x}{\sigma}\right), & x\in[-\sigma,\sigma], \\ f_{right}(x), & x\in(\sigma,1], \end{cases} \tag{1.75}$$

where

$$f_{left}(x) = \sum_{j=0}^{\alpha} \frac{f^{(j)}(-1)}{j!} \left(\frac{x}{\sigma}+1\right)^j, \quad f_{right}(x) = \sum_{j=0}^{\alpha} \frac{f^{(j)}(1)}{j!} \left(\frac{x}{\sigma}-1\right)^j. \tag{1.76}$$

Obviously $f^* \in C^\alpha[-1,1]$.

We denote by f_n^* the Fourier coefficient of $f^*(x)$

$$f_n^* = \frac{1}{2}\int_{-1}^{1} f^*(x) e^{-i\pi n x} dx. \tag{1.77}$$

Let

$$A_{ks}(f) = f^{(s)}(1) - (-1)^{k+s} f^{(s)}(-1). \tag{1.78}$$

We need some additional lemmas.

Lemma 1.5 *[60] Let $f^{(q+v)} \in AC[-1,1]$ for some $q, v, m \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, \dots, q-1. \tag{1.79}$$

Then, the following estimate holds as $n, N \to \infty$

$$f_n^* = \sum_{j=q}^{q+v} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}(2N+m+1)^k}{(j-k)!} B_n(k) + o(n^{-q-v-1}). \tag{1.80}$$

Proof. Taking into account the smoothness of f and definition of f^* with $\alpha = q+v$ (see (1.75)), we write

$$\begin{aligned} f_n^* &= \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q+v} \frac{f^{*(k)}(1) - f^{*(k)}(-1)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^{q+v+1}} \int_{-1}^{1} f^{*(q+v+1)}(x) e^{-i\pi n x} dx \\ &= \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q+v} \frac{f_{right}^{(k)}(1) - f_{left}^{(k)}(-1)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^{q+v+1}} \int_{-\frac{2N}{2N+m+1}}^{\frac{2N}{2N+m+1}} f^{*(q+v+1)}(x) e^{-i\pi n x} dx \\ &\quad + \frac{1}{2(i\pi n)^{q+v+1}} \int_{-1}^{-\frac{2N}{2N+m+1}} f_{left}^{(q+v+1)}(x) e^{-i\pi n x} dx \\ &\quad + \frac{1}{2(i\pi n)^{q+v+1}} \int_{\frac{2N}{2N+m+1}}^{1} f_{right}^{(q+v+1)}(x) e^{-i\pi n x} dx. \end{aligned} \tag{1.81}$$

Then

$$\begin{aligned} f_n^* &= \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q+v} \frac{f_{right}^{(k)}(1) - f_{left}^{(k)}(-1)}{(i\pi n)^{k+1}} \\ &\quad + \left(\frac{2N+m+1}{2N}\right)^{q+v} \frac{1}{2(i\pi n)^{q+v+1}} \int_{-1}^{1} f^{(q+v+1)}(x) e^{-i\pi n \frac{2N}{2N+m+1} x} dx, \end{aligned} \tag{1.82}$$

where

$$f_{left}^{(k)}(-1) = \left(\frac{2N+m+1}{2N}\right)^k \sum_{j=k}^{q+v}(-1)^{j-k} f^{(j)}(-1)\frac{(m+1)^{j-k}}{(j-k)!(2N)^{j-k}} \tag{1.83}$$

and

$$f_{right}^{(k)}(1) = \left(\frac{2N+m+1}{2N}\right)^k \sum_{j=k}^{q+v} f^{(j)}(1)\frac{(m+1)^{j-k}}{(j-k)!(2N)^{j-k}}. \tag{1.84}$$

Substituting these into (1.82), in view of the generalized Riemann-Lebesgue theorem (see[1]), we get

$$f_n^* = \frac{(-1)^{n+1}}{2}\sum_{k=0}^{q+v}\frac{1}{(i\pi n)^{k+1}}\left(\frac{2N+m+1}{2N}\right)^k \sum_{j=k}^{q+v}\frac{A_{kj}(f)(m+1)^{j-k}}{(j-k)!(2N)^{j-k}} + o(n^{-q-v-1}),\ n, N \to \infty. \tag{1.85}$$

Condition (1.79) yields

$$f_n^* = \frac{(-1)^{n+1}}{2}\sum_{j=q}^{q+v}\sum_{k=0}^{j}\left(\frac{2N+m+1}{2N}\right)^k \frac{A_{kj}(f)(m+1)^{j-k}}{(j-k)!(2N)^{j-k}(i\pi n)^{k+1}} + o(n^{-q-v-1}),\ n, N \to \infty \tag{1.86}$$

which completes the proof. □

Let

$$\Phi_{k,m}(e^{i\pi x}) = e^{\frac{i\pi}{2}(m-1)x}\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}},\ x \neq 2n,\ n \in \mathbb{Z}. \tag{1.87}$$

This function can be expressed by means of elementary functions.

Let m be odd

$$\Phi_{k,m}(e^{i\pi x}) = e^{\frac{i\pi}{2}(m-1)x}\sum_{r=-\infty}^{\infty}\frac{1}{(2r+x)^{k+1}},\ x \neq 2n,\ n \in \mathbb{Z}. \tag{1.88}$$

Let

$$\phi(x) := \sum_{r=-\infty}^{\infty}\frac{1}{2r+x} = \frac{\pi i}{2}\frac{e^{i\pi x}+1}{e^{i\pi x}-1}. \tag{1.89}$$

Taking into account that

$$\sum_{r=-\infty}^{\infty}\frac{1}{(2r+x)^{k+1}} = \frac{(-1)^k}{k!}\phi^{(k)}(x), \tag{1.90}$$

we get

$$\Phi_{k,m}(e^{i\pi x}) = \frac{(-1)^k}{k!}e^{\frac{i\pi}{2}(m-1)x}\phi^{(k)}(x). \tag{1.91}$$

It remains to calculate $\phi^{(k)}(x)$. Let

$$g(x) := \frac{x+1}{x-1}. \tag{1.92}$$

We apply the well known identity ([44])

$$\frac{d^k}{dx^k}\phi(x) = \frac{(i\pi)^{k+1}}{2}\sum_{j=0}^{k} S(k,j)e^{i\pi jx}g^{(j)}\left(e^{i\pi x}\right), \tag{1.93}$$

where $S(k,j)$ are the Stirling numbers of the second kind. Taking into account that

$$g^{(s)}(x) = 2\frac{(-1)^s s!}{(x-1)^{s+1}},\ s \geq 1, \tag{1.94}$$

we derive

$$\begin{aligned}\Phi_{k,m}(e^{i\pi x}) = (\pi i)^{k+1}\frac{(-1)^k}{k!}e^{\frac{i\pi}{2}(m-1)x}&\left[\sum_{j=1}^{k} S(k,j)(-1)^j j!\frac{e^{i\pi jx}}{(e^{i\pi x}-1)^{j+1}}\right.\\ &\left.+\frac{1}{2}\left(1+\frac{2}{e^{i\pi x}-1}\right)\right].\end{aligned} \tag{1.95}$$

In particular,

$$\Phi_{0,m}(e^{i\pi x}) = \frac{i\pi}{2}e^{\frac{i\pi}{2}(m-1)x}\left(1+\frac{2}{e^{i\pi x}-1}\right). \tag{1.96}$$

Similar arguments can be performed for even m. In particular, when m is even and $k=0$, we have

$$\Phi_{0,m}(e^{i\pi x}) = \frac{\pi}{2}e^{\frac{i\pi}{2}(m-1)x}\frac{1}{\sin\frac{\pi x}{2}}. \tag{1.97}$$

Lemma 1.6 *[60] Let $f^{(q+v+m)} \in AC[-1,1]$ for some $q, v \geq 0$, $m \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0,\ k = 0,\ldots,q-1. \tag{1.98}$$

Then, the following estimate holds for $|n| \leq N + c$ (c is a constant)

$$\begin{aligned}F_{n,m} - f_n^* = {}&\frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+v+1}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}\\ &\times\left(\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{\left(2r+\frac{2n}{2N+m+1}\right)^{k+1}} - e^{-\frac{i\pi(m-1)n}{2N+m+1}}\sum_{\tau=0}^{m-1}\frac{\Phi_{k,m}^{(\tau)}(-1)}{\tau!}\left(e^{\frac{2i\pi n}{2N+m+1}}+1\right)^{\tau}\right.\\ &\left.-e^{-\frac{i\pi(m-1)n}{2N+m+1}}\sum_{\tau=m}^{q-j+v+m}\frac{\Phi_{k,m}^{(\tau)}(-1)}{\tau!}\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=1}^{m}v_{\ell,s}^{-1}e^{\frac{2i\pi n(s-1)}{2N+m+1}}\right)\\ &+o\left(N^{-q-v-2}\right),\ N\to\infty.\end{aligned} \tag{1.99}$$

Proof. First, let us prove the property

$$F_{n,m} = \sum_{r=-\infty}^{\infty} f^*_{n+r(2N+m+1)} - \sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)}, \ n \in \mathbb{Z}. \tag{1.100}$$

Equations (1.38) and (1.39) imply

$$F_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) \left(e^{-\frac{2i\pi nk}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{-\frac{2i\pi(N+\ell)k}{2N+m+1}} \right). \tag{1.101}$$

Taking into account definition of f^* in (1.75) with $\alpha = q+m+v$, we write

$$f\left(\frac{k}{N}\right) = f^*\left(\frac{2k}{2N+m+1}\right) = \sum_{t=-\infty}^{\infty} f^*_t e^{\frac{2i\pi kt}{2N+m+1}}. \tag{1.102}$$

Hence

$$F_{n,m} = \frac{1}{2N+m+1} \sum_{k=-N}^{N} \sum_{t=-\infty}^{\infty} f^*_t \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right). \tag{1.103}$$

We get after some transformations

$$\begin{aligned} F_{n,m} = & \frac{1}{2N+m+1} \sum_{k=-N}^{N+m} \sum_{t=-N}^{N+m} \sum_{r=-\infty}^{\infty} f^*_{t+r(2N+m+1)} \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right) \\ & - \frac{1}{2N+m+1} \sum_{k=N+1}^{N+m} \sum_{t=-\infty}^{\infty} f^*_t \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right). \end{aligned} \tag{1.104}$$

Taking into account that

$$\frac{1}{2N+m+1} \sum_{k=-N}^{N+m} e^{\frac{2i\pi(t-n)k}{2N+m+1}} = \delta_{t,n}, \ -N \le t, n \le N+m, \tag{1.105}$$

we obtain

$$\begin{aligned} F_{n,m} = & \sum_{r=-\infty}^{\infty} f^*_{n+r(2N+m+1)} - \sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} \\ & - \frac{1}{2N+m+1} \sum_{k=N+1}^{N+m} \sum_{t=-\infty}^{\infty} f^*_t \left(e^{\frac{2i\pi(t-n)k}{2N+m+1}} - \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} \right). \end{aligned} \tag{1.106}$$

Let us show that the last term in the right-hand side of (1.106) vanishes. According to (1.40)

$$\begin{aligned} \sum_{k=N+1}^{N+m} \sum_{\ell=1}^{m} \theta_{n,\ell} e^{\frac{2i\pi(t-N-\ell)k}{2N+m+1}} &= \sum_{\ell=1}^{m} e^{\frac{2i\pi(\ell+N)(N+m)}{2N+m+1}} \sum_{s=1}^{m} v^{-1}_{\ell,s} e^{\frac{2i\pi n(s-N-m-1)}{2N+m+1}} \sum_{p=1}^{m} e^{\frac{2i\pi(t-N-\ell)(p+N)}{2N+m+1}} \\ &= \sum_{p=1}^{m} e^{\frac{2i\pi t(p+N)}{2N+m+1}} \sum_{s=1}^{m} e^{\frac{2i\pi n(s-N-m-1)}{2N+m+1}} \sum_{\ell=1}^{m} v^{-1}_{\ell,s} e^{\frac{2i\pi(\ell+N)(m-p)}{2N+m+1}} \\ &= \sum_{p=1}^{m} e^{\frac{2i\pi t(p+N)}{2N+m+1}} e^{-\frac{2i\pi n(p+N)}{2N+m+1}} = \sum_{p=1}^{m} e^{\frac{2i\pi(t-n)(p+N)}{2N+m+1}} = \sum_{k=N+1}^{N+m} e^{\frac{2i\pi(t-n)k}{2N+m+1}} \end{aligned} \tag{1.107}$$

which proves equation (1.100).

Equation (1.100) shows that

$$F_{n,m} - f_n^* = \sum_{r\neq 0} f^*_{n+r(2N+m+1)} - \sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)}. \tag{1.108}$$

Estimate (1.80) of Lemma 1.5 concludes

$$\begin{aligned}\sum_{r\neq 0} f^*_{n+r(2N+m+1)} &= \frac{(-1)^{n+1}}{2N+m+1} \sum_{j=q}^{q+v+m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \sum_{r\neq 0} \frac{(-1)^{r(m+1)}}{\left(2r+\frac{2n}{2N+m+1}\right)^{k+1}} + o\left(N^{-q-v-m-1}\right),\end{aligned} \tag{1.109}$$

$$\begin{aligned}\sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} &= \frac{(-1)^{N+\ell+1}}{2N+m+1} \sum_{j=q}^{q+v+m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{\left(2r+\frac{2(N+\ell)}{2N+m+1}\right)^{k+1}} + o\left(N^{-q-v-m-1}\right).\end{aligned} \tag{1.110}$$

Then, (1.49) leads to

$$\begin{aligned}\sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} &= \sum_{j=q}^{q+v+m} \frac{1}{(2N+m+1)N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \sum_{\ell=1}^{m} (-1)^{N+\ell+1}\theta_{n,\ell} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{\left(2r+\frac{2(N+\ell)}{2N+m+1}\right)^{k+1}} \\ &+ o\left(N^{-q-v-2}\right).\end{aligned} \tag{1.111}$$

Equations (1.40) and (1.87) imply

$$\begin{aligned}\sum_{\ell=1}^{m} \theta_{n,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)} &= \frac{(-1)^{n+1}}{2N+m+1} \sum_{j=q}^{q+v+m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times e^{-\frac{i\pi(m-1)n}{2N+m+1}} \sum_{\ell=1}^{m} \Phi_{k,m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}\right) \sum_{s=1}^{m} v^{-1}_{\ell,s} e^{\frac{2\imath\pi n(s-1)}{2N+m+1}} \\ &+ o\left(N^{-q-v-2}\right).\end{aligned} \tag{1.112}$$

The Taylor expansion

$$\Phi_{k,m}\left(e^{\frac{2\imath\pi(N+\ell)}{2N+m+1}}\right) = \sum_{\tau=0}^{v+m} \frac{1}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau} + O(N^{-v-m-1}) \tag{1.113}$$

and estimate (1.48) yield

$$\begin{aligned}\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty}f^*_{N+\ell+r(2N+m+1)} = {} & \frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+v+m}\frac{1}{N^j} \\ & \times\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}e^{-\frac{i\pi(m-1)n}{2N+m+1}} \\ & \times\sum_{\tau=0}^{v+m}\frac{1}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=1}^{m}v^{-1}_{\ell,s}e^{\frac{2\imath\pi n(s-1)}{2N+m+1}} \\ & + o\left(N^{-q-v-2}\right).\end{aligned} \tag{1.114}$$

Finally, the following relations

$$\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=1}^{m}v^{-1}_{\ell,s}e^{\frac{2\imath\pi n(s-1)}{2N+m+1}} = \left(e^{\frac{2i\pi n}{2N+m+1}}+1\right)^{\tau},\ \tau=0,\ldots,m-1, \tag{1.115}$$

imply

$$\begin{aligned}\sum_{\ell=1}^{m}\theta_{n,\ell}\sum_{r=-\infty}^{\infty}f^*_{N+\ell+r(2N+m+1)} &= \frac{(-1)^{n+1}}{2N+m+1}\sum_{j=q}^{q+v+m}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ \times e^{-\frac{i\pi(m-1)n}{2N+m+1}} & \left(\sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\left(e^{\frac{2i\pi n}{2N+m+1}}+1\right)^{\tau}\right. \\ & \left.+\sum_{\tau=m}^{v+m}\frac{1}{\tau!}\Phi^{(\tau)}_{k,m}(-1)\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau}\sum_{s=1}^{m}v^{-1}_{\ell,s}e^{\frac{2\imath\pi n(s-1)}{2N+m+1}}\right) \\ & + o\left(N^{-q-v-2}\right).\end{aligned} \tag{1.116}$$

Substitution of (1.109) and (1.116) into (1.108) completes the proof. □

Lemma 1.7 *[60] Let $f^{(q+2m)} \in AC[-1,1]$ for some $q \geq 0$, $m \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0,\ k=0,...,q-1. \tag{1.117}$$

Then, the following estimates hold as $N \to \infty$

$$F_{N-p,m} = C_{q,m}(f)\frac{(-1)^{N+p+1}}{N^{q+m+1}}\binom{m+p}{m} + O\left(N^{-q-m-2}\right),\ p \geq 0, \tag{1.118}$$

and

$$F_{-N+p,m} = -F_{N-p,m} + O\left(N^{-q-m-2}\right),\ p \geq 0, \tag{1.119}$$

where

$$C_{q,m}(f) = \sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k+1}i^k\pi^{k-m+1}(q-k)!}\Phi^{(m)}_{k,m}(-1). \tag{1.120}$$

Proof. Equation (1.100) claims that

$$F_{N-p,m} = \sum_{r=-\infty}^{\infty} f^*_{N-p+r(2N+m+1)} - \sum_{\ell=1}^{m} \theta_{N-p,\ell} \sum_{r=-\infty}^{\infty} f^*_{N+\ell+r(2N+m+1)}. \tag{1.121}$$

In view of Lemma 1.5, for $v = 2m$, and (1.49), we get

$$\begin{aligned} F_{N-p,m} = {} & \frac{(-1)^{N+1}}{2N+m+1} \sum_{j=q}^{q+2m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ & \times \left((-1)^p \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{\left(2r + \frac{2(N-p)}{2N+m+1}\right)^{k+1}} - \sum_{\ell=1}^{m} (-1)^\ell \theta_{N-p,\ell} \sum_{r=-\infty}^{\infty} \frac{(-1)^{r(m+1)}}{\left(2r + \frac{2(N+\ell)}{2N+m+1}\right)^{k+1}} \right) \\ & + o\left(N^{-q-m-2}\right). \end{aligned} \tag{1.122}$$

Equations (1.40) and (1.87) show that

$$\begin{aligned} F_{N-p,m} = {} & \frac{(-1)^{N+p+1}}{2N+m+1} \sum_{j=q}^{q+2m} \frac{e^{-\frac{i\pi(m-1)(N-p)}{2N+m+1}}}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ & \times \left(\Phi_{k,m}\left(e^{\frac{2i\pi(N-p)}{2N+m+1}}\right) - \sum_{\ell=1}^{m} \Phi_{k,m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}\right) \sum_{s=1}^{m} v^{-1}_{\ell,s} e^{\frac{2i\pi(N-p)(s-1)}{2N+m+1}} \right) \\ & + o\left(N^{-q-m-2}\right). \end{aligned} \tag{1.123}$$

Now, we simplify the expression in the brackets which we denote by S (see also (1.32))

$$\begin{aligned} S &= \Phi_{k,m}(\alpha_{-p}) - \sum_{\ell=1}^{m} \Phi_{k,m}(\alpha_\ell) \sum_{s=1}^{m} v^{-1}_{\ell,s} \alpha^{s-1}_{-p} \\ &= \sum_{j=1}^{m} \underset{z=\alpha_j}{res} \frac{\omega(\alpha_{-p})\Phi_{k,m}(z)}{\omega(z)(z-\alpha_{-p})} + \underset{z=\alpha_{-p}}{res} \frac{\omega(\alpha_{-p})\Phi_{k,m}(z)}{\omega(z)(z-\alpha_{-p})}, \end{aligned} \tag{1.124}$$

where $\omega(z) = \prod_{\ell=1}^{m}(z-\alpha_\ell)$. Hence

$$S = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\alpha_{-p})\Phi_{k,m}(z)}{\omega(z)(z-\alpha_{-p})} dz, \tag{1.125}$$

where Γ contains the points $\{\alpha_\ell\}_{\ell=1}^m$ and α_{-p}. Then,

$$\begin{aligned} S &= \frac{(i\pi)^m}{N^m 2\pi i} \frac{(m+p)!}{p!} \int_\Gamma \frac{\Phi_{k,m}(z)}{(z+1)^{m+1}} dz + O\left(N^{-m-1}\right) \\ &= \frac{(i\pi)^m \Phi^{(m)}_{k,m}(-1)}{N^m} \binom{m+p}{m} + O\left(N^{-m-1}\right), \ N \to \infty. \end{aligned} \tag{1.126}$$

Substituting this into (1.123), we get the first statement. The second one can be proved similarly. □

We put

$$R_{N,m}(f,x) = f(x) - I_{N,m}(f,x). \tag{1.127}$$

Next theorem presents the exact rate of pointwise convergence of the QP interpolation for $m \geq 1$.

Theorem 1.6 *[60] Let $f^{(q+2m)} \in AC[-1,1]$ for some $q \geq 0$, $m \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \dots, q-1. \tag{1.128}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$R_{N,m}(f,x) = \frac{D_{N,m}(f,x)}{N^{q+m+1}} + o(N^{-q-m-1}), \tag{1.129}$$

where

$$\begin{aligned} D_{N,m}(f,x) = i(-1)^N C_{q,m}(f) \Bigg[&\sin(\pi(N+1)\sigma x) \sum_{k=0}^{\acute{m}} \binom{m-k}{k} \frac{(-1)^k}{2^{2k+1} \cos^{2k+2} \frac{\pi x}{2}} \\ &- \sin(\pi N \sigma x) \sum_{k=0}^{\acute{m}-1} \binom{m-k-2}{k} \frac{(-1)^k}{2^{2k+3} \cos^{2k+4} \frac{\pi x}{2}} \Bigg], \end{aligned} \tag{1.130}$$

$\acute{m} = \left[\frac{m}{2}\right]$ and $C_{q,m}(f)$ is defined by (1.120).

Proof. According to definition of f^* (see (1.75) for $\alpha = q + 2m$), we have that $f^{*(q+2m)} \in AC[-1,1]$, and consequently

$$f^*(x) = \sum_{n=-\infty}^{\infty} f_n^* e^{i\pi n x}, \ \ x \in (-1,1). \tag{1.131}$$

Hence,

$$f(x) = \sum_{n=-\infty}^{\infty} f_n^* e^{i\pi n \sigma x}, \ \ x \in [-1,1] \tag{1.132}$$

and

$$R_{N,m}(f,x) = \sum_{n=-N}^{N} (f_n^* - F_{n,m}) e^{i\pi n \sigma x} + \sum_{|n|>N} f_n^* e^{i\pi n \sigma x}. \tag{1.133}$$

The following transformation can be easily verified (see derivation of similar transformations in [24, 55, 74])

$$
\begin{aligned}
R_{N,m}(f,x) = & \frac{F_{N+1,m}e^{i\pi N\sigma x} - F_{N,m}e^{i\pi(N+1)\sigma x}}{\left(1+e^{-i\pi\sigma x}\right)\left(1+e^{i\pi\sigma x}\right)} \\
& + \frac{F_{-N-1,m}e^{-i\pi N\sigma x} - F_{-N,m}e^{-i\pi(N+1)\sigma x}}{\left(1+e^{-i\pi\sigma x}\right)\left(1+e^{i\pi\sigma x}\right)} \\
& + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)\left(1+e^{i\pi\sigma x}\right)} \sum_{n=-N}^{N} \delta_n^1 \left(\{f_s^* - F_{s,m}\}\right) e^{i\pi\sigma n x} \\
& + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)\left(1+e^{i\pi\sigma x}\right)} \sum_{|n|>N} \delta_n^1 \left(\{f_s^*\}\right) e^{i\pi\sigma n x}.
\end{aligned} \tag{1.134}
$$

Reiteration of this transformation up to $\acute{m}+1$ times leads to the following expansion of the error

$$
\begin{aligned}
R_{N,m}(f,x) = & e^{i\pi N\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{N+1}^k \left(\{F_{s,m}\}\right)}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} \\
& - e^{i\pi(N+1)\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{N}^k \left(\{F_{s,m}\}\right)}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} \\
& + e^{-i\pi N\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{-N-1}^k \left(\{F_{s,m}\}\right)}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} \\
& - e^{-i\pi(N+1)\sigma x} \sum_{k=0}^{\acute{m}} \frac{\delta_{-N}^k \left(\{F_{s,m}\}\right)}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} + r_{N,m}(f,x),
\end{aligned} \tag{1.135}
$$

where

$$
\begin{aligned}
r_{N,m}(f,x) = & \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+1}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+1}} \sum_{n=-N}^{N} \delta_n^{\acute{m}+1} \left(\{f_s^* - F_{s,m}\}\right) e^{i\pi\sigma n x} \\
& + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+1}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+1}} \sum_{|n|>N} \delta_n^{\acute{m}+1} \left(\{f_s^*\}\right) e^{i\pi\sigma n x}.
\end{aligned} \tag{1.136}
$$

First, we show that

$$
r_{N,m}(f,x) = o(N^{-q-m-1}), \ N \to \infty, \ |x| < 1. \tag{1.137}
$$

Application of similar transformation leads to the following expansion for $r_{N,m}(f,x)$

$$
\begin{aligned}
r_{N,m}(f,x) = & \frac{\delta_{-N-1}^{\acute{m}+1} \left(\{F_{s,m}\}\right) e^{-i\pi N\sigma x} - \delta_{N}^{\acute{m}+1} \left(\{F_{s,m}\}\right) e^{i\pi(N+1)\sigma x}}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \\
& + \frac{\delta_{N+1}^{\acute{m}+1} \left(\{F_{s,m}\}\right) e^{i\pi N\sigma x} - \delta_{-N}^{\acute{m}+1} \left(\{F_{s,m}\}\right) e^{-i\pi(N+1)\sigma x}}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \\
& + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \sum_{n=-N}^{N} \delta_n^{\acute{m}+2} \left(\{f_s^* - F_{s,m}\}\right) e^{i\pi\sigma n x} \\
& + \frac{1}{\left(1+e^{-i\pi\sigma x}\right)^{\acute{m}+2}\left(1+e^{i\pi\sigma x}\right)^{\acute{m}+2}} \sum_{|n|>N} \delta_n^{\acute{m}+2} \left(\{f_s^*\}\right) e^{i\pi\sigma n x}.
\end{aligned} \tag{1.138}
$$

First, we estimate $\delta_n^{\acute{m}+2}(\{f_s^*\})$ as $|n| > N$, $N \to \infty$. Taking into account estimate (1.80) of Lemma 1.5, for $v = 2m$, we write

$$\delta_n^{\acute{m}+2}(\{f_s^*\}) = \sum_{j=q}^{q+2m} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}(2N+m+1)^k}{(j-k)!} \delta_n^{\acute{m}+2}(\{B_s(k)\}) + o(n^{-q-2m-1}). \tag{1.139}$$

According to Lemma 1.4, with $p = \acute{m}+2$, we get

$$\delta_n^{\acute{m}+2}(\{B_s(k)\}) = O(n^{-2\acute{m}-k-5}) \tag{1.140}$$

and hence,

$$\delta_n^{\acute{m}+2}(\{f_s^*\}) = O(N^{-q} n^{-2\acute{m}-5}) + o(n^{-q-2m-1}),\ |n| > N,\ N \to \infty. \tag{1.141}$$

Therefore, the last term in the right-hand side of (1.138) is $o(N^{-q-m-1})$.

Second, we estimate $\delta_n^{\acute{m}+2}(\{f_s^* - F_{s,m}\})$ for $|n| \le N$ as $N \to \infty$. According to estimate (1.99) of Lemma 1.6, for $v = m$, we derive

$$\begin{aligned}\delta_n^{\acute{m}+2}(\{F_{s,m} - f_s^*\}) &= \frac{1}{2N+m+1} \sum_{j=q}^{q+m+1} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!N^j} \\ &\times \left(\frac{(2N+m+1)^{k+1}(i\pi)^{k+1}}{2^k} \delta_n^{\acute{m}+2}\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(k)\right\}\right)\right. \\ &- \sum_{\tau=0}^{m-1} \frac{\Phi_{k,m}^{(\tau)}(-1)}{\tau!} \delta_n^{\acute{m}+2}\left(\left\{(-1)^{s+1}\left(e^{\frac{2i\pi s}{2N+m+1}}+1\right)^{\tau} e^{-\frac{i\pi(m-1)s}{2N+m+1}}\right\}\right) - \\ &\left.\sum_{\tau=m}^{q-j+2m} \frac{1}{\tau!}\Phi_{k,m}^{(\tau)}(-1) \sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^{\tau} \sum_{p=1}^{m} v_{\ell,p}^{-1} \delta_n^{\acute{m}+2}\left(\left\{(-1)^{s+1} e^{\frac{\imath\pi s(2p-m-1)}{2N+m+1}}\right\}\right)\right) \\ &+ o(N^{-q-m-2}).\end{aligned} \tag{1.142}$$

Lemmas 1.3 and 1.1 yield, when $p = \acute{m}+2$, the following estimates

$$\delta_n^{\acute{m}+2}\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(k)\right\}\right) = O\left(N^{-2\acute{m}-k-5}\right), \tag{1.143}$$

$$\delta_n^{\acute{m}+2}\left(\left\{(-1)^{s+1} e^{\frac{i\pi\beta s}{2N+m+1}}\right\}\right) = O\left(N^{-2\acute{m}-4}\right) \tag{1.144}$$

and taking into account (1.48), we get

$$\delta_n^{\acute{m}+2}(\{F_{s,m} - f_s^*\}) = o\left(N^{-q-m-2}\right),\ |n| \le N,\ N \to \infty. \tag{1.145}$$

Hence, the third term in the right-hand side of (1.138) is $o(N^{-q-m-1})$ as $N \to \infty$.

Now, we estimate the first two terms in the right-hand side of (1.138). We have

$$\delta_N^{\acute{m}+1}\left(\{F_{s,m}\}\right) = \sum_{k=0}^{2\acute{m}+2} \binom{2\acute{m}+2}{k} F_{N+\acute{m}+1-k,m}. \tag{1.146}$$

Taking into account (1.51)

$$\delta_N^{\acute{m}+1}\left(\{F_{s,m}\}\right) = \sum_{k=\acute{m}+1}^{2\acute{m}+2} \binom{2\acute{m}+2}{k} F_{N+\acute{m}+1-k,m}. \tag{1.147}$$

In view of Lemma 1.7

$$\begin{aligned}\delta_N^{\acute{m}+1}\left(\{F_{s,m}\}\right) = C_{q,m}(f)\frac{(-1)^{N+\acute{m}}}{N^{q+m+1}} \sum_{k=\acute{m}+1}^{2\acute{m}+2} (-1)^k \binom{2\acute{m}+2}{k}\binom{m+k-\acute{m}-1}{m} \\ + O\left(N^{-q-m-2}\right).\end{aligned} \tag{1.148}$$

Taking into account the identity ([73, 75])

$$\sum_{k=\acute{m}+1}^{2\acute{m}+2} (-1)^k \binom{2\acute{m}+2}{k}\binom{m+k-\acute{m}-1}{m} = 0, \tag{1.149}$$

we conclude that

$$\delta_N^{\acute{m}+1}\left(\{F_{s,m}\}\right) = O\left(N^{-q-m-2}\right). \tag{1.150}$$

Similarly, we estimate the other terms and see that (1.137) is true.

Now, we return to the first four terms in the right hand-side of (1.135) which we denote by I_1, I_2, I_3 and I_4, respectively.

In view of relations (1.51) and Lemma 1.7, for the first term in the right-hand side of (1.135), we have

$$\begin{aligned}\delta_{N+1}^{k}\left(\{F_{s,m}\}\right) &= \sum_{s=0}^{2k}\binom{2k}{s}F_{N+1+k-s,m} = \sum_{s=k+1}^{2k}\binom{2k}{s}F_{N+1+k-s,m} \\ &= \sum_{s=0}^{k-1}\binom{2k}{s+k+1}F_{N-s,m}.\end{aligned} \tag{1.151}$$

Then,

$$\begin{aligned}I_1 &= e^{i\pi N\sigma x}\sum_{k=0}^{\acute{m}} \frac{\delta_{N+1}^{k}\left(\{F_{s,m}\}\right)}{\left(1+e^{-i\pi\sigma x}\right)^{k+1}\left(1+e^{i\pi\sigma x}\right)^{k+1}} \\ &= e^{i\pi N\sigma x}\sum_{k=0}^{\acute{m}} \frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}} \sum_{s=0}^{k-1}\binom{2k}{s+k+1}F_{N-s,m}.\end{aligned} \tag{1.152}$$

Lemma 1.7 implies

$$I_1 = C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{i\pi N\sigma x}\sum_{k=1}^{\acute{m}}\frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}}\sum_{s=0}^{k-1}(-1)^s\binom{2k}{s+k+1}\binom{m+s}{m} + O(N^{-q-m-2}). \tag{1.153}$$

Application of the following identity

$$\sum_{s=0}^{k-1}(-1)^s\binom{2k}{s+k+1}\binom{m+s}{m} = (-1)^{k+1}\binom{m-k-1}{k-1} \tag{1.154}$$

deduces

$$I_1 = C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{i\pi N\sigma x}\sum_{k=0}^{\acute{m}-1}\binom{m-k-2}{k}\frac{(-1)^k}{2^{2k+4}\cos^{2k+4}\frac{\pi\sigma x}{2}} + O(N^{-q-m-2}). \tag{1.155}$$

Similarly, for the third term in the right-hand side of (1.135), we have

$$\begin{aligned}\delta_{-N-1}^k(\{F_{s,m}\}) &= \sum_{s=0}^{2k}\binom{2k}{s}F_{-N-1+k-s,m}\\ &= \sum_{s=0}^{k-1}\binom{2k}{s}F_{-N-1+k-s,m}\\ &= \sum_{s=0}^{k-1}\binom{2k}{k-1-s}F_{-N+s,m}\\ &= -\sum_{s=0}^{k-1}\binom{2k}{k+s+1}F_{N-s,m} + O(N^{-q-m-2}).\end{aligned} \tag{1.156}$$

Then,

$$\begin{aligned}I_3 &= e^{-i\pi N\sigma x}\sum_{k=0}^{\acute{m}}\frac{\delta_{-N-1}^k(\{F_{s,m}\})}{(1+e^{-i\pi\sigma x})^{k+1}(1+e^{i\pi\sigma x})^{k+1}}\\ &= -e^{-i\pi N\sigma x}\sum_{k=0}^{\acute{m}}\frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}}\sum_{s=0}^{k-1}\binom{2k}{k+s+1}F_{N-s}\\ &\quad + O(N^{-q-m-2})\\ &= -C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{-i\pi N\sigma x}\sum_{k=0}^{\acute{m}-1}\binom{m-k-2}{k}\frac{(-1)^k}{2^{2k+4}\cos^{2k+4}\frac{\pi\sigma x}{2}}\\ &\quad + O(N^{-q-m-2}).\end{aligned} \tag{1.157}$$

Now, we can write

$$I_1 + I_3 = iC_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}\sin(\pi N\sigma x)\sum_{k=0}^{\acute{m}-1}\binom{m-k-2}{k}\frac{(-1)^k}{2^{2k+3}\cos^{2k+4}\frac{\pi\sigma x}{2}} + O(N^{-q-m-2}). \tag{1.158}$$

For the second term in the right-hand side of (1.135), we figure out that

$$\delta_N^k(\{F_{s,m}\}) = \sum_{s=0}^{2k}\binom{2k}{s}F_{N+k-s,m} = \sum_{s=k}^{2k}\binom{2k}{s}F_{N+k-s,m} \\ = \sum_{s=0}^{k}\binom{2k}{k-s}F_{N-s,m}. \tag{1.159}$$

Then,

$$I_2 = -e^{i\pi(N+1)\sigma x}\sum_{k=0}^{\acute{m}}\frac{\delta_N^k(\{F_{s,m}\})}{(1+e^{-i\pi\sigma x})^{k+1}(1+e^{i\pi\sigma x})^{k+1}} \\ = -e^{i\pi(N+1)\sigma x}\sum_{k=0}^{\acute{m}}\frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}}\sum_{s=0}^{k}\binom{2k}{k-s}F_{N-s,m} \\ = -C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{i\pi(N+1)\sigma x}\sum_{k=0}^{\acute{m}}\frac{1}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}}\sum_{s=0}^{k}(-1)^s\binom{2k}{k-s}\binom{m+s}{m} \\ + O(N^{-q-m-2}). \tag{1.160}$$

Application of the identity

$$\sum_{s=0}^{k}(-1)^s\binom{2k}{k-s}\binom{m+s}{m} = (-1)^k\binom{m-k}{k} \tag{1.161}$$

implies

$$I_2 = -C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{i\pi(N+1)\sigma x}\sum_{k=0}^{\acute{m}}\binom{m-k}{k}\frac{(-1)^k}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}} \\ + O(N^{-q-m-2}). \tag{1.162}$$

Similarly

$$I_4 = C_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}e^{-i\pi(N+1)\sigma x}\sum_{k=0}^{\acute{m}}\binom{m-k}{k}\frac{(-1)^k}{2^{2k+2}\cos^{2k+2}\frac{\pi\sigma x}{2}} \\ + O(N^{-q-m-2}), \tag{1.163}$$

and, consequently,

$$I_2 + I_4 = -iC_{q,m}(f)\frac{(-1)^{N+1}}{N^{q+m+1}}\sin(\pi(N+1)\sigma x)\sum_{k=0}^{\acute{m}}\binom{m-k}{k}\frac{(-1)^k}{2^{2k+1}\cos^{2k+2}\frac{\pi\sigma x}{2}} \\ + O(N^{-q-m-2}), \tag{1.164}$$

which completes the proof. □

Remark 1.4 *Note that* $R_{N,m}(f,\pm 1) = 0$ *as* $I_{N,m}(f,x)$ *interpolates* f *on grid* k/N, $|k| \le N$ *which includes the endpoints of interval* $[-1,1]$.

Let us compare Theorem 1.6 with its classical counterparts Theorems 1.1 and 1.2.

The first important difference is smoothness requirements on function f. In Theorem 1.6 the condition is $f^{(q+2m)} \in AC[-1,1]$, $q \geq 0$, $m \geq 1$ and as bigger is the value of m as smoother must be the function. Theorems 1.1 and 1.2 require $f^{(q+1)} \in AC[-1,1]$ and $f^{(q+2)} \in AC[-1,1]$ for even and odd q, respectively. Therefore, when $m = 1$ and q is odd, the requirements are similar, but in all other cases, Theorem 1.6 puts stricter smoothness condition on f.

The second difference is requirement on the values of f and its derivatives at the endpoints of the interval. In Theorems 1.1 and 1.2 the condition is $f^{(k)}(1)-f^{(k)}(-1) = 0$, $k = 0, \ldots, q-1$ while in Theorem 1.6 the condition is stricter $f^{(k)}(1) = f^{(k)}(-1) = 0$, $k = 0, \ldots, q-1$.

The third difference is extra smoothness requirement in Theorem 1.6 compared with the convergence rate. In the classical theorems the condition $f^{(q+\alpha)} \in AC[-1,1]$, $\alpha = 1,2$ provides the convergence rate $O(N^{-q-\alpha})$, respectively. In case of the QP interpolation the condition $f^{(q+2m)} \in AC[-1,1]$ provides the rate of convergence only $O(N^{-q-m-1})$. So, we have requirement for extra $m-1$ derivatives which is due to estimate (1.49).

The fourth difference is that in case of the classical interpolation the exact constant of the main term of asymptotic error depends on $A_q(f) = f^{(q)}(1) - f^{(q)}(-1)$ while in case of the QP interpolation, we have factor $A_{kq}(f) = f^{(q)}(1) - (-1)^{k+q} f^{(q)}(-1)$.

However, when f is such that all requirements of Theorems 1.1, 1.2 and 1.6 are valid then the QP interpolation has better accuracy than the classical trigonometric interpolation. Theorems 1.1 and 1.2 state convergence rate $O(N^{-q-1})$ for even q and $O(N^{-q-2})$ for odd q, respectively. The QP interpolation has convergence rate $O(N^{-q-m-1})$, $m \geq 1$. Hence, except the case $m = 1$ and odd q when both interpolations have similar rate of convergence, in all other cases, the pointwise convergence rate of the QP interpolation exceeds the one of the classical interpolation by factor $O(N^{-m})$ for even q and $O(N^{-m+1})$ for odd q.

In particular, when f is infinitely differentiable and $f(1) \neq f(-1)$ $(q = 0)$ then the classical interpolation has convergence rate $O(N^{-1})$ while the QP interpolation converges by rate $O(N^{-m-1})$, $m \geq 1$. We see accelerated convergence by factor $O(N^{-m})$ in this important case.

1.3.3 Analysis of the case $m = 0$

We need estimate for $F_{n,0} - f_n^*$.

Lemma 1.8 *Let $f^{(q+v)} \in AC[-1,1]$ for some $q, v \geq 0$, $q+v \neq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \; k = 0, \dots, q-1. \tag{1.165}$$

Then, the following estimate holds for $|n| \leq N + c$ (c is a constant) as $N \to \infty$

$$F_{n,0} - f_n^* = \sum_{j=q}^{q+v} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(2N+1)^k}{(j-k)!} \sum_{r \neq 0} B_{n+r(2N+1)}(k) + o\left(N^{-q-v-1}\right). \tag{1.166}$$

Proof. Taking into account the smoothness of f and definition of f^* with $\alpha = q + v$ (see (1.75)) similar to (1.100), we write

$$F_{n,0} - f_n^* = \sum_{s \neq 0} f^*_{n+s(2N+1)}. \tag{1.167}$$

We apply Lemma 1.5 to (1.167) and get

$$\begin{aligned} F_{n,0} - f_n^* = & \frac{(-1)^{n+1}}{2N+1} \sum_{j=q}^{q+v} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)}{2^{j-k}(j-k)!(i\pi)^{k+1}} \sum_{s \neq 0} \frac{(-1)^s}{\left(2s + \frac{2n}{2N+1}\right)^{k+1}} \\ & + o(N^{-q-v-1}) \end{aligned} \tag{1.168}$$

which completes the proof. □

Theorem 1.7 *[60] Let $f^{(q+1)} \in AC[-1,1]$ for some $q \geq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \;\; k = 0, \dots, q-1. \tag{1.169}$$

Then, the following estimate holds for $|x| < 1$ as $N \to \infty$

$$R_{N,0}(f, x) = \frac{D_{N,0}(f, x)}{N^{q+1}} + o(N^{-q-1}), \tag{1.170}$$

where

$$D_{N,0}(f, x) = \frac{(-1)^N}{2^{q+1}} \frac{\sin \pi N x}{\cos \frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(f) 2^k}{(q-k)! i^k \pi^{k+1}} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r+1)^{k+1}}. \tag{1.171}$$

Proof. Similar to (1.133), we have

$$R_{N,0}(f,x) = \sum_{n=-N}^{N} (f_n^* - F_{n,0}) e^{i\pi n\sigma x} + \sum_{|n|>N} f_n^* e^{i\pi n \sigma x}. \tag{1.172}$$

We proceed as in the proof of Theorem 1.6 and derive

$$\begin{aligned} R_{N,0}(f,x) = {} & F_{N,0} \frac{e^{-i\pi N\sigma x} - e^{i\pi(N+1)\sigma x}}{(1+e^{-i\pi\sigma x})(1+e^{i\pi\sigma x})} + F_{-N,0} \frac{e^{i\pi N\sigma x} - e^{-i\pi(N+1)\sigma x}}{(1+e^{-i\pi\sigma x})(1+e^{i\pi\sigma x})} \\ & + \frac{1}{(1+e^{-i\pi\sigma x})(1+e^{i\pi\sigma x})} \sum_{n\le N} \delta_n^1 (\{f_s^* - F_{s,0}\}) e^{i\pi n\sigma x} \\ & + \frac{1}{(1+e^{-i\pi\sigma x})(1+e^{i\pi\sigma x})} \sum_{|n|>N} \delta_n^1 (\{f_s^*\}) e^{i\pi n\sigma x}. \end{aligned} \tag{1.173}$$

Lemma 1.5, with $v=1$, implies

$$\begin{aligned} \delta_n^1(\{f_s^*\}) = {} & \frac{1}{2^q N^q} \sum_{k=0}^{q} \frac{A_{kq}(f)(2N+1)^k}{(q-k)!} \delta_n^1(\{B_s(k)\}) \\ & + \frac{1}{2^{q+1}N^{q+1}} \sum_{k=0}^{q+1} \frac{A_{k(q+1)}(f)(2N+1)^k}{(q+1-k)!} \delta_n^1(\{B_s(k)\}) + o(n^{-q-2}). \end{aligned} \tag{1.174}$$

In view of Lemma 1.4

$$\delta_n^1(\{B_s(k)\}) = O(n^{-k-3}) \tag{1.175}$$

and, consequently,

$$\delta_n^1(\{f_s^*\}) = o(n^{-q-2}) + O(N^{-q}n^{-3}),\ |n|>N,\ N\to\infty. \tag{1.176}$$

Hence, the last term in the right-hand side of (1.173) is $o(N^{-q-1})$ as $N\to\infty$.

Lemma 1.8, with $v=1$, leads to the following relation

$$\begin{aligned} \delta_n^1(\{F_{s,0} - f_s^*\}) = {} & \sum_{k=0}^{q} \frac{A_{kq}(f)(2N+1)^k}{2^q(q-k)!N^q} \delta_n^1 \left(\left\{ \sum_{r\ne 0} B_{s+r(2N+1)}(k) \right\} \right) \\ & + \sum_{k=0}^{q+1} \frac{A_{k(q+1)}(f)(2N+1)^k}{2^{q+1}(q+1-k)!N^{q+1}} \delta_n^1 \left(\left\{ \sum_{r\ne 0} B_{s+r(2N+1)}(k) \right\} \right) + o(N^{-q-2}). \end{aligned} \tag{1.177}$$

In view of Lemma 1.3

$$\delta_n^1 \left(\left\{ \sum_{r\ne 0} B_{s+r(2N+1)}(k) \right\} \right) = O(N^{-k-3}). \tag{1.178}$$

Then,

$$\delta_n^1(\{F_{s,0} - f_s^*\}) = o(N^{-q-2}),\ |n|\le N,\ N\to\infty \tag{1.179}$$

and therefore, the fourth term in the right-hand side of (1.173) is also $o(N^{-q-1})$ as $N \to \infty$.

Now, we have

$$R_{N,0}(f,x) = F_{N,0}\frac{e^{-i\pi N\sigma x} - e^{i\pi(N+1)\sigma x}}{(1+e^{-i\pi\sigma x})(1+e^{i\pi\sigma x})} + F_{-N,0}\frac{e^{i\pi N\sigma x} - e^{-i\pi(N+1)\sigma x}}{(1+e^{-i\pi\sigma x})(1+e^{i\pi\sigma x})} + o(N^{-q-1}). \quad (1.180)$$

Using relation (see (1.167))

$$F_{\pm N,0} = \sum_{r=-\infty}^{\infty} f^{*}_{\pm N + r(2N+1)}, \quad (1.181)$$

together with estimate of Lemma 1.5, we get

$$F_{\pm N,0} = \sum_{k=0}^{q} \frac{A_{kq}(f)}{(q-k)!2^{q-k}N^{q-k}} \sum_{r=-\infty}^{\infty} B_{\pm N + r(2N+1)}(k) + O(N^{-q-2}). \quad (1.182)$$

Lemma 1.2, for $p = 0$, concludes

$$F_{\pm N,0} = \frac{(-1)^{N+1}}{2^{q+1}N^{q+1}} \sum_{k=0}^{q} \frac{2^k A_{kq}(f)}{(q-k)!(i\pi)^{k+1}} \sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r \pm 1)^{k+1}} + O(N^{-q-2}) \quad (1.183)$$

which completes the proof together with (1.180). □

Comparison with Theorems 1.1 and 1.2 shows the same smoothness requirement and rate of convergence for even q. For odd q, the rate of convergence of the classical interpolation is higher by factor $O(N^{-1})$ if function has extra smoothness $f^{(q+2)} \in AC[-1,1]$.

1.4 L_2-Convergence

In this section, we study L_2-convergence of the QP interpolation and derive exact constants of the asymptotic errors in this framework ([56, 57]).

1.4.1 Asymptotic Estimates

We reformulate Lemmas 1.5 and 1.6 for our convenience.

Lemma 1.9 *[56, 57] Let $f^{(q+m)} \in AC[-1,1]$ for some $q, m \geq 0$, $q + m \neq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \; k = 0, \ldots, q-1. \quad (1.184)$$

Then, the following estimates hold as $N \to \infty$

$$f^{*}_{n} = \frac{(-1)^{n+1}}{(2N+m+1)N^q}\mu_{q,m}\left(f, \frac{2n}{2N+m+1}\right) + o(n^{-q-1}), \; n \to \infty, \quad (1.185)$$

and

$$F_{n,m} - f_n^* = \frac{(-1)^{n+1}}{(2N+m+1)N^q}\nu_{q,m}\left(f, \frac{2n}{2N+m+1}\right) + o\left(N^{-q-1}\right), \ |n| \le N, \tag{1.186}$$

where

$$\mu_{q,m}(f,x) = \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(q-k)!(i\pi x)^{k+1}}, \tag{1.187}$$

and

$$\begin{aligned}\nu_{q,m}(f,x) = &\sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k}}{2^{q-k}(i\pi)^{k+1}(q-k)!}\\ &\times\left(\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{(2r+x)^{k+1}} - e^{-i\pi\frac{m-1}{2}x}\sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi_{k,m}^{(\tau)}(-1)\left(e^{i\pi x}+1\right)^{\tau}\right).\end{aligned} \tag{1.188}$$

Remark 1.5 *When $m = 0$, the second term in the right-hand side of (1.188) vanishes and*

$$\nu_{q,0}(f,x) = \sum_{k=0}^{q} \frac{A_{kq}(f)}{2^{q-k}(i\pi)^{k+1}(q-k)!}\sum_{r\neq 0}\frac{(-1)^r}{(2r+x)^{k+1}}. \tag{1.189}$$

Theorem 1.8 *[56, 57] Let $f^{(q+m)} \in AC[-1,1]$ for some $q, m \ge 0$, $q+m \neq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, ..., q-1. \tag{1.190}$$

Then, the following estimate holds

$$\lim_{N\to\infty} N^{q+\frac{1}{2}}||R_{N,m}(f,x)||_{L_2(-1,1)} = c_{q,m}(f), \tag{1.191}$$

where

$$\begin{aligned}c_{q,m}^2(f) = &-\frac{m+1}{8}\int_{-1}^{1}\left|\int_{-1}^{1}\nu_{q,m}(f,h)e^{i\pi(m+1)xh/2}dh - \int_{|h|>1}\mu_{q,m}(f,h)e^{i\pi(m+1)xh/2}dh\right|^2 dx\\ &+\frac{1}{2}\int_{-1}^{1}\left|\nu_{q,m}(f,x)\right|^2 dx + \frac{1}{2}\int_{|x|>1}\left|\mu_{q,m}(f,x)\right|^2 dx.\end{aligned} \tag{1.192}$$

Proof. We divide $||R_{N,m}(f,x)||^2_{L_2(-1,1)}$ into three parts as follows

$$\begin{aligned}||R_{N,m}(f,x)||^2_{L_2(-1,1)} &= \int_{-1}^{1}\left|R_{N,m}(f,x)\right|^2 dx\\ &= \frac{2N+m+1}{2N}\int_{-\frac{2N}{2N+m+1}}^{\frac{2N}{2N+m+1}}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^2 dx\\ &= \frac{2N+m+1}{2N}\int_{-1}^{1}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^2 dx\\ &- \frac{2N+m+1}{2N}\int_{\frac{2N}{2N+m+1}}^{1}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^2 dx\\ &- \frac{2N+m+1}{2N}\int_{-1}^{-\frac{2N}{2N+m+1}}\left|R_{N,m}\left(f,\frac{2N+m+1}{2N}x\right)\right|^2 dx\\ &= I_1 - I_2 - I_3,\end{aligned} \tag{1.193}$$

where

$$I_1 = \frac{2N+m+1}{2N} \int_{-1}^{1} \left| R_{N,m} \left(f, \frac{2N+m+1}{2N} x \right) \right|^2 dx, \tag{1.194}$$

$$I_2 = \frac{m+1}{2N} \int_{0}^{1} \left| R_{N,m} \left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N} x \right) \right|^2 dx, \tag{1.195}$$

and

$$I_3 = \frac{m+1}{2N} \int_{0}^{1} \left| R_{N,m} \left(f, \frac{m+1}{2N} x - \frac{2N+m+1}{2N} \right) \right|^2 dx. \tag{1.196}$$

First is estimation of I_1. According to definition of f^* (see (1.75) with $\alpha = q+m$), we write

$$f^*(x) = \sum_{n=-\infty}^{\infty} f_n^* e^{i\pi n x}, \ x \in (-1,1) \tag{1.197}$$

and

$$f(x) = \sum_{n=-\infty}^{\infty} f_n^* e^{i\pi n \frac{2N}{2N+m+1} x}, \ x \in [-1,1]. \tag{1.198}$$

From here

$$R_{N,m}(f,x) = \sum_{n=-N}^{N} \left(f_n^* - F_{n,m} \right) e^{i\pi n \frac{2N}{2N+m+1} x} + \sum_{|n|>N} f_n^* e^{i\pi n \frac{2N}{2N+m+1} x}. \tag{1.199}$$

Therefore

$$I_1 = \frac{2N+m+1}{N} \sum_{n=-N}^{N} \left| f_n^* - F_{n,m} \right|^2 + \frac{2N+m+1}{N} \sum_{|n|>N} \left| f_n^* \right|^2. \tag{1.200}$$

In view of Lemma 1.9, we obtain

$$\begin{aligned} I_1 &= \frac{1}{(2N+m+1)N^{2q+1}} \sum_{n=-N}^{N} \left| \nu_{q,m} \left(f, \frac{2n}{2N+m+1} \right) \right|^2 \\ &+ \frac{1}{(2N+m+1)N^{2q+1}} \sum_{|n|>N} \left| \mu_{q,m} \left(f, \frac{2n}{2N+m+1} \right) \right|^2 \\ &+ o\left(N^{-2q-2} \right), \ N \to \infty. \end{aligned} \tag{1.201}$$

Tending N to infinity and replacing the sums by the corresponding integrals, we get

$$\lim_{N\to\infty} N^{2q+1} I_1 = \frac{1}{2} \int_{-1}^{1} \left| \nu_{q,m}(f,x) \right|^2 dx + \frac{1}{2} \int_{|x|>1} \left| \mu_{q,m}(f,x) \right|^2 dx. \tag{1.202}$$

Second is estimation of I_2. From (1.199)

$$\begin{aligned} R_{N,m} \left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N} x \right) &= \sum_{n=-N}^{N} (-1)^n \left(f_n^* - F_{n,m} \right) e^{-i\pi x \frac{(m+1)n}{2N+m+1}} \\ &+ \sum_{|n|>N} (-1)^n f_n^* e^{-i\pi x \frac{(m+1)n}{2N+m+1}}. \end{aligned} \tag{1.203}$$

According to Lemma 1.9

$$R_{N,m}\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x\right) = \\ \frac{1}{(2N+m+1)N^q}\sum_{n=-N}^{N}\nu_{q,m}\left(f, \frac{2n}{2N+m+1}\right)e^{-i\pi x\frac{(m+1)n}{2N+m+1}} \\ - \frac{1}{(2N+m+1)N^q}\sum_{|n|>N}\mu_{q,m}\left(f, \frac{2n}{2N+m+1}\right)e^{-i\pi x\frac{(m+1)n}{2N+m+1}} \\ + O(N^{-q-1}),\ N\to\infty. \tag{1.204}$$

Tending N to infinity and replacing the sums by the corresponding integrals, we write

$$\lim_{N\to\infty} N^q R_N\left(f, \frac{2N+m+1}{2N} - \frac{m+1}{2N}x\right) = \frac{1}{2}\int_{-1}^{1}\nu_{q,m}(f,h)e^{-i\pi(m+1)xh/2}dh \\ - \frac{1}{2}\int_{|h|>1}\mu_{q,m}(f,h)e^{-i\pi(m+1)xh/2}dh. \tag{1.205}$$

Hence,

$$\lim_{N\to\infty} N^{2q+1}I_2 = \frac{m+1}{8}\int_0^1\left|\int_{-1}^{1}\nu_{q,m}(f,h)e^{-i\pi(m+1)xh/2}dh \right. \\ \left. - \int_{|h|>1}\mu_{q,m}(f,h)e^{-i\pi(m+1)xh/2}dh\right|^2 dx. \tag{1.206}$$

Similarly, we estimate I_3. From (1.199)

$$R_{N,m}\left(f, \frac{m+1}{2N}x - \frac{2N+m+1}{2N}\right) = \sum_{n=-N}^{N}(-1)^n\left(f_n^* - F_{n,m}\right)e^{i\pi x\frac{(m+1)n}{2N+m+1}} \\ + \sum_{|n|>N}(-1)^n f_n^* e^{i\pi x\frac{(m+1)n}{2N+m+1}}. \tag{1.207}$$

By the same steps

$$\lim_{N\to\infty} N^q R_N\left(f, \frac{m+1}{2N}x - \frac{2N+m+1}{2N}\right) = \frac{1}{2}\int_{-1}^{1}\nu_{q,m}(f,h)e^{i\pi(m+1)xh/2}dh \\ - \frac{1}{2}\int_{|h|>1}\mu_{q,m}(f,h)e^{i\pi(m+1)xh/2}dh \tag{1.208}$$

and

$$\lim_{N\to\infty} N^{2q+1}I_3 = \frac{m+1}{8}\int_0^1\left|\int_{-1}^{1}\nu_{q,m}(f,h)e^{i\pi(m+1)xh/2}dh \right. \\ \left. - \int_{|h|>1}\mu_{q,m}(f,h)e^{i\pi(m+1)xh/2}dh\right|^2 dx \tag{1.209}$$

which completes the proof. □

The case $q = m = 0$ needs special attention. Estimate (1.185) of Lemma 1.9 is valid for this choice. Additional smoothness requirement is needed for estimate (1.186).

Lemma 1.10 *[56, 57] Let $f' \in L_2[-1,1]$. Then, the following estimate holds for $|n| \le N$*

$$F_{n,0} - f_n^* = \frac{(-1)^{n+1}}{2N+1}\nu_{0,0}\left(f, \frac{2n}{2N+1}\right) + o\left(N^{-1}\right), \ N \to \infty, \tag{1.210}$$

where $\nu_{0,0}$ is defined by (1.189).

Proof. As we mentioned, estimate (1.185) of Lemma 1.9 is valid for $q = m = 0$

$$f_n^* = (-1)^{n+1}\frac{A_{00}(f)}{2i\pi n} + \frac{1}{2i\pi n}b_{n,N}, \tag{1.211}$$

where

$$b_{n,N} = \int_{-1}^{1} f^{*'}(x)e^{-i\pi n x}dx. \tag{1.212}$$

According to Parseval's identity and definition of f^* (see (1.75)), we have

$$\sum_{n=-\infty}^{\infty} |b_{n,N}|^2 = \int_{-1}^{1} \left|f^{*'}(x)\right|^2 dx = \frac{2N+1}{2N}\int_{-1}^{1} |f'(x)|^2 dx \tag{1.213}$$

which completes the proof in view of (1.167) and (1.211). □

Theorem 1.9 *[56, 57] Let $f' \in L_2[-1,1]$. Then the following estimate holds*

$$\lim_{N\to\infty} N^{\frac{1}{2}}||R_{N,0}(f,x)||_{L_2(-1,1)} = c_{0,0}(f), \tag{1.214}$$

where

$$\begin{aligned} c_{0,0,}^2(f) = &-\frac{1}{8}\int_{-1}^{1}\left|\int_{-1}^{1}\nu_{0,0}(f,h)e^{i\pi x h/2}dh - \int_{|h|>1}\mu_{0,0}(f,h)e^{i\pi x h/2}dh\right|^2 dx \\ &+ \frac{1}{2}\int_{-1}^{1}\left|\nu_{0,0}\left(f,x\right)\right|^2 dx + \frac{1}{2}\int_{|x|>1}\left|\mu_{0,0}\left(f,x\right)\right|^2 dx. \end{aligned} \tag{1.215}$$

Let us compare Theorems 1.8 and 1.9 with their classical counterpart Theorem 1.3. As in the case of pointwise convergence, the first difference concerns the values and derivatives of f at the endpoints of interval. In Theorems 1.8 and 1.9 it is required $f^{(k)}(1) = f^{(k)}(-1) = 0$, $k = 0, \dots, q-1$ instead of $f^{(k)}(1) = f^{(k)}(-1)$, $k = 0, \dots, q-1$ as in Theorem 1.3. The second difference is smoothness requirements on function f. Theorem 1.8 requires $f^{(q+m)} \in AC[-1,1]$

when $q+m \neq 0$ while Theorem 1.3 requires only $f^{(q)} \in AC[-1,1]$ when $q \neq 0$. The smoothness requirements are the same $f' \in L_2[-1,1]$ when $q = m = 0$.

However, on the contrary to the estimates for the pointwise convergence, these additional conditions do not lead to faster convergence rates. In both cases the convergence rate is $O(N^{-q-1/2})$. Comparison of asymptotic accuracies of both interpolations is possible by comparison of constants $c_{q,m}(f)$ of Theorems 1.8 and 1.9 with constant $A_q(f)c_q$ of Theorem 1.3. In general, it is possible only for specific f as constant $c_{q,m}(f)$ depends on f. In Section 1.6 we perform such comparisons for different m and q by numerical experiments. In an important case, when $q = 0$, such comparison is possible independently of f.

Let $q = 0$. In this case

$$\mu_{0,m}(f,x) = A_0(f)\mu^*_{0,m}(x), \tag{1.216}$$

$$\nu_{0,m}(f,x) = A_0(f)\nu^*_{0,m}(x), \tag{1.217}$$

where

$$\mu^*_{0,m}(x) = \frac{1}{i\pi x}, \tag{1.218}$$

and

$$\nu^*_{0,m}(x) = \frac{1}{i\pi}\left(\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{(2r+x)} - e^{-i\pi\frac{m-1}{2}x}\sum_{\tau=0}^{m-1}\frac{1}{\tau!}\Phi^{(\tau)}_{0,m}(-1)\left(e^{i\pi x}+1\right)^{\tau}\right). \tag{1.219}$$

Hence, constant $c_{q,m}(f)$ can be rewritten as follows

$$c_{0,m}(f) = |A_0(f)|c^*_{0,m}, \tag{1.220}$$

where

$$\begin{aligned} c^*_{0,m} = \Bigg(&-\frac{m+1}{8}\int_{-1}^{1}\left|\int_{-1}^{1}\nu^*_{0,m}(h)e^{i\pi(m+1)xh/2}dh - \int_{|h|>1}\mu^*_{0,m}(h)e^{i\pi(m+1)xh/2}dh\right|^2 dx \\ &+\frac{1}{2}\int_{-1}^{1}\left|\nu^*_{0,m}(x)\right|^2 dx + \frac{1}{2}\int_{|x|>1}\left|\mu^*_{0,m}(x)\right|^2 dx\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{1.221}$$

Note that $c^*_{0,m}$ is independent of f and the ratio $c_0/c^*_{0,m}$ (see Theorem 1.3) will show the differences in asymptotic L_2-accuracies of both interpolations independently of f.

Table 1.3 presents the values of $c^*_{0,m}$ and $c_0/c^*_{0,m}$ for different m, where $m = -1$ corresponds to the classical case. We see that by increasing m we increase (asymptotically) the accuracy of the QP interpolation compared to the classical one. For example, when $q = 0$ and $m = 7$ the QP interpolation is 3367 times asymptotically more accurate in the L_2-norm compared to the classical interpolation. Section 1.6 gives similar comparisons for moderate values of N for specific functions.

	$m=-1$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
$c^*_{0,m}$	0.33	0.19	0.036	0.011	0.0037	0.0014	0.00056	0.00023	0.000098
$c_0/c^*_{0,m}$	1	1.7	9	30	89	236	589	1435	3367

Table 1.3: Numerical values of $c^*_{0,m}$ and ratio $c_0/c^*_{0,m}$ where $m = -1$ corresponds to the classical case.

1.5 Limit Function Analysis

In this section, we study behavior of the QP interpolation at the points $x = \pm 1$ in terms of the limit functions ([57]).

Theorem 1.10 *[57] Let $f^{(q+m)} \in AC[-1,1]$ for some $q, m \geq 0$, $q + m \neq 0$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, ..., q-1. \tag{1.222}$$

Then, the following estimates hold

$$\lim_{N\to\infty} N^q R_{N,m}\left(f, \pm\left(1 - \frac{h}{N}\right)\right) = \ell_{x\to\pm1,q,m}(f,h), \ h > 0, \tag{1.223}$$

where

$$\begin{aligned}\ell_{x\to\pm1,q,m}(f,h) = &\frac{1}{2}\int_{-1}^{1} \nu_{q,m}(f,t) e^{\mp i\pi\left(\frac{m+1}{2}+h\right)t} dt \\ &- \frac{1}{2}\int_{|t|>1} \mu_{q,m}(f,t) e^{\mp i\pi\left(\frac{m+1}{2}+h\right)t} dt\end{aligned} \tag{1.224}$$

and functions $\mu_{q,m}$ and $\nu_{q,m}$ are defined in Lemma 1.9.

Proof. In view of (1.199), we write

$$\begin{aligned}R_{N,m}\left(f, \pm\left(1 - \frac{h}{N}\right)\right) = &\sum_{n=-N}^{N} (-1)^n (f^*_n - F_{n,m}) e^{\mp\frac{i\pi n}{2N+m+1}(m+1+2h)} \\ &+ \sum_{|n|>N} (-1)^n f^*_n e^{\mp\frac{i\pi n}{2N+m+1}(m+1+2h)}.\end{aligned} \tag{1.225}$$

Estimates of Lemma 1.9 imply

$$N^q R_{N,m}\left(f, \pm\left(1-\frac{h}{N}\right)\right) = \frac{1}{(2N+m+1)} \sum_{n=-N}^{N} \nu_{q,m}\left(\frac{2n}{2N+m+1}\right) e^{\frac{\mp i\pi n}{2N+m+1}(m+1+2h)} - \frac{1}{(2N+m+1)} \sum_{|n|>N} \mu_{q,m}\left(\frac{2n}{2N+m+1}\right) e^{\frac{\mp i\pi n}{2N+m+1}(m+1+2h)} + O\left(\frac{1}{N}\right), \ N \to \infty. \tag{1.226}$$

Tending N to infinity and replacing the sums by corresponding integrals, we get the required estimates. □

Similarly can be proved the next one based on estimate of Lemma 1.10.

Theorem 1.11 *[57] Let $f' \in L_2[-1,1]$. Then the following estimates hold*

$$\lim_{N\to\infty} R_{N,0}\left(f, \pm\left(1-\frac{h}{N}\right)\right) = \ell_{x\to\pm1,0,0}(f,h), \ h > 0, \tag{1.227}$$

where

$$\ell_{x\to\pm1,0,0}(f,h) = \frac{1}{2}\int_{-1}^{1} \nu_{0,0}(f,t) e^{\mp i\pi\left(\frac{1}{2}+h\right)t} dt - \frac{1}{2}\int_{|t|>1} \mu_{0,0}(f,t) e^{\mp i\pi\left(\frac{1}{2}+h\right)t} dt, \tag{1.228}$$

and

$$\nu_{0,0}(f,x) = \frac{A_0(f)}{i\pi} \sum_{r\neq 0} \frac{(-1)^r}{2r+x}, \ \mu_{0,0}(f,x) = \frac{A_0(f)}{i\pi x}. \tag{1.229}$$

Remark 1.6 *In Theorems 1.10 and 1.11, in case of $h = 0$, obviously $R_{N,m}(f, \pm 1) = 0$.*

Compared with Theorem 1.4, Theorems 1.10 and 1.11 require additional smoothness from f and also put stricter conditions on the values of f and its derivatives at the endpoints of the interval but provide the same rate of convergence $O(N^{-q})$. To understand which interpolation has better asymptotic uniform error we need comparison of

$$\ell_{q,m}(f) = \max\{\max_{h>0} |\ell_{x\to 1,q,m}(f,h)|, \max_{h>0} |\ell_{x\to -1,q,m}(f,h)|\} \tag{1.230}$$

with its classical analogue $|A_q(f)|\ell_q^*$ (see (1.18)). In general, such comparison is possible only for specific functions as $\ell_{q,m}(f)$ depends on f. We show some numerical results in Section 1.6 for specific functions. Function independent comparisons are possible when $q = 0$.

Let $q = 0$. Taking into account equations (1.216)-(1.219), we can rewrite $\ell_{q,m}(f)$ as

$$\ell_{0,m}(f) = |A_0(f)|\ell^*_{0,m}, \tag{1.231}$$

where

$$\ell^*_{0,m} = \max\{\max_{h>0} |\ell^*_{x\to 1,0,m}(h)|, \max_{h>0} |\ell^*_{x\to -1,0,m}(h)|\} \tag{1.232}$$

and (see (1.218), (1.219))

$$\ell^*_{x\to\pm1,0,m}(h) = \frac{1}{2}\int_{-1}^{1} \nu^*_{0,m}(t)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)t}dt - \int_{|t|>1} \mu^*_{0,m}(t)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)t}dt. \tag{1.233}$$

Hence, ratio $\ell^*_0/\ell^*_{0,m}$, which is independent of f, will show which interpolation has better asymptotic uniform accuracy. Table 1.4 shows the values of $\ell^*_{0,m}$ and $\ell^*_0/\ell^*_{0,m}$. We see that for $q = 0$ and $m = 7$ the QP interpolation is 4545.5 times more accurate in the uniform-norm compared to the classical interpolation. This is more than the number 3367 which we had in the framework of the L_2-norm (see Table 1.3).

	$m=-1$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
$\ell^*_{0,m}$	0.5	0.14	0.033	0.010	0.0038	0.0015	0.00059	0.00025	0.00011
$\ell^*_0/\ell^*_{0,m}$	1	3.6	15	50	132	333	848	2000	4546

Table 1.4: Numerical values of $\ell^*_{0,m}$ and $\ell^*_0/\ell^*_{0,m}$ where $m = -1$ corresponds to the classical interpolation.

1.6 Numerical Analysis

In this section, we compare the QP and the classical interpolations by series of numerical experiments.

Let

$$f(x) = (x^2-1)^q \sin(x-1),\ q \geq 0. \tag{1.234}$$

We start with analysis of the pointwise convergence. Figures 1.1 and 1.2 show the behaviors of $|R_N(f,x)|$ and $|R_{N,m}(f,x)|$ on $[-0.6, 0.6]$ for different values of q, m and $N = 256$.

As we observed above, the classical trigonometric interpolation has pointwise convergence rates $O(N^{-q-1})$ or $O(N^{-q-2})$ for even and odd q, respectively. The QP interpolation has

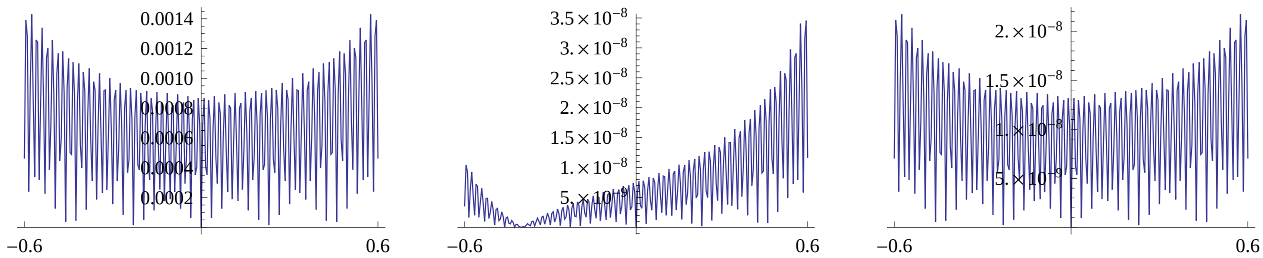

Figure 1.1: The graphs of $|R_{256}(f,x)|$ on $[-0.6, 0.6]$ for $q = 0, 1, 2$ (from left to right) while interpolating (1.234) by the classical trigonometric interpolation.

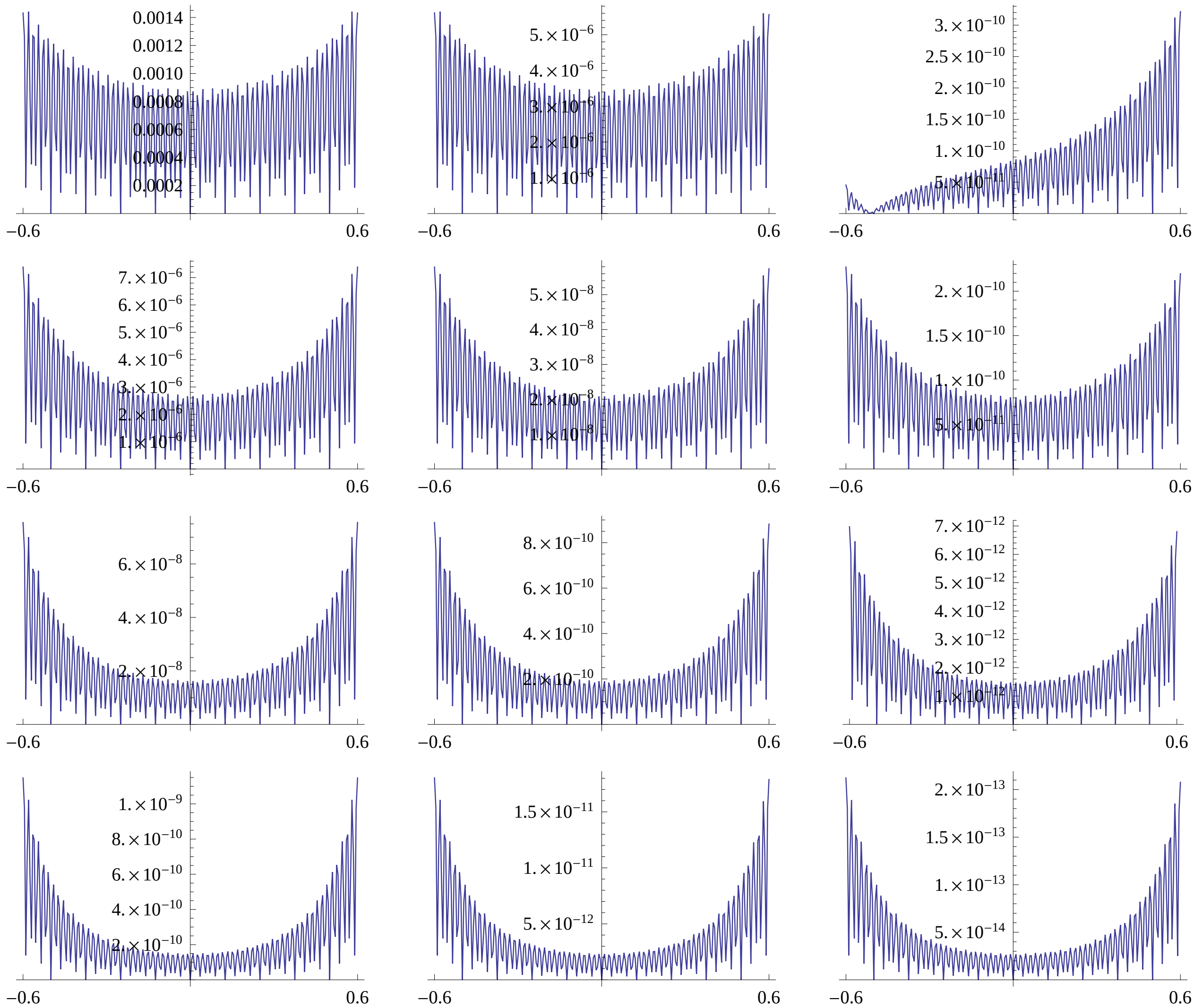

Figure 1.2: The graphs of $|R_{256,m}(f,x)|$ on $[-0.6, 0.6]$ for $q = 0, 1, 2$ (from left to right) and $m = 0, 1, 2, 3$ (from top to bottom) while interpolating (1.234) by the QP interpolation.

pointwise convergence rate $O(N^{-q-m-1})$, $m \geq 0$. Hence, the classical interpolation for odd q has asymptotically ($N >> 1$) better accuracy than the QP interpolation for $m = 0$ and the same rate of convergence for $m = 1$. For other cases the QP interpolation has better accuracy than the classical interpolation when q is fixed. Figures 1.1 and 1.2 confirm these observations for $N = 256$.

Comparison of the figures shows that for $q = 1$ the classical interpolation has better accuracy than the QP interpolation for $m = 0$. The classical interpolation for $q = 1$ is more accurate than the QP interpolation for $q = 1$ and $m = 1$. For $q = 0$ the classical interpolation has the same accuracy as the QP interpolation for $q = 0$ and $m = 0$. In all other cases, the QP interpolation is much more precise than the classical interpolation.

Now, we consider convergence in the L_2-norm. As we mentioned above, theoretical comparison of the classical and the QP interpolations is impossible independently of f when $q > 0$. Here, we perform such comparison for (1.234). The last column of Table 1.5 shows numerical values of $c_{q,m}(f)$. Let

$$c_{q,m,N}(f) = N^{q+1/2}||R_{N,m}(f,x)||_{L_2(-1,1)}. \tag{1.235}$$

Table 1.5 presents the values of $c_{q,m,N}(f)$ for $N = 16, 32, 64$ and $N = 128$ which are rather close to their limits $c_{q,m}(f)$ starting from $N = 16$ (however for this specific function).

Comparison of the QP and the classical interpolations can be performed by calculating the values of ratio $\frac{|A_q(f)|c_q}{c_{q,m}(f)}$ (see Theorem 1.3). When the ratio is greater than 1 then the QP interpolation has better asymptotic accuracy in the L_2-norm compared to the classical one. As bigger is the ratio as more accurate is the QP interpolation compared to the classical interpolation. Table 1.6 shows the values of $|A_q(f)|c_q$ and Table 1.7 shows the values of the ratio for (1.234). The values of the ratio become smaller when q becomes bigger but when q is fixed, by increasing m, we can increase the value of the ratio.

The numbers in the first row of Table 1.7 slightly differ from the numbers in the second row of Table 1.3 due to different order of rounding in calculations.

Finally, we analyze the behaviors of the limit functions. For comparison of accuracies, we need to compare the maximum values of $|A_q(f)\ell_{x\to\pm1,q}(h)|$ (Theorem 1.4) and $|\ell_{x\to\pm1,q,m}(f,h)|$ for $h \geq 0$. In case of the classical interpolation the limit functions at the points ±1 are identical

$q=0$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{q,m}(f)$
$m=0$	0.17455	0.17295	0.17212	0.17170	0.17127
$m=1$	0.03408	0.03315	0.03275	0.03257	0.03240
$m=2$	0.01044	0.01002	0.00984	0.00975	0.00968
$m=3$	0.00376	0.00355	0.00346	0.00342	0.00339
$m=4$	0.00148	0.00136	0.00132	0.00130	0.00128
$m=5$	0.00062	0.00055	0.00053	0.00052	0.00051
$m=6$	0.00027	0.00023	0.00022	0.00022	0.00021
$m=7$	0.00012	0.00010	0.000094	0.000091	0.000089
$q=1$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{q,m}(f)$
$m=0$	0.17819	0.17798	0.17785	0.17778	0.17771
$m=1$	0.06981	0.06798	0.06721	0.06687	0.06656
$m=2$	0.03228	0.03076	0.03012	0.02983	0.02956
$m=3$	0.01568	0.01455	0.01409	0.01388	0.01369
$m=4$	0.00781	0.00703	0.00672	0.00658	0.00646
$m=5$	0.00396	0.00344	0.00324	0.00316	0.00309
$m=6$	0.00203	0.00170	0.00158	0.00153	0.00148
$m=7$	0.00106	0.00085	0.00077	0.00074	0.00072
$q=2$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{q,m}(f)$
$m=0$	0.1042	0.1052	0.1059	0.1063	0.1067
$m=1$	0.1014	0.1019	0.1022	0.1024	0.1026
$m=2$	0.0766	0.0749	0.0739	0.0736	0.0733
$m=3$	0.0523	0.0492	0.0479	0.0474	0.0469
$m=4$	0.0339	0.0307	0.0294	0.0288	0.0283
$m=5$	0.0213	0.0185	0.0174	0.0169	0.0165
$m=6$	0.0132	0.0109	0.0100	0.0097	0.0094
$m=7$	0.0080	0.0063	0.0057	0.0054	0.0052
$q=3$	$N=16$	$N=32$	$N=64$	$N=128$	$c_{q,m}(f)$
$m=0$	0.2372	0.2359	0.2348	0.2341	0.2333
$m=1$	0.1176	0.1102	0.1076	0.1065	0.1056
$m=2$	0.1193	0.1209	0.1222	0.1230	0.1239
$m=3$	0.1216	0.1209	0.1209	0.1209	0.1211
$m=4$	0.1087	0.1033	0.1011	0.1002	0.0994
$m=5$	0.0885	0.0798	0.0763	0.0748	0.0735
$m=6$	0.0678	0.0577	0.0538	0.0521	0.0507
$m=7$	0.0497	0.0398	0.0361	0.0346	0.0334

Table 1.5: Numerical values of $c_{N,m,q}(f)$ and $c_{q,m}(f)$ for (1.234).

q	0	1	2	3
$\lvert A_q(f)\rvert c_q$	0.30	0.15	0.14	0.24

Table 1.6: Numerical values of $|A_q(f)|c_q$ for (1.234).

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$
$q=0$	1.8	9.3	31	89	235	590	1427	3371
$q=1$	0.8	2.3	5.1	11	23	49	101	208
$q=2$	1.3	1.4	1.9	3.0	4.9	8.5	15	27
$q=3$	1.0	2.3	1.9	2.0	2.4	3.3	4.7	7.2

Table 1.7: Numerical values of ratio $\frac{|A_q(f)|c_q}{c_{q,m}(f)}$ for (1.234).

while for the QP interpolation they have, in general, different behavior. Figure 1.3 shows the graphs of $|A_q(f)\ell_{x\to\pm1,q}(h)|$ for (1.234).

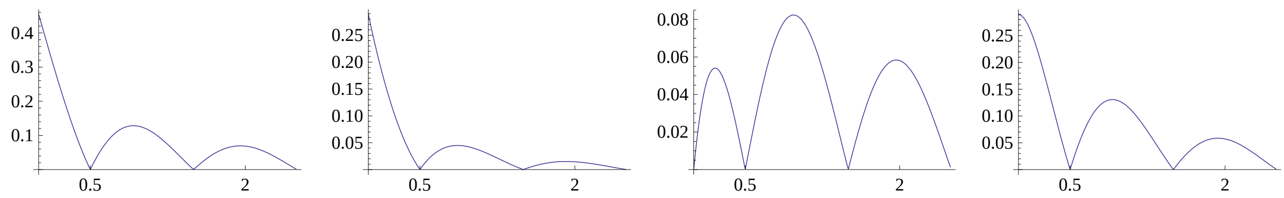

Figure 1.3: The graphs of $|A_q(f)\ell_{x\to\pm1,q}(h)|$ on $[0, 2.5]$ for $q = 0, 1, 2, 3$ (from left to right) for (1.234).

Figures 1.4 and 1.5 show the graphs of $|\ell_{x\to\pm1,q,m}(f, h)|$ for $m = 5$ and $q = 0, 1, 2, 3$ (from left to right) for (1.234). Ratio $|A_q(f)|\ell_q^*/\ell_{q,m}(f)$ compares accuracies of the classical and the QP interpolations. As bigger is the value of the ratio as more accurate is the QP interpolation compared to the classical one. Table 1.8 presents the values of the ratio for different q and m. We see that for $q = 2$ and $m = 0, 1, 2$ the classical interpolation is more precise than the QP interpolation for this specific example. We also see that as bigger is the value of q as less impressive is the QP interpolation even for big values of m. The numbers in the first column of Table 1.8 differ from the numbers in the second row of Table 1.4 due to different order of rounding in the calculations.

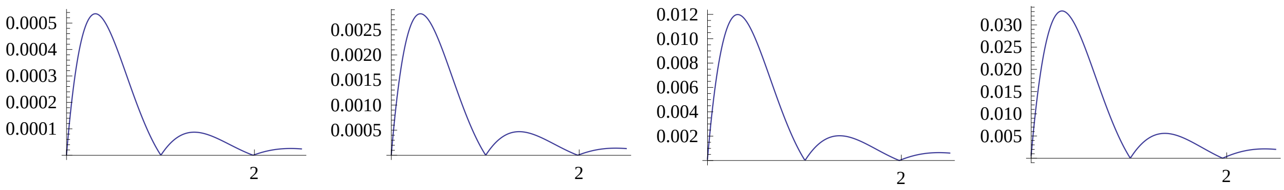

Figure 1.4: The graphs of $|\ell_{x\to 1,q,m}(f,h)|$ for $q = 0, 1, 2, 3$ (from left to right), $m = 5$ and (1.234).

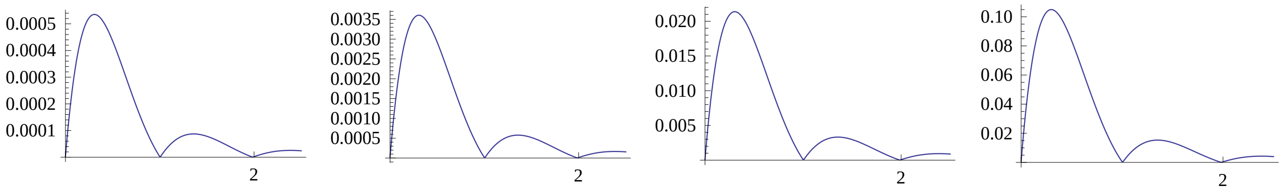

Figure 1.5: The graphs of $|\ell_{x\to -1,q,m}(f,h)|$ for $q = 0, 1, 2, 3$ (from left to right), $m = 5$ and (1.234).

	$q=0$	$q=1$	$q=2$	$q=3$
$m=0$	3.5	1.7	0.6	2.2
$m=1$	15	3.9	0.6	2.9
$m=2$	47	8.5	0.8	1.6
$m=3$	132	18	1.3	1.6
$m=4$	346	39	2.2	2.1
$m=5$	833	81	3.8	2.9

Table 1.8: Ratio $|A_q(f)|\ell_q^*/\ell_{q,m}(f)$.

CHAPTER 2

Convergence acceleration by rational corrections

In this chapter, we consider convergence acceleration of the QP interpolation by rational corrections in terms of $e^{i\pi\sigma x}$. This approach leads to QP Rational (QPR) approximation. We analyze its pointwise convergence and limit functions behavior deriving exact constants of the main terms of asymptotic errors.

Section 2.1 recounts some results concerning the classical rational trigonometric interpolation. In Section 2.2, we derive the QPR approximation where rational functions are utilized as corrections of the error of the QP interpolation. Section 2.3 studies pointwise convergence and Section 2.4 explores the behavior of the limit functions of the QPR approximation. Section 2.5 presents some numerical results. We recap details from [59, 63].

2.1 The Classical Rational Trigonometric Interpolation

Different approaches are known for convergence acceleration of the classical interpolation when 2-periodic extension of the interpolated function is discontinuous or has low smoothness on the real line. A broadly studied approach is error correction by rational functions (in terms of $e^{i\pi x}$) which is applicable both for the truncated Fourier series and interpolation.

We recap details from [35, 36]. Consider a finite sequence of complex numbers $\lambda = \{\lambda_k\}_{|k|=1}^{p}$. By $\delta_n^k\left(\lambda, \{y_s\}_{s=-\infty}^{\infty}\right)$, we denote modified finite differences defined by the following recurrent relations

$$\begin{aligned}
&\delta_n^0\left(\lambda, \{y_s\}_{s=-\infty}^{\infty}\right) = \delta_n^0\left(\lambda, \{y_s\}\right) = y_n, \\
&\delta_n^k\left(\lambda, \{y_s\}_{s=-\infty}^{\infty}\right) = \delta_n^k\left(\lambda, \{y_s\}\right) = \delta_n^{k-1}\left(\lambda, \{y_s\}\right) + \lambda_{-k}\delta_{n-1}^{k-1}\left(\lambda, \{y_s\}\right) \\
&\qquad + \lambda_k\left(\delta_{n+1}^{k-1}\left(\lambda, \{y_s\}\right) + \lambda_{-k}\delta_n^{k-1}\left(\lambda, \{y_s\}\right)\right), \ k \geq 1
\end{aligned} \tag{2.1}$$

for some sequence $\{y_s\}_{s=-\infty}^{\infty}$. For $\lambda \equiv 1$, from (1.52), we see that

$$\delta_n^k(\lambda, \{y_s\}) = \delta_n^k(\{y_s\}). \tag{2.2}$$

Following [35], we write the Rational Trigonometric (RT) interpolation as follows

$$\begin{aligned} I_N^p(f,x) &= I_N(f,x) \\ &+ (e^{-i\pi Nx} - e^{i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\lambda_{-k}\delta_N^{k-1}(\lambda, \{\check{f}_s\})}{\prod_{s=1}^{k}(1+\lambda_{-s}e^{i\pi x})(1+\lambda_s e^{-i\pi x})} \\ &+ (e^{i\pi Nx} - e^{-i\pi(N+1)x}) \sum_{k=1}^{p} \frac{\lambda_{k}\delta_{-N}^{k-1}(\lambda, \{\check{f}_s\})}{\prod_{s=1}^{k}(1+\lambda_{-s}e^{i\pi x})(1+\lambda_s e^{-i\pi x})} \end{aligned} \tag{2.3}$$

with the error

$$\begin{aligned} R_N^p(f,x) &= f(x) - I_N^p(f,x) \\ &= \frac{1}{\prod_{s=1}^{p}(1+\lambda_{-s}e^{i\pi x})(1+\lambda_s e^{-i\pi x})} \sum_{|n|=N+1}^{\infty} \delta_n^p(\lambda, \{f_s\}) e^{i\pi nx} \\ &+ \frac{1}{\prod_{s=1}^{p}(1+\lambda_{-s}e^{i\pi x})(1+\lambda_s e^{-i\pi x})} \sum_{n=-N}^{N} \delta_n^p(\lambda, \{f_s - \check{f}_s\}) e^{i\pi nx}. \end{aligned} \tag{2.4}$$

Rational corrections contain unknown parameters λ_k which determination is a crucial issue for realization of the rational interpolations. An approach which is applicable for cases when interpolated function is smooth on $[-1,1]$ is

$$\lambda_k = \lambda_{-k} = 1 - \frac{\tau_k}{N}, \ k = 1, \ldots, p, \tag{2.5}$$

where new parameters τ_k are independent of N and f ([35]). Below, we present some convergence theorems that outline behavior of interpolations with such choice of parameters. Another approach is determination of λ_k from the following system of equations

$$\delta_n^p(\lambda, \{\check{f}_s\}) = 0, \ n = N-p+1, \cdots, N+p. \tag{2.6}$$

Such interpolations are known as the Fourier-Pade (FP) interpolations ([36]).

Next theorems describe asymptotic behavior of the classical RT interpolation when parameters λ_k are chosen as in (2.5). Let

$$\prod_{s=1}^{p}(1+\tau_s x) = \sum_{s=0}^{p} \gamma_s(\tau) x^s, \tag{2.7}$$

where $\tau = \{\tau_1, \ldots, \tau_p\}$.

Theorem 2.1 *[35] Let $q \geq 0$ be even and $f^{(q+2p+1)} \in AC[-1,1]$ for some $p \geq 1$. Let*

$$A_k(f) = 0,\ k = 0, \ldots, q-1 \tag{2.8}$$

and parameters λ_k be chosen as in (2.36)*. Then, the following estimate holds for $|x| < 1$*

$$R_N^p(f,x) = A_q(f)\frac{(-1)^{N+p+\frac{q}{2}}}{2^{2p+1}\pi^{q+1}q!N^{2p+q+1}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q,p} + o(N^{-2p-q-1}),\ N \to \infty, \tag{2.9}$$

where

$$\psi_{j,p} = \sum_{s=0}^{p}(-1)^s\gamma_s(\tau)\sum_{k=0}^{p}\gamma_k(\tau)(2p-k-s+j)!\phi_{2p-k-s+j+1} \tag{2.10}$$

and

$$\phi_j = \sum_{s=-\infty}^{\infty}\frac{(-1)^s}{(2s+1)^j}. \tag{2.11}$$

Theorem 2.2 *[35] Let $q \geq 1$ be odd and $f^{(q+2p+2)} \in AC[-1,1]$ for some $p \geq 1$. Let*

$$A_k(f) = 0,\ k = 0, \ldots, q-1 \tag{2.12}$$

and parameters λ_k be chosen as in (2.36)*. Then, the following estimate holds for $|x| < 1$*

$$\begin{aligned} R_N^p(f,x) = {} & A_q(f)\frac{(-1)^{N+p+\frac{q+1}{2}+1}}{2^{2p+2}\pi^{q+1}q!N^{2p+q+2}}\frac{\sin\frac{\pi x}{2}\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+2}\frac{\pi x}{2}}\psi_{q+1,p} \\ & + A_{q+1}(f)\frac{(-1)^{N+p+\frac{q+1}{2}}}{2^{2p+1}\pi^{q+2}(q+1)!N^{2p+q+2}}\frac{\sin\frac{\pi x}{2}(2N+1)}{\cos^{2p+1}\frac{\pi x}{2}}\psi_{q+1,p} \\ & + o(N^{-2p-q-2}),\ N \to \infty. \end{aligned} \tag{2.13}$$

Comparison with Theorems 1.1 and 1.2 shows that for enough smooth functions the RT interpolations are asymptotically more precise than the classical interpolation and improvement is by the factor $O(N^{-2p})$ as $N \to \infty$.

Estimates of Theorems 2.1 and 2.2 outline an approach for determination of parameters τ_k. Determination of theses parameters ([35]) from the following equations

$$\psi_{q+w,p} = 0,\ w = 0, \ldots, p-1 \tag{2.14}$$

for even values of q, and

$$\psi_{q+w+1,p} = 0,\ w = 0, \ldots, p-1 \tag{2.15}$$

for odd q, leads to more accurate interpolations with the rates of convergence

$$R_N^p(f,x) = o(N^{-2p-q-1}) \tag{2.16}$$

and

$$R_N^p(f,x) = o(N^{-2p-q-2}) \tag{2.17}$$

for even and odd q, respectively. Table 2.1 shows the optimal values of parameters τ_k derived from (2.14) when $q=0$. Optimal values for τ_k are known also when $q>0$ (see [35]).

p	τ_k
$p=1$	$\tau_1 = 1.570796$
$p=2$	$\tau_1 = 1.290966,\ \tau_2 = 5.733849$
$p=3$	$\tau_1 = 1.164993,\ \tau_2 = 4.712389,\ \tau_3 = 10.58977$
$p=4$	$\tau_1 = 1.088781,\ \tau_2 = 4.208368,\ \tau_3 = 8.857598,\ \tau_4 = 15.750661$
$p=5$	$\tau_1 = 1.035985,\ \tau_2 = 3.893407,\ \tau_3 = 7.937983,\ \tau_4 = 13.381556,\ \tau_5 = 21.092649$

Table 2.1: Optimal values of τ_k derived from (2.14) when $q=0$.

Next theorem studies the behavior of the RT interpolation at the endpoints of the interval in terms of the limit functions.

Theorem 2.3 *Let $f^{(q+2p)} \in AC[-1,1]$ for some $p \geq 1$, $q \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1),\ k = 0,\dots,q-1. \tag{2.18}$$

Let λ_k be chosen as in (2.5). Then, the following estimate holds

$$\lim_{N\to\infty} N^q R_N^p\left(f, \pm\left(1-\frac{h}{N}\right)\right) = A_q(f)\ell^p_{x\to\pm1,q}(h),\ h \geq 0, \tag{2.19}$$

where

$$\begin{aligned}\ell^p_{x\to\pm1,q}(h) = & \frac{(-1)^{p+1}}{2(i\pi)^{q+1}q!}\frac{1}{\prod_{s=1}^p(\tau_s^2+\pi^2h^2)}\sum_{s=0}^p(-1)^s\gamma_s(\tau)\sum_{k=0}^p\gamma_k(\tau)(2p-k-s+q)! \\ & \times\left(\int_{|t|>1}\frac{e^{\mp i\pi ht}}{t^{2p-s-k+q+1}}dt - \int_{-1}^1 e^{\mp i\pi ht}\sum_{r\neq0}\frac{(-1)^r}{(2r+t)^{2p-k-s+q+1}}dt\right).\end{aligned} \tag{2.20}$$

Proof. We use representation of error as in (2.4). First, we observe that

$$\lim_{N\to\infty} N^{2p}\prod_{s=1}^p\left(1+\theta_{-s}e^{i\pi x}\right)\left(1+\theta_s e^{-i\pi x}\right) = \prod_{s=1}^p(\tau_s^2+\pi^2h^2). \tag{2.21}$$

Then, we use asymptotic expansion of the Fourier coefficients

$$f_n = \sum_{m=q}^{q+2p} A_m(f) B_n(m) + o(n^{-2p-q-1}) \tag{2.22}$$

and consequently

$$\delta_n^p(\lambda, \{f_s\}) = \sum_{m=q}^{q+2p} A_m(f) \delta_n^p(\theta, \{B_s(m)\}) + o(n^{-2p-q-1}), \ n \to \infty. \tag{2.23}$$

Taking into account that

$$\begin{aligned} \delta_n^p(\theta, \{B_s(m)\}) &= \frac{(-1)^{n+p+1}}{2(i\pi n)^{m+1} n^{2p} m!} \sum_{s=0}^{p} (-1)^s \frac{\gamma_s(\tau)}{N^s n^{-s}} \sum_{k=0}^{p} \frac{\gamma_k(\tau)}{N^k n^{-k}} (2p-k-s+m)! \\ &+ \frac{1}{N^{2p}} O(n^{-m-2}), \ |n| > N, \ N \to \infty, \end{aligned} \tag{2.24}$$

we get

$$\begin{aligned} \delta_n^p(\theta, \{f_s\}) = A_q(f) &\frac{(-1)^{n+p+1}}{2(i\pi n)^{q+1} n^{2p} q!} \sum_{s=0}^{p} (-1)^s \frac{\gamma_s(\tau)}{N^s n^{-s}} \sum_{k=0}^{p} \frac{\gamma_k(\tau)}{N^k n^{-k}} (2p-k-s+q)! \\ &+ \frac{1}{N^{2p}} O(n^{-q-2}) \\ &+ o(n^{-2p-q-1}), \ |n| > N, \ N \to \infty, \end{aligned} \tag{2.25}$$

and similarly

$$\begin{aligned} \delta_n^p(\theta, \{\check{f}_s - f_s\}) = A_q(f) &\frac{(-1)^{n+p+1}}{2(i\pi N)^{q+1} N^{2p} q!} \sum_{s=0}^{p} (-1)^s \gamma_s(\tau) \sum_{k=0}^{p} \gamma_k(\tau) (2p-k-s+q)! \\ &\times \sum_{r \neq 0} \frac{(-1)^r}{\left(2r + \frac{n}{N}\right)^{2p-k-s+q+1}} \\ &+ o(N^{-2p-q-1}), \ |n| \le N, \ N \to \infty, \end{aligned} \tag{2.26}$$

which complete the proof. □

Remark 2.1 *Estimate of Theorem 2.3 is valid also for* $q = 0$ *and* $h > 0$*. Case* $h = 0$ *must be considered separately. It can be verified that*

$$\lim_{N \to \infty} R_N^p(f, \pm 1) = A_0(f) \ell_{x \to \pm 1, 0}^p(0), \tag{2.27}$$

where

$$\ell_{x \to \pm 1, 0}^p(0) = \pm \frac{1}{2}. \tag{2.28}$$

Taking into account that

$$|\ell^p_{x\to 1,q}(h)| = |\ell^p_{x\to -1,q}(h)|, \tag{2.29}$$

we put

$$\ell^{p,*}_q = \max_{h\geq 0} |\ell^p_{x\to 1,q}(h)| \tag{2.30}$$

which characterizes the asymptotic ($N \to \infty$) uniform error of the classical RT interpolation.

Estimate of Theorem 2.3 outlines another approach for determination of parameters τ_k by minimization of constant $\ell^{p,*}_q$. Tables 2.2 and 2.3 shows the values of τ_k that minimize $\ell^{p,*}_q$ for $p = 1$ and $p = 2$, respectively. The last columns of the tables present the values of $\ell^*_q/\ell^{p,*}_q$ which show how much the classical RT interpolation is asymptotically accurate with optimal parameters compared to the classical interpolation.

q	τ_k	$\ell^{p,*}_q$	$\ell^*_q/\ell^{p,*}_q$
$q=1$	$\tau_1 = 1.6970$	0.023	6.96
$q=2$	$\tau_1 = 2.2274$	0.0020	5.5
$q=3$	$\tau_1 = 3.6886$	0.00062	10.65
$q=4$	$\tau_1 = 4.2055$	0.000061	11.80

Table 2.2: Values of τ_k that minimize $\ell^{p,*}_q$ for $p = 1$.

q	τ_k	$\ell^{p,*}_q$	$\ell^*_q/\ell^{p,*}_q$
$q=1$	$\tau_1 = 0.6072,\ \tau_2 = 3.7356$	0.0075	21.33
$q=2$	$\tau_1 = 1.1716,\ \tau_2 = 4.3508$	0.00045	24.44
$q=3$	$\tau_1 = 2.1719,\ \tau_2 = 6.3818$	0.000098	67.35
$q=4$	$\tau_1 = 2.5793,\ \tau_2 = 6.8302$	$1.0 \cdot 10^{-5}$	72.0

Table 2.3: Values of τ_k that minimize $\ell^{p,*}_q$ for $p = 2$.

Equation (2.28) shows that for $q = 0$ the minimal value for $\ell^{p,*}_q$ can't be smaller than 0.5 which equals to ℓ^*_q. Hence, for $q = 0$ the RT interpolation has better accuracy compared to the classical interpolation away from the endpoints of the interval and the same bad accuracy at the points $x = \pm 1$.

2.2 QPR Approximation

In this section, we introduce convergence acceleration of the QP interpolation by rational corrections (in terms of $e^{i\pi\sigma x}$) proceeding as in case of the classical rational trigonometric interpolation.

We recap details from [59].

Assume $f \in C^{\alpha}[-1,1]$, $\alpha \geq 1$. We take into account definition of f^* (see (1.75)) and use the representation of the error of the QP interpolation derived in (1.133)

$$R_{N,m}(f,x) = \sum_{n=-N}^{N} (f_n^* - F_{n,m}) e^{i\pi n\sigma x} + \sum_{|n|>N} f_n^* e^{i\pi n\sigma x}. \tag{2.31}$$

The following transformation is easy to verify (see [55, 76] for similar transformations)

$$\begin{aligned} R_{N,m}(f,x) = {} & \lambda_{-1} \frac{F_{-N-1,m} e^{-i\pi\sigma N x} - F_{N,m} e^{i\pi\sigma(N+1)x}}{(1+\lambda_{-1}e^{i\pi\sigma x})(1+\lambda_1 e^{-i\pi\sigma x})} \\ & + \lambda_1 \frac{F_{N+1,m} e^{i\pi\sigma N x} - F_{-N,m} e^{-i\pi\sigma(N+1)x}}{(1+\lambda_{-1}e^{i\pi\sigma x})(1+\lambda_1 e^{-i\pi\sigma x})} \\ & + \frac{1}{(1+\lambda_{-1}e^{i\pi\sigma x})(1+\lambda_1 e^{-i\pi\sigma x})} \sum_{n=-N}^{N} \delta_n^1 \left(\lambda, \{f_s^* - F_{s,m}\}\right) e^{i\pi n\sigma x} \\ & + \frac{1}{(1+\lambda_{-1}e^{i\pi\sigma x})(1+\lambda_1 e^{-i\pi\sigma x})} \sum_{|n|>N} \delta_n^1 \left(\lambda, \{f_s^*\}\right) e^{i\pi n\sigma x}. \end{aligned} \tag{2.32}$$

Reiteration of it up to p times leads to the following expansion of the error

$$\begin{aligned} R_{N,m}(f,x) = {} & \sum_{k=1}^{p} \lambda_{-k} \frac{\delta_{-N-1}^{k-1}\left(\lambda, \{F_{s,m}\}\right) e^{-i\pi\sigma N x} - \delta_{N}^{k-1}\left(\lambda, \{F_{s,m}\}\right) e^{i\pi\sigma(N+1)x}}{\prod_{j=1}^{k} (1+\lambda_{-j}e^{i\pi\sigma x})(1+\lambda_j e^{-i\pi\sigma x})} \\ & + \sum_{k=1}^{p} \lambda_{k} \frac{\delta_{N+1}^{k-1}\left(\lambda, \{F_{s,m}\}\right) e^{i\pi\sigma N x} - \delta_{-N}^{k-1}\left(\lambda, \{F_{s,m}\}\right) e^{-i\pi\sigma(N+1)x}}{\prod_{j=1}^{k} (1+\lambda_{-j}e^{i\pi\sigma x})(1+\lambda_j e^{-i\pi\sigma x})} \\ & + \frac{1}{\prod_{j=1}^{p} (1+\lambda_{-j}e^{i\pi\sigma x})(1+\lambda_j e^{-i\pi\sigma x})} \sum_{n=-N}^{N} \delta_n^p \left(\lambda, \{f_s^* - F_{s,m}\}\right) e^{i\pi n\sigma x} \\ & + \frac{1}{\prod_{j=1}^{p} (1+\lambda_{-j}e^{i\pi\sigma x})(1+\lambda_j e^{-i\pi\sigma x})} \sum_{|n|>N} \delta_n^p \left(\lambda, \{f_s^*\}\right) e^{i\pi n\sigma x}, \end{aligned} \tag{2.33}$$

where the first two terms can be assumed as corrections of the error.

This observation leads to the following QP Rational (QPR) approximation

$$
\begin{aligned}
I^p_{N,m}(f,x) &= I_{N,m}(f,x) \\
&+ \sum_{k=1}^{p} \lambda_{-k} \frac{\delta^{k-1}_{-N-1}\left(\lambda,\{F_{s,m}\}\right) e^{-i\pi\sigma N x} - \delta^{k-1}_{N}\left(\lambda,\{F_{s,m}\}\right) e^{i\pi\sigma(N+1)x}}{\prod_{j=1}^{k}\left(1+\lambda_{-j}e^{i\pi\sigma x}\right)\left(1+\lambda_j e^{-i\pi\sigma x}\right)} \\
&+ \sum_{k=1}^{p} \lambda_{k} \frac{\delta^{k-1}_{N+1}\left(\lambda,\{F_{s,m}\}\right) e^{i\pi\sigma N x} - \delta^{k-1}_{-N}\left(\lambda,\{F_{s,m}\}\right) e^{-i\pi\sigma(N+1)x}}{\prod_{j=1}^{k}\left(1+\lambda_{-j}e^{i\pi\sigma x}\right)\left(1+\lambda_j e^{-i\pi\sigma x}\right)}
\end{aligned}
\tag{2.34}
$$

with the error

$$
\begin{aligned}
R^p_{N,m}(f,x) &= \frac{1}{\prod_{j=1}^{p}\left(1+\lambda_{-j}e^{i\pi\sigma x}\right)\left(1+\lambda_j e^{-i\pi\sigma x}\right)} \sum_{n=-N}^{N} \delta^p_n\left(\lambda,\{f^*_s - F_{s,m}\}\right) e^{i\pi\sigma n x} \\
&+ \frac{1}{\prod_{j=1}^{p}\left(1+\lambda_{-j}e^{i\pi\sigma x}\right)\left(1+\lambda_j e^{-i\pi\sigma x}\right)} \sum_{|n|>N} \delta^p_n\left(\lambda,\{f^*_s\}\right) e^{i\pi\sigma n x}.
\end{aligned}
\tag{2.35}
$$

In [59] we called $I^p_{N,m}$ as QPR interpolation but taking into account that $I_{N,m}(f,x_k) \neq f(x_k)$ we changed its name to avoid confusion.

The QPR approximation is undefined until parameters λ_k are unknown. Hence, determination of these parameters is a crucial problem for realization of the QPR approximation.

We assume two approaches for determination of those parameters. First, we assume that

$$
\lambda_{-k} = \lambda_k = 1 - \frac{\tau_k}{N},\ k = 1,\ldots,p, \tag{2.36}
$$

where τ_k are some new parameters independent of N and f. In the next sections, we study the behavior of the QPR approximation for this choice of parameters λ_k and discuss the problem of determination of τ_k based on asymptotic estimates.

Second, we determine parameters λ_k from the following system of equations.

$$
\delta^p_n(\lambda,\{F_{s,m}\}) = 0,\ n = N - p + \left[\frac{m}{2}\right] + 1,\cdots,N + p + \left[\frac{m}{2}\right]. \tag{2.37}
$$

In Section 2.5, we consider some results of numerical experiments that realize this approach. We will call it as QP Fourier-Pade (FP) interpolation.

2.3 Pointwise Convergence

In this section, we study pointwise convergence of the QPR approximation and derive exact constants of the main term of asymptotic error. Throughout the section, we assume that parameters λ_k are determined by (2.36). Let $\gamma_k(\tau)$ be the coefficients of the following polynomial

$$\prod_{s=1}^{p}(1+\tau_s x)=\sum_{k=0}^{p}\gamma_k(\tau)x^k, \tag{2.38}$$

where $\tau=\{\tau_1,\dots,\tau_p\}$.

2.3.1 Preliminaries

Consider a vector $\mu=\{\mu_k\}_{k=1}^p$ of complex numbers. By $\Delta_n^k\left(\mu,\{y_s\}_{s=-\infty}^{\infty}\right)$, we denote generalized finite differences defined by the following recurrent relations

$$\begin{aligned}&\Delta_n^0\left(\mu,\{y_s\}_{s=-\infty}^{\infty}\right)=\Delta_n^0\left(\mu,\{y_s\}\right)=y_n,\\&\Delta_n^k\left(\mu,\{y_s\}_{s=-\infty}^{\infty}\right)=\Delta_n^k\left(\mu,\{y_s\}\right)=\Delta_n^{k-1}\left(\mu,\{y_s\}\right)+\mu_k\Delta_{n-1}^{k-1}\left(\mu,\{y_s\}\right),\ k\geq 1\end{aligned} \tag{2.39}$$

for some sequence $\{y_s\}_{s=-\infty}^{\infty}$. When $\mu\equiv 1$ (see (1.53))

$$\Delta_n^k\left(\mu,\{y_s\}\right)=\Delta_n^k\left(\{y_s\}\right). \tag{2.40}$$

It is easy to verify that

$$\Delta_n^1\left(\mu,\{y_s\}\right)=y_n+\mu_1 y_{n-1}, \tag{2.41}$$

$$\Delta_n^2\left(\mu,\{y_s\}\right)=\Delta_n^1\left(\mu,\{y_s\}\right)+\mu_2\Delta_{n-1}^1\left(\mu,\{y_s\}\right)=y_n+(\mu_1+\mu_2)y_{n-1}+\mu_1\mu_2 y_{n-2} \tag{2.42}$$

and

$$\Delta_n^3\left(\mu,\{y_s\}\right)=y_n+(\mu_1+\mu_2+\mu_3)y_{n-1}+(\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3)y_{n-2}+\mu_1\mu_2\mu_3 y_{n-3}. \tag{2.43}$$

In general, we can prove by the mathematical induction that

$$\Delta_n^p\left(\mu,\{y_s\}\right)=\sum_{t=0}^{p}\gamma_t(\mu)y_{n-t}, \tag{2.44}$$

where $\gamma_t(\mu)$ are the coefficients of the following polynomial

$$\sum_{t=0}^{p}\gamma_t(\mu)x^t=\prod_{t=1}^{p}(1+\mu_t x). \tag{2.45}$$

Consider vectors $\lambda=\{\lambda_k\}_{|k|=1}^p$, $\lambda^+=\{\lambda_k\}_{k=1}^p$ and $\lambda^-=\{\lambda_{-k}\}_{k=1}^p$. Similar to (2.44), we can show that

$$\delta_n^p\left(\lambda,\{y_s\}\right)=\sum_{k=0}^{p}\gamma_k(\lambda^+)\sum_{t=0}^{p}\gamma_t(\lambda^-)y_{n+k-t}, \tag{2.46}$$

where

$$\prod_{k=1}^{p}(1+\lambda_k x) = \sum_{k=0}^{p}\gamma_k(\lambda^+)x^k, \tag{2.47}$$

$$\prod_{k=1}^{p}(1+\lambda_{-k} x) = \sum_{k=0}^{p}\gamma_k(\lambda^-)x^k. \tag{2.48}$$

Let us modify (2.44) in view of (2.36). For $p=1$

$$\Delta_n^1\left(\lambda^+,\{y_s\}\right) = y_n + \left(1-\frac{\tau_1}{N}\right)y_{n-1} = \Delta_n^1\left(\{y_s\}\right) - \frac{\tau_1}{N}\Delta_{n-1}^0\left(\{y_s\}\right). \tag{2.49}$$

For $p=2$

$$\begin{aligned}\Delta_n^2\left(\lambda^+,\{y_s\}\right) &= \Delta_n^1\left(\lambda^+,\{y_s\}\right) + \left(1-\frac{\tau_2}{N}\right)\Delta_{n-1}^1\left(\lambda^+,\{y_s\}\right)\\ &= \Delta_n^1\left(\{y_s\}\right) - \frac{\tau_1}{N}y_{n-1} + \left(1-\frac{\tau_2}{N}\right)\left(\Delta_{n-1}^1\left(\{y_s\}\right) - \frac{\tau_1}{N}y_{n-2}\right)\\ &= \Delta_n^2\left(\{y_s\}\right) - \frac{\tau_1+\tau_2}{N}\Delta_{n-1}^1\left(\{y_s\}\right) + \frac{\tau_1\tau_2}{N^2}\Delta_{n-2}^0\left(\{y_s\}\right).\end{aligned} \tag{2.50}$$

In general, we can prove by the mathematical induction the following expansion ([74])

$$\Delta_n^p\left(\lambda^+,\{y_s\}\right) = \sum_{t=0}^{p}\gamma_t(\lambda^+)y_{n-t} = \sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\Delta_{n-t}^{p-t}\left(\{y_s\}\right). \tag{2.51}$$

Now, let us modify (2.46) in view of (2.36). According to (2.44) and (2.51), we get (note that $\lambda^+ = \lambda^-$)

$$\delta_n^p\left(\lambda,\{y_s\}\right) = \sum_{k=0}^{p}\gamma_k(\lambda^+)\sum_{t=0}^{p}\gamma_t(\lambda^-)y_{n+k-t} = \sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}\gamma_k(\lambda^+)\Delta_{n+k-t}^{p-t}\left(\{y_s\}\right). \tag{2.52}$$

Similar to (2.51), we can show that

$$\sum_{k=0}^{p}\gamma_k(\lambda^+)y_{n+k} = \sum_{k=0}^{p}(-1)^k\frac{\gamma_k(\tau)}{N^k}\Delta_{n+p}^{p-k}\left(\{y_s\}\right). \tag{2.53}$$

Then, (2.52) implies

$$\delta_n^p\left(\lambda,\{y_s\}\right) = \sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^k\frac{\gamma_k(\tau)}{N^k}\Delta_{n+p}^{p-k}\left(\left\{\Delta_{\ell-t}^{p-t}\left(\{y_s\}\right)\right\}\right). \tag{2.54}$$

This leads to the following needed expansion

$$\delta_n^p\left(\lambda,\{y_s\}\right) = \sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^k\frac{\gamma_k(\tau)}{N^k}\Delta_{n+p-t}^{2p-k-t}\left(\{y_s\}\right) \tag{2.55}$$

as

$$\Delta_{n+p}^{p-k}\left(\left\{\Delta_{\ell-t}^{p-t}\left(\{y_s\}\right)\right\}\right) = \Delta_{n+p-t}^{2p-k-t}\left(\{y_s\}\right). \tag{2.56}$$

Then

$$\delta_n^w\left(\{\delta_\ell^p\left(\lambda,\{y_s\}\right)\}\right)=\sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^k\frac{\gamma_k(\tau)}{N^k}\Delta_{n+w}^{2w}\left(\left\{\Delta_{\ell+p-t}^{2p-k-t}\left(\{y_s\}\right)\right\}\right). \tag{2.57}$$

Taking into account that

$$\Delta_{n+w}^{2w}\left(\left\{\Delta_{\ell+p-t}^{2p-k-t}\left(\{y_s\}\right)\right\}\right)=\Delta_{n+w+p-t}^{2w+2p-k-t}\left(\{y_s\}\right), \tag{2.58}$$

we work out that

$$\delta_n^w\left(\{\delta_\ell^p\left(\lambda,\{y_s\}\right)\}\right)=\sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^k\frac{\gamma_k(\tau)}{N^k}\Delta_{n+w+p-t}^{2w+2p-k-t}\left(\{y_s\}\right). \tag{2.59}$$

We will frequently use the latest formula.

Let

$$\psi_{p,m,j}^{\pm}(\tau)=\sum_{t=0}^{p}(-1)^t\gamma_t(\tau)\sum_{k=0}^{p}\gamma_k(\tau)(2p-k-t+j)!\sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r\pm 1)^{2p-k-t+j+1}}. \tag{2.60}$$

Now, we prove some lemmas.

Lemma 2.1 *[59] Let λ_k be chosen as in* (2.36)*, and $\beta\in\mathbb{R}$ be a constant. Then, the following estimate holds for $n\in\mathbb{Z}$, $p,w\geq 0$*

$$\begin{aligned}\delta_n^w\left(\left\{\delta_\ell^p\left(\lambda,\left\{(-1)^s e^{\frac{i\pi\beta s}{2N+m+1}}\right\}_{s=-\infty}^{\infty}\right)\right\}_{\ell=-\infty}^{\infty}\right)&=\left(\frac{\pi\beta}{2}\right)^{2w}\frac{(-1)^n e^{\frac{i\pi\beta n}{2N+m+1}}}{N^{2w+2p}}h_p(\beta,\tau)\\&\quad+O(N^{-2w-2p-1}),\ N\to\infty,\end{aligned} \tag{2.61}$$

where

$$h_p(\beta,\tau)=\left(\frac{\pi\beta}{2}\right)^{2p}\sum_{t=0}^{p}\gamma_t(\tau)\sum_{k=0}^{p}(-1)^k\gamma_k(\tau)\left(\frac{2}{i\pi\beta}\right)^{k+t}. \tag{2.62}$$

Proof. From (2.59), we have

$$\begin{aligned}\delta_n^w\left(\left\{\delta_\ell^p\left(\lambda,\left\{(-1)^s e^{\frac{i\pi\beta s}{2N+m+1}}\right\}\right)\right\}\right)&=\sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^k\frac{\gamma_k(\tau)}{N^k}\\&\quad\times\Delta_{n+w+p-t}^{2w+2p-k-t}\left(\left\{(-1)^s e^{\frac{i\pi\beta s}{2N+m+1}}\right\}\right).\end{aligned} \tag{2.63}$$

Then, in view of (1.62)

$$\begin{aligned}\Delta_{n+w+p-t}^{2w+2p-k-t}\left(\left\{(-1)^s e^{\frac{i\pi\beta s}{2N+m+1}}\right\}\right)&=\frac{(-1)^{n+w+p+t}(i\pi\beta)^{2w+2p-k-t}}{(2N+m+1)^{2w+2p-k-t}}e^{\frac{i\pi\beta(n+w+p-t)}{2N+m+1}}\\&\quad+O\left(N^{-2w-2p+k+t-1}\right)\\&=(-1)^{n+w+p+t}e^{\frac{i\pi\beta n}{2N+m+1}}\left(\frac{i\pi\beta}{2N}\right)^{2w+2p-k-t}+O(N^{-2w-2p+k+t-1})\end{aligned} \tag{2.64}$$

which completes the proof together with (2.63). □

Lemma 2.2 *[59] Let λ_k be chosen as in* (2.36)*. Then, the following estimate holds for $p, w, j \geq 0$ as $N \to \infty$*

$$\delta_{\pm N}^{w}\left(\left\{\delta_{\ell}^{p}\left(\lambda,\left\{\sum_{r=-\infty}^{\infty}B_{s+r(2N+m+1)}(j)\right\}_{s=-\infty}^{\infty}\right)\right\}_{\ell=-\infty}^{\infty}\right)=\frac{(-1)^{N+p+w+1}}{2(i\pi N)^{j+1}N^{2w+2p}j!} \times \psi^{\pm}_{p,m,j+2w}(\tau)+O(N^{-2w-2p-j-2}), \tag{2.65}$$

where $\psi^{\pm}_{p,m,j+2w}(\tau)$ is defined by (2.60)

Proof. Equation (2.59) claims that

$$\delta_{\pm N}^{w}\left(\left\{\delta_{\ell}^{p}\left(\lambda,\left\{\sum_{r=-\infty}^{\infty}B_{s+r(2N+m+1)}(j)\right\}\right)\right\}\right)=\sum_{t=0}^{p}(-1)^{t}\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^{k}\frac{\gamma_k(\tau)}{N^k} \times \Delta^{2w+2p-k-t}_{\pm N+w+p-t}\left(\left\{\sum_{r=-\infty}^{\infty}B_{s+r(2N+m+1)}(j)\right\}\right). \tag{2.66}$$

Then in view of (1.67)

$$\Delta^{2w+2p-k-t}_{\pm N+w+p-t}\left(\left\{\sum_{r=-\infty}^{\infty}B_{s+r(2N+m+1)}(j)\right\}\right)=\frac{(-1)^{N+p-k+w+1}(2w+2p-k-t+j)!}{2(i\pi N)^{j+1}N^{2w+2p-k-t}j!} \times \sum_{r=-\infty}^{\infty}\frac{(-1)^{r(m+1)}}{(2r\pm 1)^{2w+2p-k-t+j+1}}+O\left(N^{-j-2w-2p+k+t-2}\right). \tag{2.67}$$

Finally, substitution of (2.67) into (2.66) completes the proof. □

Lemma 2.3 *[59] Let λ_k be chosen as in* (2.36) *and $p, w, j \geq 0$. Then, the following estimate holds for $|n| \leq N$ as $N \to \infty$*

$$\delta_{n}^{w}\left(\left\{\delta_{\ell}^{p}\left(\lambda,\left\{\sum_{r\neq 0}B_{s+r(2N+m+1)}(j)\right\}_{s=-\infty}^{\infty}\right)\right\}_{\ell=-\infty}^{\infty}\right)=\frac{(-1)^{n+w+p+1}}{2(i\pi N)^{j+1}N^{2w+2p}j!} \times \sum_{t=0}^{p}(-1)^{t}\gamma_t(\tau)\sum_{k=0}^{p}\gamma_k(\tau)\sum_{r\neq 0}\frac{(-1)^{r(m+1)}(2w+2p-k-t+j)!}{\left(2r+\frac{2n}{2N+m+1}\right)^{2w+2p-k-t+j+1}} + O\left(N^{-2w-2p-j-2}\right). \tag{2.68}$$

Proof. Equation (2.59) implies

$$\delta_{n}^{w}\left(\left\{\delta_{\ell}^{p}\left(\lambda,\left\{\sum_{r\neq 0}B_{s+r(2N+m+1)}(j)\right\}\right)\right\}\right)=\sum_{t=0}^{p}(-1)^{t}\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^{k}\frac{\gamma_k(\tau)}{N^k} \times \Delta^{2w+2p-k-t}_{n+w+p-t}\left(\left\{\sum_{r\neq 0}B_{s+r(2N+m+1)}(j)\right\}\right). \tag{2.69}$$

In view of (1.71)

$$\Delta_{n+w+p-t}^{2w+2p-k-t}\left(\left\{\sum_{r\neq 0} B_{s+r(2N+m+1)}(j)\right\}\right) = \frac{(-1)^{n+w+p+k+1}(2w+2p-k-t+j)!}{2(i\pi N)^{j+1}N^{2w+2p-k-t}j!} \times \sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{\left(2r+\frac{2(n+w+p-t)}{2N+m+1}\right)^{2w+2p-k-t+j+1}} + O\left(N^{-j-2w-2p+k+t-2}\right) \tag{2.70}$$

which completes the proof together with (2.69). □

Lemma 2.4 *[59] Let λ_k be chosen as in* (2.36)*. Then, the following estimate holds*

$$\delta_n^w\left(\left\{\delta_\ell^p\left(\lambda,\{B_s(j)\}_{s=-\infty}^{\infty}\right)\right\}_{\ell=-\infty}^{\infty}\right) = \frac{(-1)^{n+w+p+1}}{2(i\pi n)^{j+1}n^{2w+2p}j!}\sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t n^{-t}}\sum_{k=0}^{p}\frac{\gamma_k(\tau)}{N^k n^{-k}} \times (2w+2p-k-t+j)! + O\left(N^{-2p}n^{-2w-j-2}\right),\ |n|>N,\ p,w,j\geq 0,\ N\to\infty. \tag{2.71}$$

Proof. Equation (2.59) implies

$$\delta_n^w\left(\{\delta_\ell^p\left(\lambda,\{B_s(j)\}\right)\}\right) = \sum_{t=0}^{p}(-1)^t\frac{\gamma_t(\tau)}{N^t}\sum_{k=0}^{p}(-1)^k\frac{\gamma_k(\tau)}{N^k}\Delta_{n+w+p-t}^{2w+2p-k-t}\left(\{B_s(j)\}\right). \tag{2.72}$$

According to (1.74)

$$\Delta_{n+w+p-t}^{2w+2p-k-t}\left(\{B_s(j)\}\right) = \frac{(-1)^{n+w+p+k+1}}{2(i\pi n)^{j+1}n^{2w+2p-k-t}}\frac{(2w+2p-k-t+j)!}{j!} + O\left(n^{-2w-2p+k+t-j-2}\right) \tag{2.73}$$

which completes the proof together with (2.72). □

2.3.2 Asymptotic Estimates

Lemma 2.5 *[59] Let $f^{(q+2p+m)}\in AC[-1,1]$ for some $q\geq 0$, $p,m\geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0,\ k=0,\ldots,q-1. \tag{2.74}$$

Let parameters λ_k be chosen as in (2.36)*. Then, the following estimates hold as $N\to\infty$*

$$\delta_{\pm N}^p\left(\lambda,\{F_{n,m}\}_{n=-\infty}^{\infty}\right) = \frac{(-1)^{N+1}}{2^{q+1}N^{q+2p+1}}\sum_{k=0}^{q}\frac{A_{kq}(f)(m+1)^{q-k}2^k}{(q-k)!(i\pi)^{k+1}} \times\left[(-1)^p\frac{\psi_{p,m,k}^{\pm}(\tau)}{k!} - \sum_{t=0}^{m-1}\frac{\Phi_{k,m}^{(t)}(-1)}{t!}\sum_{\mu=0}^{t}\binom{t}{\mu}e^{\pm\frac{i\pi N(2\mu-m+1)}{2N+m+1}}h_p(2\mu-m+1,\tau)\right] + O\left(N^{-q-2p-2}\right), \tag{2.75}$$

$$\delta^p_{\pm(N+1)}\left(\lambda, \{F_{n,m}\}_{n=-\infty}^{\infty}\right) = -\delta^p_{\pm N}\left(\lambda, \{F_{n,m}\}_{n=-\infty}^{\infty}\right) + O\left(N^{-q-2p-2}\right) \tag{2.76}$$

and

$$\delta^p_{-N}\left(\lambda, \{F_{n,m}\}_{n=-\infty}^{\infty}\right) = (-1)^{m+1}\delta^p_{N}\left(\lambda, \{F_{n,m}\}_{n=-\infty}^{\infty}\right) + O\left(N^{-q-2p-2}\right). \tag{2.77}$$

Proof. We proceed as in the proof of Lemma 1.6 and derive

$$\begin{aligned} F_{n,m} = &\sum_{j=q}^{q+2p+m} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(2N+m+1)^k}{(j-k)!(m+1)^{k-j}} \sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(k) \\ &+ \frac{(-1)^n e^{-\frac{i\pi n(m-1)}{2N+m+1}}}{2N+m+1} \sum_{j=q}^{q+2p+m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \left[\sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \left(e^{\frac{2i\pi n}{2N+m+1}} + 1\right)^t \right. \\ &\left. + \sum_{t=m}^{m+2p} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \sum_{\ell=1}^{m} \left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}} + 1\right)^t \sum_{s=1}^{m} v^{-1}_{\ell,s} e^{\frac{2i\pi n(s-1)}{2N+m+1}} \right] + o\left(N^{-q-2p-2}\right). \end{aligned} \tag{2.78}$$

Then,

$$\begin{aligned} \delta^p_{\pm N}\left(\lambda, \{F_{n,m}\}\right) = &\sum_{j=q}^{q+2p+m} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(2N+m+1)^k}{(j-k)!(m+1)^{k-j}} \\ &\times \delta^p_{\pm N}\left(\lambda, \left\{\sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(k)\right\}\right) \\ &+ \frac{1}{2N+m+1} \sum_{j=q}^{q+2p+m} \frac{1}{N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!} \\ &\times \left[\sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} \delta^p_{\pm N}\left(\lambda, \left\{(-1)^n e^{\frac{i\pi n(2\mu-m+1)}{2N+m+1}}\right\}\right) \right. \\ &\left. + \sum_{t=m}^{m+2p} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \sum_{\ell=1}^{m} \left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}} + 1\right)^t \sum_{s=1}^{m} v^{-1}_{\ell,s} \delta^p_{\pm N}\left(\lambda, \left\{(-1)^n e^{\frac{i\pi n(2s-1-m)}{2N+m+1}}\right\}\right) \right] \\ &+ o\left(N^{-q-2p-2}\right). \end{aligned} \tag{2.79}$$

Lemmas 2.1 and 2.2 provide estimates

$$\begin{aligned} \delta^p_{\pm N}\left(\lambda, \left\{(-1)^n e^{\frac{i\pi n(2\mu-m+1)}{2N+m+1}}\right\}\right) = &\frac{(-1)^N e^{\pm\frac{i\pi N(2\mu-m+1)}{2N+m+1}}}{N^{2p}} h_p(2\mu-m+1,\tau) \\ &+ O(N^{-2p-1}), \end{aligned} \tag{2.80}$$

and

$$\begin{aligned} \delta^p_{\pm N}\left(\lambda, \left\{\sum_{r=-\infty}^{\infty} B_{n+r(2N+m+1)}(k)\right\}\right) = &\frac{(-1)^{N+p+1}}{2(i\pi N)^{k+1} N^{2p} k!} \psi^{\pm}_{p,m,k}(\tau) \\ &+ O(N^{-2p-k-2}). \end{aligned} \tag{2.81}$$

Substitution of (2.80) and (2.81) into (2.79) completes the proof.

Estimates (2.76) and (2.77) can be proved similarly. □

Theorem 2.4 *[59] Let $f^{(q+2p+m)} \in AC[-1,1]$ for some $q \ge 0$, $p, m \ge 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ \ k = 0, \dots, q-1. \tag{2.82}$$

Let parameters λ_k be chosen as in (2.36). Then, the following estimate holds for $|x| < 1$

$$R_{N,m}^p(f,x) = \frac{D_{N,m}^p(f,x)}{N^{q+2p+1}} + o(N^{-q-2p-1}), \ N \to \infty. \tag{2.83}$$

where

$$\begin{aligned} D_{N,m}^p(f,x) = & \frac{(-1)^N}{2^{q+2p+1}} \frac{\sin\left(\pi\sigma(N+\frac{1}{2})x - \frac{\pi m}{2}\right)}{\cos^{2p+1}\frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(f)(m+1)^{q-k}2^k}{(q-k)! i^{k-m} \pi^{k+1}} \\ & \times \left(\frac{(-1)^p}{k!} \psi_{p,m,k}^+(\tau) - \sum_{t=0}^{m-1} \frac{\Phi_{k,m}^{(t)}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu - m + 1, \tau) \right), \end{aligned} \tag{2.84}$$

and $\psi_{p,m,k}^+$ is defined by (2.60), $\Phi_{k,m}$ by (1.87), and h_p by (2.62).

Proof. Application of the Abel transformation to (2.35) concludes

$$\begin{aligned} R_{N,m}^p(f,x) = & e^{-i\pi\sigma N x} \frac{\delta_{-N-1}^p(\lambda, \{F_{s,m}\})}{c(x)} - e^{i\pi\sigma(N+1)x} \frac{\delta_N^p(\lambda, \{F_{s,m}\})}{c(x)} \\ & + e^{i\pi\sigma N x} \frac{\delta_{N+1}^p(\lambda, \{F_{s,m}\})}{c(x)} - e^{-i\pi\sigma(N+1)x} \frac{\delta_{-N}^p(\lambda, \{F_{s,m}\})}{c(x)} \\ & + \frac{1}{c(x)} \sum_{|n| \le N} \delta_n^1(\{\delta_\ell^p(\lambda, \{f_s^* - F_{s,m}\})\}) e^{i\pi\sigma n x} \\ & + \frac{1}{c(x)} \sum_{|n| > N} \delta_n^1(\{\delta_\ell^p(\lambda, \{f_s^*\})\}) e^{i\pi\sigma n x}, \end{aligned} \tag{2.85}$$

where

$$c(x) = 4\cos^2\left(\frac{\pi\sigma x}{2}\right) \prod_{j=1}^{p} \left(1 + \lambda_{-j} e^{i\pi\sigma x}\right)\left(1 + \lambda_j e^{-i\pi\sigma x}\right) \tag{2.86}$$

and

$$\lim_{N\to\infty} c(x) = 2^{2p+2} \cos^{2p+2}\left(\frac{\pi x}{2}\right). \tag{2.87}$$

According to estimate (1.80) of Lemma 1.5, we figure out that

$$\begin{aligned} \delta_n^1(\{\delta_\ell^p(\lambda, \{f_s^*\})\}) = & \sum_{j=q}^{q+2p+m} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(m+1)^{j-k}(2N+m+1)^k}{(j-k)!} \\ & \times \delta_n^1(\{\delta_\ell^p(\lambda, \{B_s(k)\})\}) + o(n^{-q-2p-m-1}). \end{aligned} \tag{2.88}$$

Lemma 2.4 shows that

$$\delta_n^1\left(\{\delta_\ell^p\left(\lambda,\{B_s(k)\}\right)\}\right)=O\left(N^{-2p}n^{-k-3}\right) \tag{2.89}$$

and consequently

$$\delta_n^1\left(\{\delta_\ell^p\left(\lambda,\{f_s^*\}\right)\}\right)=O(N^{-q-2p}n^{-3})+o(n^{-q-2p-m-1}),\ \ |n|>N,\ \ N\to\infty. \tag{2.90}$$

Hence, the last term in the right-hand side of (2.85) is $o\left(N^{-q-2p-1}\right)$.

Similarly, according to estimate (1.99) of Lemma 1.6

$$\begin{aligned}
\delta_n^1\left(\{\delta_\ell^p\left(\lambda,\{F_{s,m}-f_s^*\}\right)\}\right)&=\sum_{j=q}^{q+2p+m}\frac{1}{2^jN^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}(2N+m+1)^k}{(j-k)!}\\
&\times\delta_n^1\left(\left\{\delta_\ell^p\left(\lambda,\left\{\sum_{r\neq 0}B_{s+r(2N+m+1)}(k)\right\}\right)\right\}\right)\\
&+\frac{1}{2N+m+1}\sum_{j=q}^{q+2p+m}\frac{1}{N^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(m+1)^{j-k}}{2^{j-k}(i\pi)^{k+1}(j-k)!}\\
&\times\left[\sum_{t=0}^{m-1}\frac{\Phi_{k,m}^{(t)}(-1)}{t!}\sum_{\mu=0}^{t}\binom{t}{\mu}\delta_n^1\left(\left\{\delta_\ell^p\left(\lambda,\left\{(-1)^se^{\frac{i\pi s(2\mu-m+1)}{2N+m+1}}\right\}\right)\right\}\right)\right.\\
&+\sum_{t=m}^{q-j+m+2p}\frac{\Phi_{k,m}^{(t)}(-1)}{t!}\sum_{\ell=1}^{m}\left(e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1\right)^t\\
&\left.\times\sum_{\tau=1}^{m}v_{\ell,\tau}^{-1}\delta_n^1\left(\left\{\delta_r^p\left(\lambda,\left\{(-1)^se^{\frac{i\pi s(2\tau-1-m)}{2N+m+1}}\right\}\right)\right\}\right)\right]+o\left(N^{-q-2p-2}\right).
\end{aligned} \tag{2.91}$$

Then, in view of Lemma 2.3

$$\delta_n^1\left(\left\{\delta_\ell^p\left(\lambda,\left\{\sum_{r\neq 0}B_{s+r(2N+m+1)}(k)\right\}\right)\right\}\right)=O\left(N^{-2p-k-3}\right). \tag{2.92}$$

According to Lemma 2.1

$$\delta_n^1\left(\left\{\delta_\ell^p\left(\lambda,\left\{(-1)^se^{\frac{i\pi s(2\mu-m+1)}{2N+m+1}}\right\}\right)\right\}\right)=O\left(N^{-2p-2}\right). \tag{2.93}$$

Finally, taking into account that $e^{\frac{2i\pi(N+\ell)}{2N+m+1}}+1=O(1/N)$, we get

$$\delta_n^1\left(\{\delta_\ell^p\left(\lambda,\{F_{s,m}-f_s^*\}\right)\}\right)=o\left(N^{-q-2p-2}\right),\ |n|\le N,\ N\to\infty. \tag{2.94}$$

Therefore, the fifth term of (2.85) is also $o\left(N^{-q-2p-1}\right)$.

As a result, we have

$$
\begin{aligned}
c(x)R^p_{N,m}(f,x) &= e^{-i\pi\sigma Nx}\delta^p_{-N-1}\left(\lambda,\{F_{s,m}\}\right)) - e^{i\pi\sigma(N+1)x}\delta^p_N\left(\lambda,\{F_{s,m}\}\right) \\
&+ e^{i\pi\sigma Nx}\delta^p_{N+1}\left(\lambda,\{F_{s,m}\}\right) - e^{-i\pi\sigma(N+1)x}\delta^p_{-N}\left(\lambda,\{F_{s,m}\}\right) \\
&+ o\left(N^{-q-2p-1}\right).
\end{aligned} \tag{2.95}
$$

Estimates (2.76) and (2.77) provide

$$
\begin{aligned}
c(x)R^p_{N,m}(f,x) = \delta^p_N\left(\lambda,\{F_{s,m}\}\right)&\left((-1)^m\left(e^{-i\pi\sigma Nx} + e^{-i\pi\sigma(N+1)x}\right)\right. \\
&\left.- \left(e^{i\pi\sigma Nx} + e^{i\pi\sigma(N+1)x}\right)\right) + o\left(N^{-q-2p-1}\right)
\end{aligned} \tag{2.96}
$$

which concludes the proof in view of Lemma 2.5. □

We have the same estimate for $m = 0$ with additional smoothness for f in the next theorem.

Theorem 2.5 *[59] Let $f^{(q+2p+1)} \in AC[-1,1]$ for some $q \geq 0$, $p \geq 1$ and*

$$
f^{(k)}(-1) = f^{(k)}(1) = 0, \quad k = 0,\dots,q-1. \tag{2.97}
$$

Let parameters λ_k be chosen as in (2.36). Then, the following estimate holds for $|x| < 1$

$$
R^p_{N,0}(f,x) = \frac{D^p_{N,0}(f,x)}{N^{q+2p+1}} + o\left(N^{-q-2p-1}\right), \ N \to \infty, \tag{2.98}
$$

where

$$
D^p_{N,0}(f,x) = \frac{(-1)^{N+p}}{2^{q+2p+1}}\frac{\sin(\pi Nx)}{\cos^{2p+1}\frac{\pi x}{2}}\sum_{k=0}^{q}\frac{A_{kq}(f)2^k}{i^k\pi^{k+1}(q-k)!k!}\psi^+_{p,0,k}(\tau) \tag{2.99}
$$

and $\psi^+_{p,0,k}(\tau)$ is defined by (2.60).

Proof. We use (2.85) for $m = 0$

$$
\begin{aligned}
R^p_{N,0}(f,x) &= e^{-i\pi\sigma Nx}\frac{\delta^p_{-N-1}\left(\lambda,\{F_{s,0}\}\right)}{c(x)} - e^{i\pi\sigma(N+1)x}\frac{\delta^p_N\left(\lambda,\{F_{s,0}\}\right)}{c(x)} \\
&+ e^{i\pi\sigma Nx}\frac{\delta^p_{N+1}\left(\lambda,\{F_{s,0}\}\right)}{c(x)} - e^{-i\pi\sigma(N+1)x}\frac{\delta^p_{-N}\left(\lambda,\{F_{s,0}\}\right)}{c(x)} \\
&+ \frac{1}{c(x)}\sum_{n=-N}^{N}\delta^1_n\left(\{\delta^p_\ell\left(\lambda,\{f^*_s - F_{s,0}\}\right)\}\right)e^{i\pi\sigma nx} \\
&+ \frac{1}{c(x)}\sum_{|n|>N}\delta^1_n\left(\{\delta^p_\ell\left(\lambda,\{f^*_s\}\right)\}\right)e^{i\pi\sigma nx}.
\end{aligned} \tag{2.100}
$$

Periodicity of $F_{n,0}$ ($F_{n+r(2N+1),0} = F_{n,0}$, $r \in \mathbb{Z}$) yields

$$\begin{aligned} \delta^p_{-N-1}\left(\lambda, \{F_{s,0}\}\right) &= \delta^p_{N}\left(\lambda, \{F_{s,0}\}\right), \\ \delta^p_{N+1}\left(\lambda, \{F_{s,0}\}\right) &= \delta^p_{-N}\left(\lambda, \{F_{s,0}\}\right). \end{aligned} \tag{2.101}$$

Hence

$$\begin{aligned} R^p_{N,0}(f,x) = {} & \delta^p_{N}\left(\lambda, \{F_{s,0}\}\right) \frac{e^{-i\pi\sigma N x} - e^{i\pi\sigma(N+1)x}}{c(x)} \\ & + \delta^p_{-N}\left(\lambda, \{F_{s,0}\}\right) \frac{e^{i\pi\sigma N x} - e^{-i\pi\sigma(N+1)x}}{c(x)} \\ & + \frac{1}{c(x)} \sum_{n=-N}^{N} \delta^1_n\left(\{\delta^p_\ell\left(\lambda, \{f^*_s - F_{s,0}\}\right)\}\right) e^{i\pi\sigma n x} \\ & + \frac{1}{c(x)} \sum_{|n|>N} \delta^1_n\left(\{\delta^p_\ell\left(\lambda, \{f^*_s\}\right)\}\right) e^{i\pi\sigma n x}. \end{aligned} \tag{2.102}$$

Estimate (1.80) of Lemma 1.5, with $v = 2p+1$, figure out that

$$\begin{aligned} \delta^1_n\left(\{\delta^p_\ell\left(\lambda, \{f^*_s\}\right)\}\right) = {} & \sum_{j=q}^{q+2p+1} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(2N+1)^k}{(j-k)!} \delta^1_n\left(\{\delta^p_\ell\left(\lambda, \{B_s(k)\}\right)\}\right) \\ & + o\left(n^{-q-2p-2}\right). \end{aligned} \tag{2.103}$$

Then, from Lemma 2.4

$$\delta^1_n\left(\{\delta^p_\ell\left(\lambda, \{B_s(k)\}\right)\}\right) = O(N^{-2p} n^{-k-3}) \tag{2.104}$$

and from (2.103)

$$\delta^1_n\left(\{\delta^p_\ell\left(\lambda, \{f^*_s\}\right)\}\right) = O(N^{-q-2p} n^{-3}) + o(n^{-q-2p-2}),\ |n| > N,\ N \to \infty, \tag{2.105}$$

which shows that the last term in the right-hand side of (2.102) is $o\left(N^{-q-2p-1}\right)$ as $N \to \infty$.

Similarly, estimate (1.166) of Lemma 1.8, with $v = 2p+1$, implies

$$\begin{aligned} \delta^1_n\left(\{\delta^p_\ell\left(\lambda, \{F_{s,0} - f^*_s\}\right)\}\right) = {} & \sum_{j=q}^{q+2p+1} \frac{1}{2^j N^j} \sum_{k=0}^{j} \frac{A_{kj}(f)(2N+1)^k}{(j-k)!} \\ & \times \delta^1_n\left(\left\{\delta^p_\ell\left(\lambda, \left\{\sum_{r\neq 0} B_{s+r(2N+1)}(k)\right\}\right)\right\}\right) \\ & + o\left(N^{-q-2p-2}\right). \end{aligned} \tag{2.106}$$

In view of Lemma 2.3

$$\delta^1_n\left(\left\{\delta^p_\ell\left(\lambda, \left\{\sum_{r\neq 0} B_{s+r(2N+1)}(k)\right\}\right)\right\}\right) = O\left(N^{-2p-k-3}\right) \tag{2.107}$$

and

$$\delta_n^1\left(\{\delta_\ell^p\left(\lambda,\{F_{s,0}-f_s^*\}\right)\}\right)=o\left(N^{-q-2p-2}\right),\ |n|\le N,\ N\to\infty \tag{2.108}$$

which shows that the third term is also $o\left(N^{-q-2p-1}\right)$ as $N\to\infty$.

Therefore,

$$\begin{aligned}R_{N,0}^p(f,x)&=\delta_N^p\left(\lambda,\{F_{s,0}\}\right)\frac{e^{-i\pi\sigma Nx}-e^{i\pi\sigma(N+1)x}}{c(x)}\\&+\delta_{-N}^p\left(\lambda,\{F_{s,0}\}\right)\frac{e^{i\pi\sigma Nx}-e^{-i\pi\sigma(N+1)x}}{c(x)}\\&+o\left(N^{-q-2p-1}\right).\end{aligned} \tag{2.109}$$

Equation (1.100) shows that

$$F_{n,0}=\sum_{r=-\infty}^{\infty}f_{n+r(2N+1)}^*,\ n\in\mathbb{Z}. \tag{2.110}$$

Applying Lemma 1.5, with $v=2p+1$, we get $(n,N\to\infty)$

$$F_{n,0}=\sum_{j=q}^{q+2p+1}\frac{1}{2^jN^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(2N+1)^k}{(j-k)!}\sum_{r=-\infty}^{\infty}B_{n+r(2N+1)}(k)+o\left(N^{-q-2p-2}\right), \tag{2.111}$$

and

$$\begin{aligned}\delta_{\pm N}^p\left(\lambda,\{F_{n,0}\}\right)&=\sum_{j=q}^{q+2p+1}\frac{1}{2^jN^j}\sum_{k=0}^{j}\frac{A_{kj}(f)(2N+1)^k}{(j-k)!}\\&\times\delta_{\pm N}^p\left(\lambda,\left\{\sum_{r=-\infty}^{\infty}B_{n+r(2N+1)}(k)\right\}\right)+o\left(N^{-q-2p-2}\right).\end{aligned} \tag{2.112}$$

Then, Lemma 2.2 with $w=0$ implies

$$\delta_{\pm N}^p\left(\lambda,F_{n,0}\right)=\frac{(-1)^{N+p+1}}{2^{q+1}N^{q+2p+1}}\sum_{k=0}^{q}\frac{A_{kq}(f)2^k}{(i\pi)^{k+1}(q-k)!k!}\psi_{p,0,k}^{\pm}(\tau)+O\left(N^{-q-2p-2}\right) \tag{2.113}$$

which completes the proof in view of (2.109) as $\psi_{p,0,k}^-(\tau)=-\psi_{p,0,k}^+(\tau)$. $\square$

Comparison with Theorems 1.6 and 1.7 asserts that if interpolated function has enough smoothness then the QPR approximation, with $2p>m$, is more accurate (asymptotically) than the QP interpolation and improvement is by factor $O(N^{-2p+m})$.

When $q=0$ then $D_{N,m}^p(f,x)$ can be rewritten in the form

$$D_{N,m}^p(f,x)=A_0(f)D_{N,m}^{p,*}(x), \tag{2.114}$$

where $D^{p,*}_{N,m}(x)$ is independent of f and

$$
\begin{aligned}
D^{p,*}_{N,m}(x) &= i^m \frac{(-1)^N}{\pi 2^{2p+1}} \frac{\sin\left(\pi\sigma(N+\frac{1}{2})x - \frac{\pi m}{2}\right)}{\cos^{2p+1}\frac{\pi x}{2}} \\
&\times \left((-1)^p \psi^{+}_{p,m,0}(\tau) - \sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{0,m}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu-m+1,\tau)\right).
\end{aligned} \tag{2.115}
$$

Let us show that the expression in the brackets vanishes for odd m independently of the choice of parameters τ_k which lead to more accurate approximation as in the next theorem.

Theorem 2.6 *Let $f^{(2p+m)} \in AC[-1,1]$ for some $p \geq 1$, odd $m \geq 1$ and*

$$f(-1) \neq f(1). \tag{2.116}$$

Let parameters λ_k be chosen as in (2.36)*. Then, the following estimate holds for $|x| < 1$*

$$R^p_{N,m}(f,x) = o(N^{-2p-1}), \ \ N \to \infty. \tag{2.117}$$

Proof. Let m be odd and $n = (m-1)/2$. We apply Theorem 2.4 with $q = 0$ and get estimate (2.83) with $D^p_{N,m}$ written as in (2.114). We show that the expression in the brackets vanishes for odd m.

Let us show that $\psi^{+}_{p,m,0}(\tau) = 0$ for odd m. Observing that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(2r+1)^{2s+1}} = 0, \tag{2.118}$$

we get

$$
\begin{aligned}
\psi^{+}_{p,m,0}(\tau) &= \sum_{t=0}^{[\frac{p}{2}]} \gamma_{2t}(\tau) \sum_{k=0}^{p} \gamma_k(\tau)(2p-k-2t)! \sum_r \frac{1}{(2r+1)^{2p-k-2t+1}} \\
&- \sum_{t=0}^{[\frac{p+1}{2}]} \gamma_{2t+1}(\tau) \sum_{k=0}^{p} \gamma_k(\tau)(2p-k-2t-1)! \sum_r \frac{1}{(2r+1)^{2p-k-2t}} \\
&= \sum_{t=0}^{[\frac{p}{2}]} \gamma_{2t}(\tau) \sum_{k=0}^{[\frac{p+1}{2}]} \gamma_{2k+1}(\tau)(2p-2k-2t-1)! \sum_r \frac{1}{(2r+1)^{2p-2k-2t}} \\
&- \sum_{t=0}^{[\frac{p+1}{2}]} \gamma_{2t+1}(\tau) \sum_{k=0}^{[\frac{p}{2}]} \gamma_{2k}(\tau)(2p-2k-2t-1)! \sum_r \frac{1}{(2r+1)^{2p-2k-2t}} = 0.
\end{aligned} \tag{2.119}
$$

Then, we have $(2n = m-1)$

$$
\begin{aligned}
S &= \sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{0,m}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu-m+1,\tau) \\
&= \pi^{2p} \sum_{s=0}^{p} \gamma_s(\tau) \sum_{k=0}^{p} (-1)^k \gamma_k(\tau) \frac{1}{(i\pi)^{k+s}} \sum_{\mu=0}^{2n} (-1)^{\mu+n} (\mu-n)^{2p-k-s} c_\mu,
\end{aligned} \tag{2.120}
$$

where

$$c_\mu = \sum_{t=\mu}^{2n} \frac{\Phi_{0,m}^{(t)}(-1)}{t!} \binom{t}{\mu}, \ \mu = 0, \ldots, 2n. \tag{2.121}$$

We will prove that

$$c_n = 0 \tag{2.122}$$

and

$$c_{n+\ell} = -c_{n-\ell}, \ \ell = 1, \ldots, n. \tag{2.123}$$

In that case

$$\begin{aligned} S &= \pi^{2p} \sum_{s=0}^{p} \gamma_s(\tau) \sum_{k=0}^{p} (-1)^k \gamma_k(\tau) \frac{1}{(i\pi)^{k+s}} \sum_{\mu=0}^{n-1} (-1)^{\mu+n} (\mu - n)^{2p-k-s} c_\mu \\ &+ \pi^{2p} \sum_{s=0}^{p} \gamma_s(\tau) \sum_{k=0}^{p} (-1)^k \gamma_k(\tau) \frac{1}{(i\pi)^{k+s}} \sum_{\mu=n+1}^{2n} (-1)^{\mu+n} (\mu - n)^{2p-k-s} c_\mu \\ &= \pi^{2p} \sum_{s=0}^{p} (-1)^s \gamma_s(\tau) \sum_{k=0}^{p} \gamma_k(\tau) \frac{1}{(i\pi)^{k+s}} \sum_{\mu=1}^{n} (-1)^\mu \mu^{2p-k-s} c_{n-\mu} \\ &+ \pi^{2p} \sum_{s=0}^{p} \gamma_s(\tau) \sum_{k=0}^{p} (-1)^k \gamma_k(\tau) \frac{1}{(i\pi)^{k+s}} \sum_{\mu=1}^{n-1} (-1)^\mu \mu^{2p-k-s} c_{n+\mu} = 0 \end{aligned} \tag{2.124}$$

which will complete the proof.

Now, we prove properties (2.122) and (2.123). We use (1.96) and get

$$\begin{aligned} \Phi_{0,m}(x) &= \frac{i\pi}{2} x^n \left(1 + \frac{2}{x-1}\right) \\ &= \frac{\pi i}{2} x^n + \pi i \frac{x^n}{x-1} \\ &= \frac{\pi i}{2} x^n + \frac{\pi i}{x-1} + \pi i \sum_{\ell=1}^{n} \binom{n}{\ell} (x-1)^{\ell-1}, \end{aligned} \tag{2.125}$$

$$\frac{\Phi_{0,m}^{(t)}(x)}{t!} = \frac{\pi i}{2} \binom{n}{t} x^{n-t} + \pi i \frac{(-1)^t}{(x-1)^{t+1}} + \pi i \sum_{\ell=t}^{n-1} \binom{n}{\ell+1} \binom{\ell}{t} (x-1)^{\ell-t} \tag{2.126}$$

and

$$\frac{\Phi_{0,m}^{(t)}(-1)}{t!} = \frac{\pi i}{2} \binom{n}{t} (-1)^{n-t} - \pi i \frac{1}{2^{t+1}} + \pi i \sum_{\ell=t}^{n-1} \binom{n}{\ell+1} \binom{\ell}{t} (-2)^{\ell-t}. \tag{2.127}$$

We have

$$\begin{aligned} c_n &= \sum_{t=n}^{2n} \frac{\Phi_{0,m}^{(t)}(-1)}{t!} \binom{t}{n} \\ &= \frac{\pi i}{2} \sum_{t=n}^{2n} \binom{t}{n} \binom{n}{t} (-1)^{n-t} - \frac{\pi i}{2} \sum_{t=n}^{2n} \binom{t}{n} \frac{1}{2^t} \\ &= 0, \end{aligned} \tag{2.128}$$

where we used the identities ([77])

$$\sum_{t=0}^{n} \binom{t+n}{n} \frac{1}{2^t} = 2^n, \tag{2.129}$$

and

$$\sum_{k=m}^{n} (-1)^{k+n} \binom{n}{k} \binom{k}{m} = \delta_{mn}. \tag{2.130}$$

For the remaining, we need some identities. First, we use ([78])

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{m} x^k = \binom{n}{m} x^m (1+x)^{n-m} \tag{2.131}$$

Taking $x = -\frac{1}{2}$, we get

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{m} \frac{1}{2^k} = \binom{n}{m} \frac{(-1)^m}{2^n}. \tag{2.132}$$

Second, we use ([78])

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{r} x^k = (-1)^r \binom{n}{r} x^r (1-x)^{n-r}. \tag{2.133}$$

Integration over the interval $(0, 1)$ leads to the next identity

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k} \binom{k}{r} = (-1)^r \binom{n}{r} \frac{(n-r)!r!}{(n+1)!} = \frac{(-1)^r}{n+1}. \tag{2.134}$$

Now, let us prove that

$$(-1)^{n+\mu} \sum_{\ell=n-\mu}^{n-1} (-1)^{\ell} \binom{\ell}{n-\mu} \binom{n}{\ell+1} = 1. \tag{2.135}$$

We have

$$\begin{aligned}
(-1)^{n+\mu} \sum_{\ell=n-\mu}^{n-1} (-1)^{\ell} \binom{\ell}{n-\mu} \binom{n}{\ell+1} &= (-1)^{n+\mu}(n+1) \sum_{\ell=n-\mu}^{n} \frac{(-1)^{\ell}}{\ell+1} \binom{\ell}{n-\mu} \binom{n}{\ell} \\
&\quad - (-1)^{n+\mu} \sum_{\ell=n-\mu}^{n} (-1)^{\ell} \binom{\ell}{n-\mu} \binom{n}{\ell} \\
&= (-1)^{n+\mu}(n+1) \frac{(-1)^{n+\mu}}{n+1} - 0 \\
&= 1.
\end{aligned} \tag{2.136}$$

Third, we use ([78])

$$\sum_{t=0}^{n} \binom{t}{r} x^t = \frac{x^r}{(1-x)^{r+1}} + \binom{n}{r} x^n - x^n \sum_{t=0}^{r} \binom{n}{r-t} \frac{x^t}{(1-x)^{t+1}}. \tag{2.137}$$

Taking $x = 1/2$, we work out that

$$\sum_{t=0}^{n} \binom{t}{r} \frac{1}{2^{t+1}} = 1 + \binom{n}{r} \frac{1}{2^{n+1}} - \frac{1}{2^n} \sum_{t=0}^{r} \binom{n}{t}. \tag{2.138}$$

Then,

$$\begin{aligned} c_{n+\ell} &= \sum_{t=n+\ell}^{2n} \frac{\Phi_{0,m}^{(t)}(-1)}{t!} \binom{t}{n+\ell} \\ &= -\pi i \sum_{t=n+\ell}^{2n} \binom{t}{n+\ell} \frac{1}{2^{t+1}}, \end{aligned} \tag{2.139}$$

and

$$\begin{aligned} c_{n-\ell} &= \sum_{t=n-\ell}^{2n} \frac{\Phi_{0,m}^{(t)}(-1)}{t!} \binom{t}{n-\ell} \\ &= -\pi i \sum_{t=n-\ell}^{2n} \binom{t}{n-\ell} \frac{1}{2^{t+1}} \\ &+ \pi i \sum_{t=n-\ell}^{2n} \binom{t}{n-\ell} \sum_{k=t}^{n-1} (-2)^{k-t} \binom{n}{k+1} \binom{k}{t} \\ &= -\pi i \sum_{t=n-\ell}^{2n} \binom{t}{n-\ell} \frac{1}{2^{t+1}} \\ &+ \pi i \sum_{k=n-\ell}^{n-1} (-1)^k 2^k \binom{n}{k+1} \sum_{t=n-\ell}^{k} \frac{(-1)^t}{2^t} \binom{t}{n-\ell} \binom{k}{t} \\ &= -\pi i \sum_{t=n-\ell}^{2n} \binom{t}{n-\ell} \frac{1}{2^{t+1}} \\ &+ \pi i (-1)^{n+\ell} \sum_{k=n-\ell}^{n-1} (-1)^k \binom{n}{k+1} \binom{k}{n-\ell} \\ &= -\pi i \sum_{t=n-\ell}^{2n} \binom{t}{n-\ell} \frac{1}{2^{t+1}} + \pi i. \end{aligned} \tag{2.140}$$

Therefore

$$\begin{aligned} \frac{c_{n+\ell} + c_{n-\ell}}{\pi i} &= - \sum_{t=n+\ell}^{2n} \binom{t}{n+\ell} \frac{1}{2^{t+1}} - \sum_{t=n-\ell}^{2n} \binom{t}{n-\ell} \frac{1}{2^{t+1}} + 1 \\ &= - \left[1 + \binom{2n}{n+\ell} \frac{1}{2^{2n+1}} - \frac{1}{2^{2n}} \sum_{t=0}^{n+\ell} \binom{2n}{t} \right] \\ &- \left[1 + \binom{2n}{n-\ell} \frac{1}{2^{2n+1}} - \frac{1}{2^{2n}} \sum_{t=0}^{n-\ell} \binom{2n}{t} \right] + 1. \end{aligned} \tag{2.141}$$

Taking into account that

$$\sum_{t=0}^{2n} \binom{2n}{t} = 2^{2n}, \tag{2.142}$$

we conclude

$$\begin{aligned}
\frac{c_{n+\ell}+c_{n-\ell}}{\pi i} &= -1-\binom{2n}{n+\ell}\frac{1}{2^{2n+1}}+\frac{1}{2^{2n}}\sum_{t=0}^{2n}\binom{2n}{t}-\frac{1}{2^{2n}}\sum_{t=n+\ell+1}^{2n}\binom{2n}{t}\\
&-\binom{2n}{n-\ell}\frac{1}{2^{2n+1}}+\frac{1}{2^{2n}}\sum_{t=0}^{2n}\binom{2n}{t}-\frac{1}{2^{2n}}\sum_{t=n-\ell+1}^{2n}\binom{2n}{t}\\
&=1-\binom{2n}{n+\ell}\frac{1}{2^{2n+1}}-\frac{1}{2^{2n}}\sum_{t=n+\ell+1}^{2n}\binom{2n}{t}-\binom{2n}{n-\ell}\frac{1}{2^{2n+1}}\\
&-\frac{1}{2^{2n}}\sum_{t=n-\ell+1}^{2n}\binom{2n}{t} \qquad (2.143)\\
&=1-\binom{2n}{n+\ell}\frac{1}{2^{2n+1}}-\frac{1}{2^{2n}}\sum_{t=0}^{n-\ell-1}\binom{2n}{t}-\binom{2n}{n-\ell}\frac{1}{2^{2n+1}}\\
&-\frac{1}{2^{2n}}\sum_{t=n-\ell+1}^{2n}\binom{2n}{t}\\
&=-\binom{2n}{n+\ell}\frac{1}{2^{2n+1}}-\binom{2n}{n-\ell}\frac{1}{2^{2n+1}}+\frac{1}{2^{2n}}\binom{2n}{n-\ell}=0
\end{aligned}$$

which completes the proof of property (2.123). □

Remark 2.2 *Estimate of Theorem 2.6 can be improved to*

$$R^p_{N,2m+1}(f,x)=O(N^{-2p-2}),\ m\geq 0,\ N\to\infty \qquad (2.144)$$

if f has additional smoothness.

When $q=0$ but m is even or $q>0$, then additional accuracy can be achieved by appropriate selection of parameters τ_k by vanishing the constant in the brackets of (2.115) (if possible). We seek solution of the following system of non-linear equations

$$\begin{aligned}
&(-1)^p\frac{\psi^+_{p,m,k}(\tau)}{k!}-\sum_{t=0}^{m-1}\frac{\Phi^{(t)}_{k,m}(-1)}{t!}\sum_{\mu=0}^{t}\binom{t}{\mu}i^{2\mu-m+1}h_p(2\mu-m+1,\tau)=0,\\
&\qquad k=0,\dots,q,\ 2p>m
\end{aligned} \qquad (2.145)$$

with unknowns τ_k, $k=1,\dots,p$. Then, Theorems 2.4 and 2.5 will provide with improved estimate

$$R^p_{N,m}(f,x)=o(N^{-q-2p-1}) \qquad (2.146)$$

or

$$R^p_{N,m}(f,x)=O(N^{-q-2p-2}) \qquad (2.147)$$

if f has additional smoothness.

Tables (2.4)-(2.7) present some solutions of system (2.145) for different values of q, p, m.

$p = 1$	$p = 2$	$p = 3$
$\tau_1 = -1.57080$	$\tau_1 = 1.51095$	$\tau_1 = 1.63829$
	$\tau_2 = 11.5976$	$\tau_{2,3} = 5.37030 \pm 4.90193i$

Table 2.4: Optimal τ_k derived from system (2.145) for $q = m = 0$.

$p = 2$	$p = 3$	$p = 4$
$\tau_1 = 0.28300$	$\tau_1 = 5.49110$	$\tau_1 = -0.02878,\ \tau_2 = 1.17797$
$\tau_2 = -2.20334$	$\tau_{2,3} = -0.10654 \pm 0.90375i$	$\tau_3 = 7.11678,\ \tau_4 = 20.18171$

Table 2.5: Optimal τ_k derived from system (2.145) for $q = 0$ and $m = 2$.

$p = 2$	$p = 3$
$\tau_1 = 0.4659439$	$\tau_1 = 6.34640$
$\tau_2 = -4.418383$	$\tau_{2,3} = -0.0813412 \pm 0.9085957i$

Table 2.6: Optimal τ_k derived from system (2.145) for $q = 2$ and $m = 1$.

$p = 3$	$p = 4$
$\tau_1 = 7.74184$	$\tau_1 = 0.97037,\ \tau_2 = -0.30496$
$\tau_{2,3} = 1.44363 \pm 1.09981i$	$\tau_3 = -6.66081,\ \tau_4 = -13.85632$

Table 2.7: Optimal τ_k derived from system (2.145) for $q = 2$ and $m = 2$.

Comparison with the results of Section 2.1 leads to the following observations. When q is odd then the classical RT interpolation with optimal τ_k has better pointwise convergence rate (for $|x| < 1$) than the optimal QPR approximation (τ_k derived from (2.145)) and improvement is by factor $O(N^{-1})$. When q is even, the convergence rates are identical.

However, as we will show in the next section, the QPR approximation has much better accuracy on the entire interval compared to the classical one due to more accurate approximation at the endpoints.

2.4 Limit Function Analysis

In this section, we study behavior of the QPR approximation at the points $x = \pm 1$ in terms of the limit functions.

Theorem 2.7 *Let $f^{(q+2p+m)} \in AC[-1,1]$ for some $q, m \geq 0$, $p \geq 1$ and*

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \;\; k = 0, \ldots, q-1. \tag{2.148}$$

Let parameters λ_k be chosen as in (2.36). Then, the following estimate holds

$$\lim_{N\to\infty} N^q R^p_{N,m}\left(f, \pm\left(1 - \frac{h}{N}\right)\right) = \ell^p_{x\to\pm 1,q,m}(f,h), \;\; h \geq 0, \tag{2.149}$$

where

$$\begin{aligned}\ell^p_{x\to\pm1,q,m}(f,h) &= \frac{1}{\prod_{s=1}^p(\tau_s^2 + \pi^2(\frac{m+1}{2}+h)^2)} \\ &\times \left(\int_{-1}^1 \nu^p_{q,m}(f,x)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)x}dx - \int_{|x|>1}\mu^p_{q,m}(f,x)e^{\mp i\pi\left(\frac{m+1}{2}+h\right)x}dx\right)\end{aligned} \tag{2.150}$$

and

$$\begin{aligned}\mu^p_{q,m}(f,x) = \sum_{\ell=0}^q \frac{A_{\ell q}(f)(m+1)^{q-\ell}}{2^{q+1-\ell}(i\pi)^{\ell+1}(q-\ell)!}\sum_{t=0}^p(-1)^{p+t}\gamma_t(\tau)\sum_{k=0}^p\gamma_k(\tau) \\ \times \frac{(\ell+2p-k-t)!}{\ell!}\frac{1}{x^{2p-k-t+\ell+1}},\end{aligned} \tag{2.151}$$

$$\begin{aligned}\nu^p_{q,m}(f,x) &= \sum_{\ell=0}^q \frac{A_{\ell q}(f)(m+1)^{q-\ell}}{2^{q+1-l}(i\pi)^{\ell+1}(q-\ell)!} \\ &\times\left(\sum_{t=0}^p(-1)^{p+t}\gamma_t(\tau)\sum_{k=0}^p\gamma_k(\tau)\frac{(2p-k-t+\ell)!}{\ell!}\sum_{r\neq0}\frac{(-1)^{r(m+1)}}{(2r+x)^{2p-k-t+\ell+1}}\right. \\ &\left.-\sum_{t=0}^{m-1}\frac{\Phi^{(t)}_{\ell,m}(-1)}{t!}\sum_{\mu=0}^t\binom{t}{\mu}e^{\frac{i\pi(2\mu-m+1)x}{2}}h_p(2\mu-m+1,\tau)\right).\end{aligned} \tag{2.152}$$

Proof. We use representation (2.35) and notice that for $x = \pm1\left(1 - \frac{h}{N}\right)$

$$\lim_{N\to\infty} N^{2p}\prod_{s=1}^p\left(1+\theta_{-s}e^{i\pi\sigma x}\right)\left(1+\theta_s e^{-i\pi\sigma x}\right) = \prod_{s=1}^p(\tau_s^2 + \pi^2(m+1+2h)^2/4). \tag{2.153}$$

Lemmas 1.5 and 2.4 imply

$$\begin{aligned}\delta_n^p\left(\lambda,\{f_s^*\}\right) = &\sum_{j=q}^{q+2p+m} \frac{(-1)^{n+p+1}}{2^{j+1}N^{j+2p}} \sum_{\ell=0}^{j} \frac{A_{\ell j}(f)(m+1)^{j-\ell}(2N+m+1)^{\ell}}{(i\pi n)^{\ell+1}\ell!(j-\ell)!} \\ &\times \sum_{t=0}^{p}(-1)^t \frac{\gamma_t(\tau)}{(n/N)^{p-t}} \sum_{k=0}^{p} \frac{\gamma_k(\tau)}{(n/N)^{p-k}}(\ell+2p-k-t)! \\ &+ O\left(N^{-q-2p}n^{-2}\right) + o(n^{-2p-q-1}).\end{aligned} \tag{2.154}$$

Then, Lemmas 1.6, 1.8, 2.1 and 2.3 yield

$$\begin{aligned}\delta_n^p\left(\lambda,\{F_{s,m}-f_s^*\}\right) = &\frac{(-1)^{n+1}}{N^{q+2p+1}} \sum_{\ell=0}^{q} \frac{A_{\ell q}(f)(m+1)^{q-\ell}}{2^{q-\ell+1}(i\pi)^{\ell+1}(q-\ell)!} \\ &\times \Bigg(\sum_{t=0}^{p}(-1)^{p+t}\gamma_t(\tau)\sum_{k=0}^{p}\gamma_k(\tau)\frac{(2p-k-t+\ell)!}{\ell!} \\ &\times \sum_{r\neq 0} \frac{(-1)^{r(m+1)}}{\left(2r+\frac{2n}{2N+m+1}\right)^{2p-k-t+\ell+1}} \\ &- \sum_{t=0}^{m-1} \frac{\Phi_{\ell,m}^{(t)}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} e^{\frac{i\pi(2\mu-m+1)n}{2N+m+1}} h_p((2\mu-m+1),\tau)\Bigg) \\ &+ o\left(N^{-q-2p-1}\right).\end{aligned} \tag{2.155}$$

These estimates complete the proof. □

Let

$$\ell_{q,m}^p(f) = \max\{\max_{h\geq 0}|\ell_{x\to 1,q,m}^p(f,h)|, \max_{h\geq 0}|\ell_{x\to -1,q,m}^p(f,h)|\}. \tag{2.156}$$

The ratio

$$|A_q(f)|\ell_q^{p,*}/\ell_{q,m}^p(f) \tag{2.157}$$

compares the uniform errors of the classical RT interpolation and the QPR approximation for specific f and τ_k. Such comparison, independently of f, is possible to perform when $q=0$. In this important case

$$\ell_{x\to\pm 1,q,m}^p(f,h) = A_0(f)\ell_{x\to\pm 1,q,m}^{p,*}(h), \tag{2.158}$$

where

$$\begin{aligned}\ell_{x\to\pm 1,q,m}^{p,*}(h) = &\frac{1}{\prod_{s=1}^{p}(\tau_s^2+\pi^2(\frac{m+1}{2}+h)^2)} \\ &\times \left(\int_{-1}^{1}\nu_m^{p,*}(x)e^{\mp i\pi(m+1+2h)x/2dx} - \int_{|x|>1}\mu_m^{p,*}(x)e^{\mp i\pi(m+1+2h)x/2}dx\right)\end{aligned} \tag{2.159}$$

and

$$\mu_m^{p,*}(x) = \frac{1}{2i\pi}\sum_{t=0}^{p}(-1)^{p+t}\gamma_t(\tau)\sum_{k=0}^{p}\gamma_k(\tau)(2p-k-t)!\frac{1}{x^{2p-k-t+1}}, \tag{2.160}$$

$$\begin{aligned}\nu_m^{p,*}(x) = \frac{1}{2i\pi}\Bigg(&\sum_{t=0}^{p}(-1)^{p+t}\gamma_t(\tau)\sum_{k=0}^{p}\gamma_k(\tau)(2p-k-t)!\sum_{r\neq 0}\frac{(-1)^{r(m+1)}}{(2r+x)^{2p-k-t+1}}\\ &-\sum_{t=0}^{m-1}\frac{\Phi_{0,m}^{(t)}(-1)}{t!}\sum_{\mu=0}^{t}\binom{t}{\mu}e^{\frac{i\pi(2\mu-m+1)x}{2}}h_p(2\mu-m+1,\tau)\Bigg).\end{aligned} \tag{2.161}$$

Let

$$\ell_{0,m}^{p,*} = \max\{\max_{h\geq 0}|\ell_{x\to 1,0,m}^{p,*}(h)|, \max_{h\geq 0}|\ell_{x\to -1,0,m}^{p,*}(h)|\}. \tag{2.162}$$

Minimization of $\ell_{0,m}^{p,*}$ by appropriate selection of τ_k leads to the QPR approximation with minimal uniform error on $[-1,1]$ for $q=0$. Tables 2.8 and 2.9 show the corresponding optimal values of τ_k for $p=1$ and $p=2$, respectively. Recall that for $q=0$ the classical RT interpolation has uniform error 0.5 and the results in the tables show higher accuracy of the QPR approximation compared to the classical RT interpolation. The QPR approximation has higher accuracy also compared to the QP interpolation (see Table 1.4).

	τ_k	$\ell_{0,m}^{1,*}$
$m=0$	$\tau_1 = 1.1964$	0.0025
$m=1$	$\tau_1 = 1.8212$	0.00028

Table 2.8: Optimal values of τ_k that minimize $\ell_{0,m}^{1,*}$.

	τ_k	$\ell_{0,m}^{2,*}$
$m=0$	$\tau_1 = 0.8116,\ \tau_2 = 3.3885$	0.0001
$m=1$	$\tau_1 = 1.3083,\ \tau_2 = 4.8660$	0.00002
$m=2$	$\tau_1 = 0.6602,\ \tau_2 = 1.4190$	0.00007
$m=3$	$\tau_1 = 0,\ \tau_2 = 0$	0.0003

Table 2.9: Optimal values of τ_k that minimize $\ell_{0,m}^{2,*}$.

2.5 Numerical Analysis

Let

$$f(x) = \sin(x - 1) \tag{2.163}$$

for which $f(1) \neq f(-1)$. Above observations outline the choice of parameters that lead to more efficient QPR approximation. First, we use odd values of m as in this case we have additional accuracy inside the interval of interpolation. Second, this additional accuracy we get for any choice of parameters τ_k. Thus, we select them such (see Tables 2.8 and 2.9) to minimize the uniform error. The corresponding graphs we show in Figure 2.1.

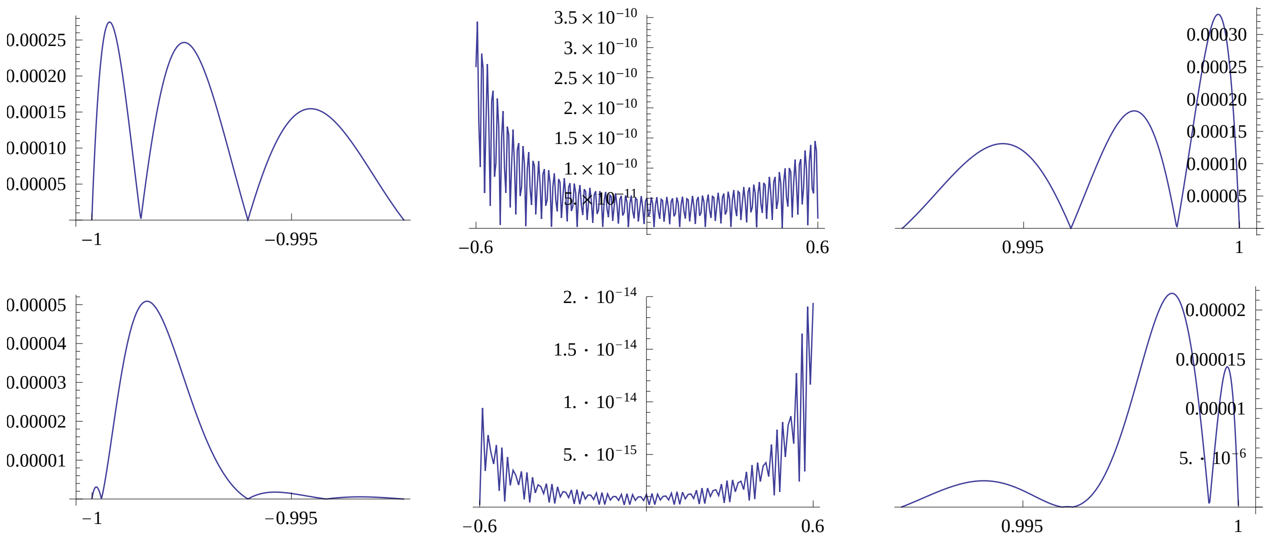

Figure 2.1: The graphs of the absolute errors $|R^p_{256,1}(f,x)|$ on $[-0.6, 0.6]$ and at the points $x = \pm 1$ for $p = 1, 2$ (from top to bottom) while approximating (2.163). Parameters τ_k are selected according to Tables 2.8 and 2.9.

For comparison, Figure 2.2 shows the corresponding (with the same N and $p = 1, 2$) results for the classical RT interpolation. As we observed above it is impossible to minimize the uniform error in case of the classical RT interpolation more than the value of the classical interpolation by appropriate choice of parameters τ_k. Thus, we select these parameters such to minimize the pointwise error inside the interval of interpolation (see Table 2.1).

Comparison of these figures show (however for this specific f) similar behavior inside the interval of interpolation and much better accuracy for the QPR approximation at the endpoints of the interval which is in accordance to theoretical observations.

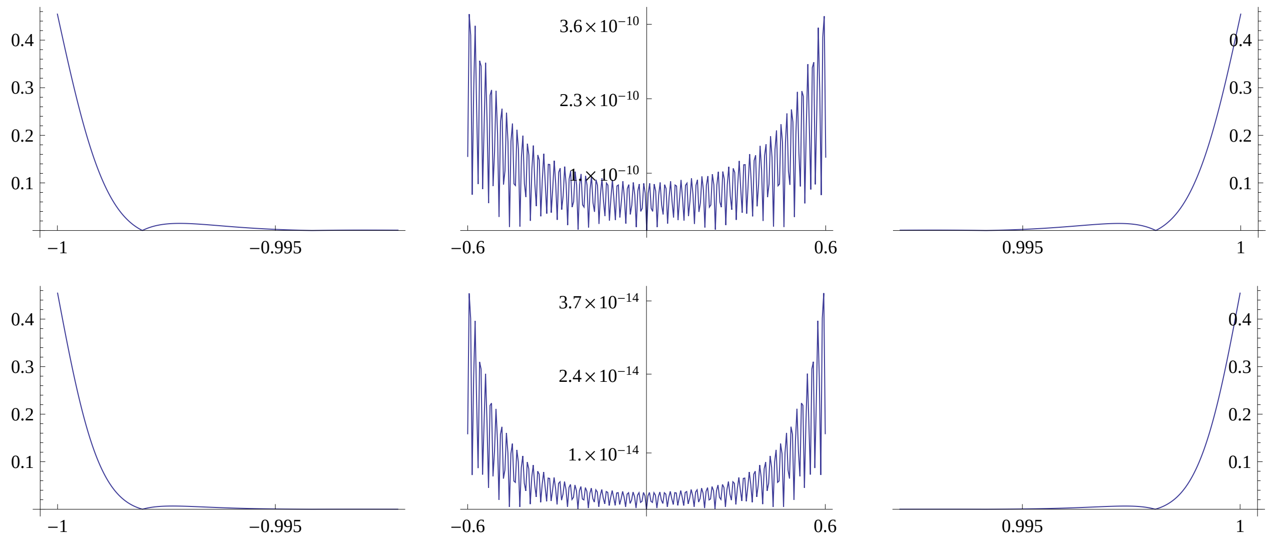

Figure 2.2: The graphs of the absolute errors $|R^p_{256}(f,x)|$ on $[-0.6, 0.6]$ and at the points $x = \pm 1$ for $p = 1, 2$ (from top to bottom) while approximating (2.163). Parameters τ_k are selected according to Table 2.1.

Now, let us discuss results that correspond to λ_k determined from system (2.37). Theory of this approach will be considered elsewhere. We present only some results of numerical experiments and bring comparison with the classical RT interpolation with the similar selection of parameters (see system (2.6)).

Figure 2.3 shows the graphs of $|R^p_{256,m}(f,x)|$ where parameters λ_k are determined from system (2.37) for $m = 1$ and $p = 1, 2, 3$.

Figure 2.4 shows the graphs of $|R^p_{256}(f,x)|$ where parameters λ_k are determined from system (2.6) for $p = 1, 2, 3$.

Comparison of the figures shows that the QP FP approximation has higher accuracy both inside and at the endpoints of the interval than the classical FP interpolation. Comparison with Figure 2.1 shows that the QP FP approximation has better accuracy away from the endpoints compared to the QPR approximation and worse accuracy at the endpoints.

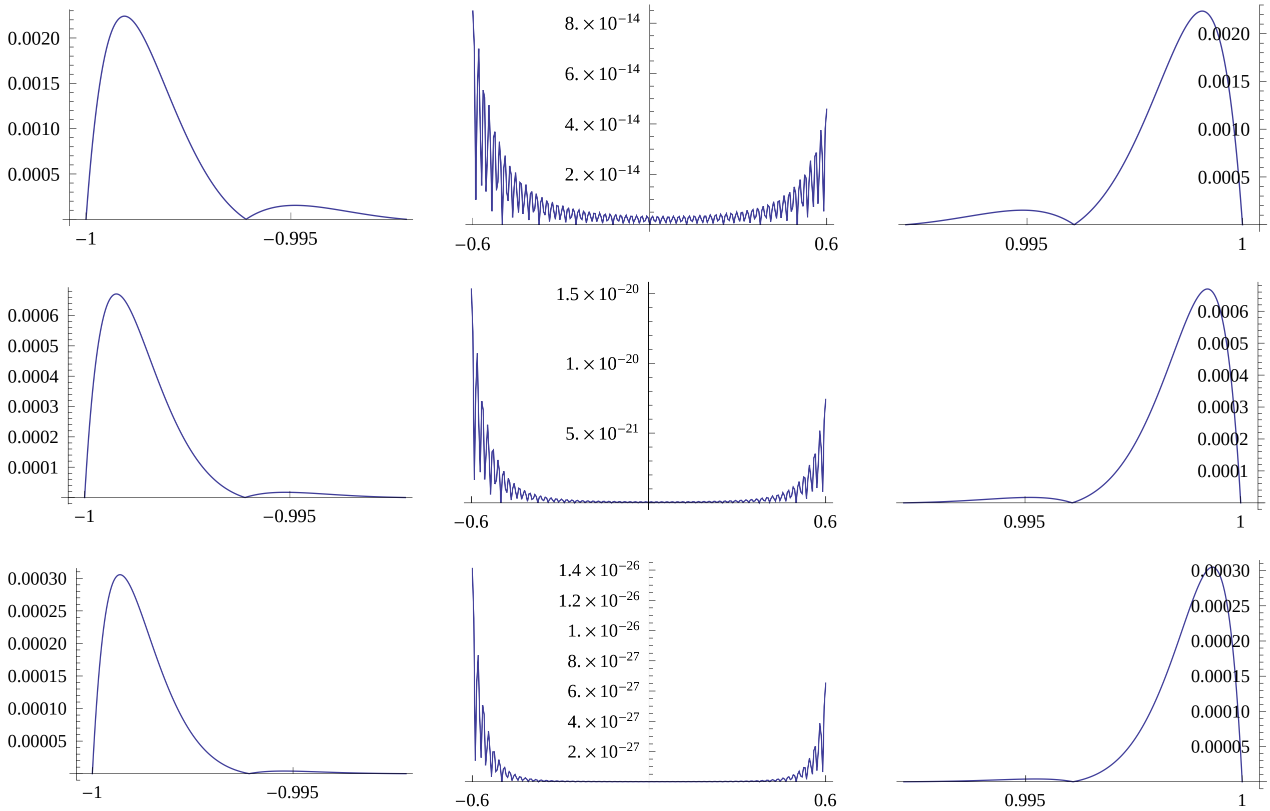

Figure 2.3: The graphs of the absolute errors while approximating (2.163) by the QP FP interpolation with $N = 256$, $m = 1$ and $p = 1, 2, 3$ (from top to bottom)

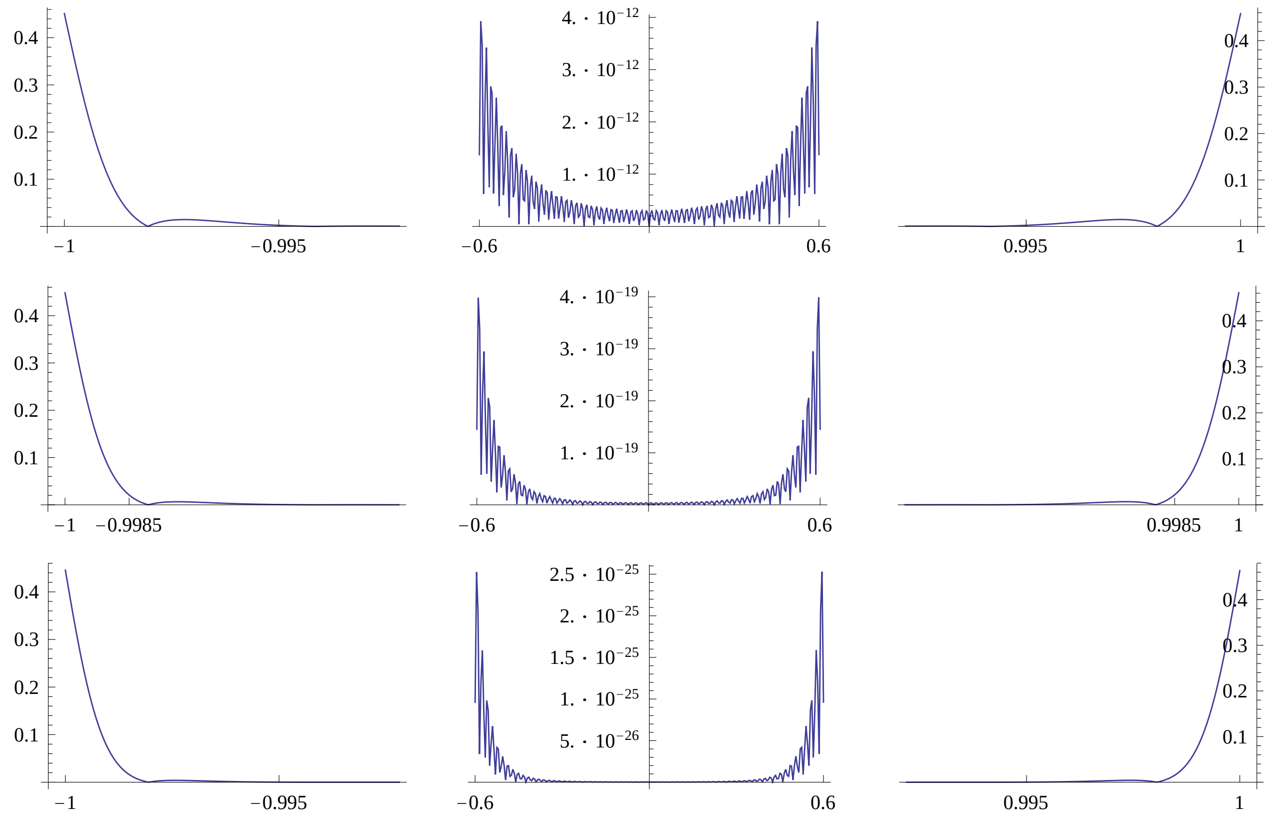

Figure 2.4: The graphs of the absolute errors while interpolating (2.163) by the classical FP interpolation with $N = 256$ and $p = 1, 2, 3$ (from top to bottom)

CHAPTER 3

Convergence acceleration by polynomial corrections

In this section, we investigate convergence acceleration of the QP interpolation and QPR approximation by polynomial corrections. Polynomial correction is a linear combination of some standard polynomials with derivatives of f in the coefficients. Subtraction of this polynomial from f reduces a new function with better convergence properties. Section 3.1 deals with construction of correction polynomial. Section 3.2 introduces the QP Polynomial (QPP) interpolation and QPR Polynomial (QPRP) approximation and studies their convergence. Section 3.3 discusses the problem of derivatives approximation via discrete Fourier coefficients. In Section 3.4, we consider some numerical results.

We recap details from [58, 63, 64].

3.1 Construction of Correction Polynomials

As previous chapters showed, the convergence rates of the QP interpolation and the QPR approximation essentially depended on the property

$$f^{(k)}(-1) = f^{(k)}(1) = 0, \ k = 0, \dots, q-1. \tag{3.1}$$

We will show how the same rates of convergence can be achieved for functions without this property by application of polynomial corrections.

In this section, we construct two different sets of polynomials $\xi_{k,q}(x)$ and $\eta_{k,q}(x)$, $k = 0, \dots, q-1$ with the following properties

$$\xi_{k,q}^{(s)}(1) - \xi_{k,q}^{(s)}(-1) = \delta_{k,s}, \ \xi_{k,q}^{(s)}(1) + \xi_{k,q}^{(s)}(-1) = 0, \ k, s = 0, \dots, q-1 \tag{3.2}$$

and

$$\eta_{k,q}^{(s)}(1) + \eta_{k,q}^{(s)}(-1) = \delta_{k,s}, \ \eta_{k,q}^{(s)}(1) - \eta_{k,q}^{(s)}(-1) = 0, \ k, s = 0, \dots, q-1. \tag{3.3}$$

We put

$$\begin{aligned} \xi_{q-1,q}(x) &= \frac{x^2(x^2-1)^{q-1}}{2^q(q-1)!}, \\ \eta_{q-1,q}(x) &= \frac{x(x^2-1)^{q-1}}{2^q(q-1)!} \end{aligned} \tag{3.4}$$

if q is even, and

$$\begin{aligned} \xi_{q-1,q}(x) &= \frac{x(x^2-1)^{q-1}}{2^q(q-1)!}, \\ \eta_{q-1,q}(x) &= \frac{x^2(x^2-1)^{q-1}}{2^q(q-1)!} \end{aligned} \tag{3.5}$$

if q is odd.

Then, we consider the following recurrence relations

$$\xi_{k,q}(x) = \xi_{k,q-1}(x) - \left(\xi_{k,q-1}^{(q-1)}(1) + \xi_{k,q-1}^{(q-1)}(-1)\right)\eta_{q-1,q}(x), \tag{3.6}$$

$$\eta_{k,q}(x) = \eta_{k,q-1}(x) - \left(\eta_{k,q-1}^{(q-1)}(1) - \eta_{k,q-1}^{(q-1)}(-1)\right)\xi_{q-1,q}(x) \tag{3.7}$$

if $q-k$ is even, and

$$\xi_{k,q}(x) = \xi_{k,q-1}(x) - \left(\xi_{k,q-1}^{(q-1)}(1) - \xi_{k,q-1}^{(q-1)}(-1)\right)\xi_{q-1,q}(x), \tag{3.8}$$

$$\eta_{k,q}(x) = \eta_{k,q-1}(x) - \left(\eta_{k,q-1}^{(q-1)}(1) + \eta_{k,q-1}^{(q-1)}(-1)\right)\eta_{q-1,q}(x) \tag{3.9}$$

if $q-k$ is odd.

Let us show some of these polynomials. When $q = 1$

$$\xi_{0,1}(x) = \frac{x}{2}, \;\; \eta_{0,1}(x) = \frac{x^2}{2}. \tag{3.10}$$

When $q = 2$

$$\begin{aligned} \xi_{0,2}(x) &= -\frac{1}{4}x\left(-3+x^2\right), \\ \xi_{1,2}(x) &= \frac{1}{4}x^2\left(-1+x^2\right) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \eta_{0,2}(x) &= \frac{x^2}{2} - \frac{1}{2}x^2\left(-1+x^2\right), \\ \eta_{1,2}(x) &= \frac{1}{4}x\left(-1+x^2\right). \end{aligned} \tag{3.12}$$

When $q = 3$

$$
\begin{aligned}
\xi_{0,3}(x) &= \frac{1}{16} x \left(15 - 10x^2 + 3x^4\right), \\
\xi_{1,3}(x) &= \frac{1}{16} \left(-9x^2 + 14x^4 - 5x^6\right), \\
\xi_{2,3}(x) &= \frac{1}{16} x \left(-1 + x^2\right)^2,
\end{aligned} \tag{3.13}
$$

and

$$
\begin{aligned}
\eta_{0,3}(x) &= \frac{1}{2} x^2 \left(3 - 3x^2 + x^4\right), \\
\eta_{1,3}(x) &= \frac{1}{16} \left(-7x + 10x^3 - 3x^5\right), \\
\eta_{2,3}(x) &= \frac{1}{16} x^2 \left(-1 + x^2\right)^2.
\end{aligned} \tag{3.14}
$$

Now, let us consider the main representation of f:

$$
f(x) = G(x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x), \tag{3.15}
$$

where

$$
\begin{aligned}
A_k^-(f) &= f^{(k)}(1) - f^{(k)}(-1), \\
A_k^+(f) &= f^{(k)}(1) + f^{(k)}(-1).
\end{aligned} \tag{3.16}
$$

Taking into account the properties of functions $\xi_{k,q}$ and $\eta_{k,q}$, we see that

$$
G^{(k)}(1) = G^{(k)}(-1) = 0, \ k = 0, \ldots, q-1. \tag{3.17}
$$

3.2 QPP Interpolation and QPRP Approximation

In this section, we assume that the exact values of $A_k^-(f)$ and $A_k^+(f)$ are known.

Approximation of G, in (3.15), by the QP interpolation, leads to the following QP Polynomial (QPP) interpolation

$$
I_{N,m,q}(f,x) = I_{N,m}(G,x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x) \tag{3.18}
$$

with the error

$$
R_{N,m,q}(f,x) = f(x) - I_{N,m,q}(f,x), \tag{3.19}
$$

where the discrete Fourier coefficients of G can be calculated from (3.15).

Now, all theorems of Chapter 1 concerning the convergence of the QP interpolation can be reformulated for the QPP interpolation. For example, the next theorem presents the behavior of the error of the QPP interpolation in terms of the pointwise convergence.

Theorem 3.1 *[58] Let $f^{(q+2m)} \in AC[-1,1]$ for some $q, m \geq 1$. Then, the following estimate holds for $|x| < 1$ as $N \to \infty$*

$$R_{N,m,q}(f,x) = \frac{D_{N,m,q}(G,x)}{N^{q+m+1}} + o(N^{-q-m-1}), \tag{3.20}$$

where

$$\begin{aligned} D_{N,m,q}(G,x) = i(-1)^N C_{q,m}(G) \Bigg[&\sin(\pi(N+1)\sigma x) \sum_{k=0}^{\acute{m}} \binom{m-k}{k} \frac{(-1)^k}{2^{2k+1}\cos^{2k+2}\frac{\pi x}{2}} \\ &- \sin(\pi N \sigma x) \sum_{k=0}^{\acute{m}-1} \binom{m-k-2}{k} \frac{(-1)^k}{2^{2k+3}\cos^{2k+4}\frac{\pi x}{2}} \Bigg], \end{aligned} \tag{3.21}$$

$\acute{m} = \left[\frac{m}{2}\right]$ *and $C_{q,m}(G)$ is defined by* (1.120)*.*

It is easy to verify that

$$A_{\ell,q}(G) = A_{\ell,q}(f) - \sum_{k=0}^{q-1} A_k^-(f) A_{\ell,q}(\xi_{k,q}) - \sum_{k=0}^{q-1} A_k^+(f) A_{\ell,q}(\eta_{k,q}) \tag{3.22}$$

and, in general, $A_{\ell,q}(G) \neq A_{\ell,q}(f)$.

Approximation of G, in (3.15), by the QPR approximation, leads to the following QPR Polynomial (QPRP) approximation

$$I_{N,m,q}^p(f,x) = I_{N,m}^p(G,x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x) \tag{3.23}$$

with the error

$$R_{N,m,q}^p(f,x) = f(x) - I_{N,m,q}^p(f,x), \tag{3.24}$$

where again the discrete Fourier coefficients of G can be calculated based on (3.15).

All theorems of Chapter 2 concerning the convergence of the QPR approximation can be reformulated for the QPRP approximation by changing f with G. For example, the theorem concerning the pointwise convergence has the following formulation.

Theorem 3.2 *[58] Let $f^{(q+2p+m)} \in AC[-1,1]$ for some $q, p, m \geq 1$. Let parameters λ_k be chosen as in* (2.36)*. Then, the following estimate holds for $|x| < 1$*

$$R_{N,m,q}^p(f,x) = \frac{D_{N,m,q}^p(G,x)}{N^{q+2p+1}} + o(N^{-q-2p-1}), \ N \to \infty, \tag{3.25}$$

where

$$D^p_{N,m,q}(G,x) = \frac{(-1)^N}{2^{q+2p+1}} \frac{\sin\left(\pi\sigma(N+\frac{1}{2})x - \frac{\pi m}{2}\right)}{\cos^{2p+1}\frac{\pi x}{2}} \sum_{k=0}^{q} \frac{A_{kq}(G)(m+1)^{q-k}2^k}{(q-k)!i^{k-m}\pi^{k+1}}$$
$$\times \left[\frac{(-1)^p}{k!}\psi^+_{p,m,k}(\tau) - \sum_{t=0}^{m-1} \frac{\Phi^{(t)}_{k,m}(-1)}{t!} \sum_{\mu=0}^{t} \binom{t}{\mu} i^{2\mu-m+1} h_p(2\mu-m+1,\tau)\right]. \tag{3.26}$$

3.3 Approximation of Derivatives

Let us show how the approximations to A_k^+ and A_k^- can be performed if the exact values are unknown. We need construction of some polynomials.

First, we consider polynomials $B_k(x)$, $k = 0, \ldots, q-1$ with the property

$$B_k^{(s)}(1) - B_k^{(s)}(-1) = \delta_{k,s}. \tag{3.27}$$

These are well-known ([41]) 2-periodic Bernoulli polynomials defined by the recurrence relations

$$B_0(x) = \frac{x}{2},\ B_k(x) = \int B_{k-1}(x)dx,\ x \in [-1,1],\ \int_{-1}^{1} B_k(x)dx = 0. \tag{3.28}$$

Here are some of the Bernoulli polynomials

$$\begin{aligned} B_1(x) &= \frac{x^2}{4} - \frac{1}{12}, \\ B_2(x) &= \frac{x^3}{12} - \frac{x}{12}, \\ B_3(x) &= \frac{x^4}{48} - \frac{x^2}{24} + \frac{7}{720}. \end{aligned} \tag{3.29}$$

Knowledge of these polynomials leads to the Lanczos representation ([45])

$$f(x) = F^-(x) + \sum_{k=0}^{q-1} A_k^-(f) B_k(x), \tag{3.30}$$

where F is a 2-periodic and relatively smooth function on the real line $F^- \in C^{q-1}(\mathbb{R})$ if $f \in C^{q-1}[-1,1]$. Approximations to $A_k^-(f)$ can be derived from the Lanczos representation by calculation of the discrete Fourier coefficients

$$\check{f}_n = \frac{1}{2N} \sum_{k=-N}^{N-1} f\left(\frac{k}{N}\right) e^{-i\pi n \frac{k}{N}} \tag{3.31}$$

of functions in the both sides of (3.30), we get

$$\check{f}_n - A_0^-(f)\check{B}_n(0) = \check{F}_n^- + \sum_{k=1}^{q-1} A_k^-(f)\check{B}_n(k). \tag{3.32}$$

Taking into account the fast decay of $\check{F}_n^-$, $n \sim N$ as $N \to \infty$ compared to other coefficients, we get the following system for determination of approximations $\widetilde{A}_k^-(f,N)$ to $A_k^-(f)$

$$\check{f}_n - A_0^-(f)\check{B}_n(0) = \sum_{k=1}^{q-1} \tilde{A}_k^-(f,N)\check{B}_n(k), \ n = n_1, \dots, n_{q-1}. \tag{3.33}$$

Investigation of the error $\widetilde{A}_k^-(f,N) - A_k^-(f)$ can be performed as in [44], where similar system is considered.

Now, we consider polynomials $E_k(x)$, $k = 0, \dots, q-1$ with the property

$$E_k^{(s)}(1) + E_k^{(s)}(-1) = \delta_{k,s}. \tag{3.34}$$

These polynomials can be constructed by the following recurrence relations

$$E_0(x) = \frac{1}{2}, \ E_k(x) = \int E_{k-1}(x)dx, \ x \in [-1,1], \ E_k(1) + E_k(-1) = 0. \tag{3.35}$$

Here are some of them

$$\begin{aligned} E_1(x) &= \frac{x}{2}, \\ E_2(x) &= \frac{x^2}{4} - \frac{1}{4}, \\ E_3(x) &= \frac{x^3}{12} - \frac{x}{4}. \end{aligned} \tag{3.36}$$

Similar to the Lanczos representation, we consider the following one

$$f(x) = F^+(x) + \sum_{k=0}^{q-1} A_k^+(f)E_k(x). \tag{3.37}$$

By multiplication of the both sides of (3.37) by $e^{i\pi x/2}$ and taking into account that $F^+(x)e^{i\pi x/2}$ has the same properties as $F^-(x)$, we calculate the discrete Fourier coefficients of the both sides, disregard the coefficients of F^+ and get the following system for determination of approximate values $\widetilde{A}_k^+(f,N)$ of $A_k^+(f)$

$$\check{f}_n^\dagger - A_0^+(f)\check{E}_n^\dagger(0) = \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\check{E}_n^\dagger(k), \ n = n_1, \dots, n_{q-1}, \tag{3.38}$$

where

$$\check{f}_n^\dagger = \frac{1}{2N} \sum_{k=-N}^{N-1} f\left(\frac{k}{N}\right) e^{-i\pi\left(n+\frac{1}{2}\right)\frac{k}{N}}. \tag{3.39}$$

Investigation of the error $\widetilde{A}_k^+(f,N) - A_k^+(f)$ can be performed as in [44].

We put

$$\widetilde{I}_{N,m,q}(f,x) = I_{N,m}(\widetilde{G},x) + \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x) \\ + A_0^-(f)\xi_{0,q}(x) + A_0^+(f)\eta_{0,q}(x) \tag{3.40}$$

with the error

$$\widetilde{R}_{N,m,q}(f,x) = f(x) - \widetilde{I}_{N,m,q}(f,x), \tag{3.41}$$

and

$$\widetilde{I}^p_{N,m,q}(f,x) = I^p_{N,m}(\widetilde{G},x) + \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x) \\ + A_0^-(f)\xi_{0,q}(x) + A_0^+(f)\eta_{0,q}(x) \tag{3.42}$$

with the error

$$\widetilde{R}^p_{N,m,q}(f,x) = f(x) - \widetilde{I}^p_{N,m,q}(f,x). \tag{3.43}$$

Here

$$\widetilde{G}(x) = f(x) - \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) - \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x) \\ - A_0^-(f)\xi_{0,q}(x) - A_0^+(f)\eta_{0,q}(x). \tag{3.44}$$

Theoretical investigation of interpolation $\widetilde{I}_{N,m,q}$ and approximation $\widetilde{I}^p_{N,m,q}$ will be carried out elsewhere. In the next section, we consider these approaches only numerically.

3.4 Numerical Analysis

Let

$$f(x) = \sin(x-1). \tag{3.45}$$

Figure 3.1 shows the graph of $|R_{256,m}|$. Figures 3.2 and 3.3 show the graphs of $|R_{256,m,q}|$ and $|\widetilde{R}_{256,m,q}|$, respectively. Comparison of Figures 3.1 and 3.2 shows that polynomial corrections improve the accuracy of the corresponding interpolations. Comparison of Figures 3.2 and 3.3 shows that approximation of jumps makes the approaches slightly less accurate than in case of utilization of the exact values.

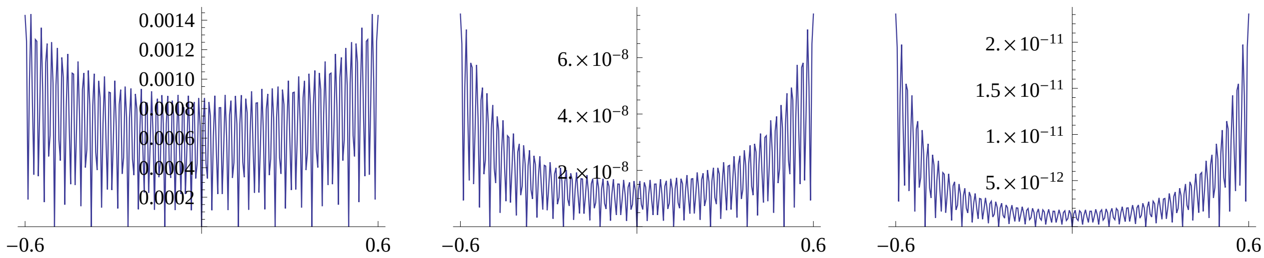

Figure 3.1: The graphs of $|R_{256,m}(f,x)|$ for $m = 0, 2, 4$ (from left to right) while interpolating (3.45) by the QP interpolation.

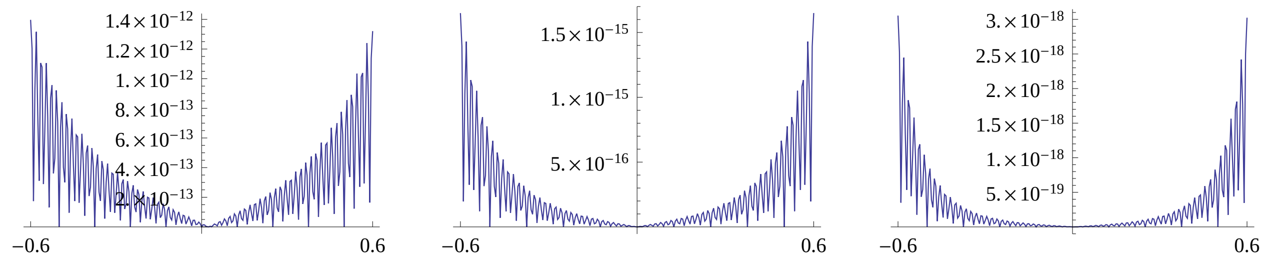

Figure 3.2: The graphs of $|R_{256,m,q}(f,x)|$ for $m = 0, 2, 4$ (from left to right) and $q = 3$ while interpolating (3.45) by the QPP interpolation with the exact values of A_k^+ and A_k^-.

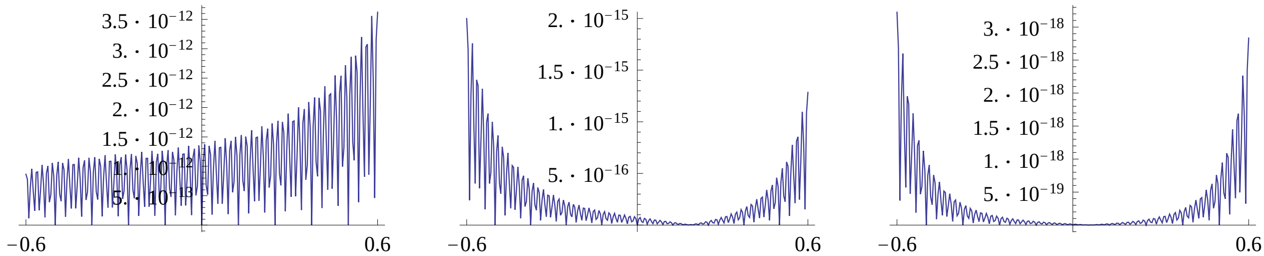

Figure 3.3: The graphs of $|\widetilde{R}_{256,m,q}(f,x)|$ for $m = 0, 2, 4$ (from left to right) and $q = 3$ while interpolating (3.45) by the QPP interpolation with the approximate values of A_k^+ and A_k^- derived from systems (3.33) and (3.38) with $n_1 = -N$, $n_2 = N - 1$.

Conclusion

Chapter 1 was devoted to the Quasi-Periodic (QP) interpolation and its comparison with the classical trigonometric interpolation.

- We sought the QP interpolation in the form

$$I_{N,m}(f,x) = \sum_{k=-N}^{N} f\left(\frac{k}{N}\right) a_k(x), \ x \in [-1,1], \ m \geq 0 \tag{3.46}$$

  and found the unique unknowns a_k from condition

$$I_{N,m}\left(e^{i\pi n\sigma x}, x\right) \equiv e^{i\pi n\sigma x}, \ |n| \leq N, \ x \in [-1,1], \ \sigma = \frac{2N}{2N+m+1}. \tag{3.47}$$

  As a result, the QP interpolation had the following explicit representation

$$I_{N,m}(f,x) = \sum_{n=-N}^{N} F_{n,m} e^{i\pi n\sigma x}, \tag{3.48}$$

  where $F_{n,m}$ were defined by (1.38).

  Theorem 1.5 proves that $I_{N,m}$ interpolates f on grid $x_k = k/N$, $|k| \leq N$, i.e.

$$I_{N,m}(f, x_k) = f(x_k). \tag{3.49}$$

- Theorems 1.6 and 1.7 show that the QP interpolation has pointwise convergence rate $O(N^{-q-m-1})$. Comparison with Theorems 1.1 and 1.2 shows that the QP interpolation has better convergence rate than the classical interpolation (if the requirements of the theorems are satisfied) and improvement is by factor $O(N^{-m})$ for even q and $O(N^{-m+1})$ for odd q.

  In particular, if f is rather smooth (for example, infinitely differentiable) on $[-1,1]$, but $f(1) \neq f(-1)$ then improvement is by factor $O(N^{-m})$, $m \geq 0$.

- In general, conditions of the theorems concerning the QP interpolation are stricter than those for the classical interpolation: additional smoothness requirement and more conditions on f and its derivatives at the endpoints of interval.

- Theorems 1.8 and 1.9 show that the QP interpolation has the same convergence rate $O(N^{-q-\frac{1}{2}})$ in the L_2-norm as the classical interpolation (Theorem 1.3). More detailed comparison of both interpolations can be performed by exploring constants $|A_q(f)|c_q$ and $c_{q,m}(f)$. In general, such comparison is impossible to perform independently of f. Numerical results obtained for a specific function (1.234) show (see Table 1.7) that by increasing m the accuracy of the QP interpolation can be increased tremendously and as smaller is q as more efficient is the QP interpolation.

  Comparison of those constants independently of f can be performed when $q = 0$ ($f(1) \neq f(-1)$). In this case $c_{0,m}(f) = |A_0(f)|c^*_{0,m}$ (see (1.220) and (1.221)). Hence, quotient $c_0/c^*_{0,m}$ shows which interpolation is asymptotically more precise in the L_2-norm. Table 1.3 presents the values of the ratio showing that by increasing m the difference in accuracies is growing enormously and this observation is independent of f.

- Theorems 1.10 and 1.11 reveal the behavior of the QP interpolation at the endpoints of interval in terms of the limit functions. Comparison with Theorem 1.4 shows the same rate of convergence $O(N^{-q})$ as the classical interpolation has. More complete comparison can be performed by exploring constants $|A_q(f)|\ell^*_q$ and $\ell_{q,m}(f)$ (see (1.230)). Again, constant $\ell_{q,m}(f)$ depends on f and comparison is possible, in general, for specific functions which we do it in Table 1.8 for (1.234). As smaller is q and bigger is m than more accurate is the QP interpolation compared to the classical interpolation in the uniform norm.

  When $q = 0$, comparison is possible to perform independently of f as $\ell_{0,m}(f) = |A_0(f)|\ell^*_{0,m}$ (see (1.231)-(1.233)). Hence, ratio $\ell^*_0/\ell^*_{0,m}$ shows which interpolation has better asymptotic uniform accuracy. Table 1.4 shows the values of the ratio. We see that as big is m as more accurate is the QP interpolation compared to the classical one.

Chapter 2 was devoted to convergence acceleration of the QP interpolation by rational correction functions which led to the QP Rational (QPR) approximation.

- The QPR approximation is a sum of the QP interpolation and some rational functions (in terms of $e^{i\pi\sigma x}$) as corrections of error

$$I_{N,m}^{p}(f,x) = I_{N,m}(f,x) + Rational\ Functions. \tag{3.50}$$

Rational corrections contain some unknown parameters λ_k (see (2.34)) which have essential impact on the convergence properties. We consider two approaches for their determination.

- The first approach is

$$\lambda_{-k} = \lambda_k = 1 - \frac{\tau_k}{N}, \ k = 1, \ldots, p, \tag{3.51}$$

  where new parameters τ_k are independent of N and f. Theorems 2.4 and 2.5 show that in this case, the QPR approximation has pointwise convergence rate $O(N^{-q-2p-1})$ and improvement compared to the QP interpolation is by factor $O(N^{-2p+m})$ if $2p > m$, and it is true independently of determination of parameters τ_k (we can take, for example, $\tau_k = k$). Then, Theorem 2.6 proves that for $q = 0$ ($f(1) \neq f(-1)$) and odd m the QPR approximation has improved convergence rate $o(N^{-2p-1})$ or $O(N^{-2p-2})$ if f is enough smooth. This is true independently of parameters τ_k.

- Estimates of Theorems 2.4 and 2.5 provide with the approach for determination of parameters τ_k by vanishing or minimizing (if possible) expression in (2.145). Tables 2.4, 2.5, 2.6 and 2.7 show the values of τ_k that vanish that expression for some q and m. For these values, the convergence rate is $o(N^{-2p-1})$ or $O(N^{-2p-2})$ if f has additional smoothness.

- Theorem 2.3 explores behavior of the classical RT interpolation at the endpoints $x = \pm 1$ in terms of the limit functions. Theorem shows convergence rate $O(N^{-q})$ when $x = \pm\left(1 - \frac{h}{N}\right)$, $h > 0$ as $N \to \infty$, independently of determination of τ_k. Maximum value $|A_q(f)|\ell_q^{p,*}$ of the exact constant of asymptotic error allows determination of parameters τ_k by minimization of $\ell_q^{p,*}$. Corresponding results are presented in Tables 2.2 and 2.3 which show better accuracy of the classical RT interpolation compared to the classical interpolation in the uniform norm when $q > 0$ (see Table 1.2). In case of $q = 0$, both interpolations provide the same uniform norm as $\ell_0^{*} = \ell_0^{p,*} = 0.5$ independently of the choice of parameters τ_k.

- Theorem 2.7 reveals the behavior of the QPR approximation at the endpoints $x = \pm 1$ in terms of the limit functions showing the same convergence rate as the classical RT interpolation has, independently of determination of parameters τ_k. More complete comparison can be performed by analysis of constants $|A_q(f)|\ell_q^{p,*}$ and $\ell_{q,m}^p(f)$. In general, such comparison can be performed only when f and parameters τ_k are specified.

  When $q = 0$, such comparison is possible independently of f and τ_k. In this case

$$\ell_{0,m}^p(f) = |A_0(f)|\ell_{0,m}^{p,*}, \tag{3.52}$$

  where $\ell_{0,m}^{p,*}$ is defined by (2.158)-(2.162). Then, ratio $\ell_0^{p,*}/\ell_{0,m}^{p,*}$ (where $\ell_0^{p,*} = 0.5$) will perform the comparison of both interpolations independently of f. Determination of parameters τ_k, optimal in the sense of the limit functions, can be performed by minimization of $\ell_{0,m}^{p,*}$. Tables 2.8 and 2.9 present optimal values for τ_k and the corresponding values of $\ell_{0,m}^{p,*}$ showing better accuracy compared to the classical RT interpolation. Comparison with the results of Table 1.4 shows also better accuracy compared to the QP interpolation in the uniform norm.

  As we mentioned above for odd m and $q = 0$, the QPR approximation provides better pointwise accuracy independently of the values of parameters τ_k. It is reasonable to get additional accuracy on entire interval using optimal τ_k in the sense of the limit functions.

- The second approach of determination of parameters λ_k is solution of the following system

$$\delta_n^p(\lambda, \{F_{s,m}\}) = 0, \ n = N - p + \left[\frac{m}{2}\right] + 1, \cdots, N + p + \left[\frac{m}{2}\right]. \tag{3.53}$$

  We present some numerical results showing better accuracy of this approach compared to its classical analog.

Chapter 3 was devoted to convergence acceleration of the QP interpolation and the QPR approximation by polynomial corrections which led to QP Polynomial (QPP) interpolation and QPR Polynomial (QPRP) approximation.

- We saw in Chapters 1 and 2 that convergence properties of the QP and QPR methods were depended essentially on condition

$$f^{(k)}(1) = f^{(k)}(-1) = 0, \ k = 0, \dots, q-1. \tag{3.54}$$

In Chapter 3 we introduce polynomial corrections that made this condition redundant.

- Assume f is such that $f(1) \neq f(-1)$. We constructed a families of polynomials $\xi_{k,q}(x)$ and $\eta_{k,q}(x)$, $k = 0, \ldots, q-1$ satisfying conditions

$$\xi_{k,q}^{(s)}(1) - \xi_{k,q}^{(s)}(-1) = \delta_{k,s},\ \xi_{k,q}^{(s)}(1) + \xi_{k,q}^{(s)}(-1) = 0,\ k, s = 0, \ldots, q-1 \tag{3.55}$$

and

$$\eta_{k,q}^{(s)}(1) + \eta_{k,q}^{(s)}(-1) = \delta_{k,s},\ \eta_{k,q}^{(s)}(1) - \eta_{k,q}^{(s)}(-1) = 0,\ k, s = 0, \ldots, q-1 \tag{3.56}$$

and wrote the following representation of f (see (3.15))

$$f(x) = G(x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x), \tag{3.57}$$

where

$$A_k^-(f) = f^{(k)}(1) - f^{(k)}(-1),\ A_k^+(f) = f^{(k)}(1) + f^{(k)}(-1). \tag{3.58}$$

Important property of function G is that

$$G^{(k)}(1) = G^{(k)}(-1) = 0,\ k = 0, \ldots, q-1 \tag{3.59}$$

and application of the QP interpolation or the QPR approximation to G provides the same convergence rate as if f satisfied condition (3.54). Application of the QP interpolation and the QPR approximation to G leads to the following QPP interpolation and QPRP approximation

$$I_{N,m,q}(f,x) = I_{N,m}(G,x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x), \tag{3.60}$$

$$I_{N,m,q}^p(f,x) = I_{N,m}^p(G,x) + \sum_{k=0}^{q-1} A_k^-(f)\xi_{k,q}(x) + \sum_{k=0}^{q-1} A_k^+(f)\eta_{k,q}(x), \tag{3.61}$$

respectively.

All theorems of Chapters 1 and 2 can be reformulated for the QPP interpolation and the QPRP approximation (see, for example, Theorems 3.1 and 3.2) by omitting condition (3.54).

- It was assumed that in the above representation the exact values of $A_k^-(f)$ and $A_k^+(f)$ were known exactly. We considered approach of their approximation based only on the discrete Fourier coefficients of f by solution of systems of linear equations (3.33) and (3.38) with unknowns $\widetilde{A}_k^-(f,N)$ and $\widetilde{A}_k^+(f,N)$.

- Based on the approximations $\widetilde{A}_k^-(f,N)$ and $\widetilde{A}_k^+(f,N)$, we considered another representation of f

$$
\begin{aligned}
f(x) = \widetilde{G}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x) \\
+ A_0^-(f)\xi_{0,q}(x) + A_0^+(f)\eta_{0,q}(x)
\end{aligned}
\tag{3.62}
$$

with the corresponding QPP interpolation and QPRP approximation, with approximated "jumps",

$$
\begin{aligned}
\widetilde{I}_{N,m,q}(f,x) = I_{N,m}(\widetilde{G},x) + \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x) \\
+ A_0^-(f)\xi_{0,q}(x) + A_0^+(f)\eta_{0,q}(x),
\end{aligned}
\tag{3.63}
$$

$$
\begin{aligned}
\widetilde{I}^p_{N,m,q}(f,x) = I^p_{N,m}(\widetilde{G},x) + \sum_{k=1}^{q-1} \widetilde{A}_k^-(f,N)\xi_{k,q}(x) + \sum_{k=1}^{q-1} \widetilde{A}_k^+(f,N)\eta_{k,q}(x) \\
+ A_0^-(f)\xi_{0,q}(x) + A_0^+(f)\eta_{0,q}(x),
\end{aligned}
\tag{3.64}
$$

respectively.

We presented some numerical results without theoretical investigations.

Notations

$AC[-1,1]$ — Absolutely Continuous functions

$A_k(f)$ — see (1.5)

$A_{ks}(f)$ — see (1.78)

$A_k^-(f)$ — $A_k^-(f) = A_k(f)$, see (3.16)

$A_k^+(f)$ — see (3.16)

$\widetilde{A}_k^-(f,N)$ — approximation of $A_k^-(f)$ from system (3.33)

$\widetilde{A}_k^+(f,N)$ — approximation of $A_k^+(f)$ from system (3.38)

α_ℓ — see (1.32)

$B_k(x)$ — Bernoulli polynomials defined by (3.28)

$B_n(j)$ — Fourier coefficient of Bernoulli polynomial $B_j(x)$, see (1.57)

$\check{B}_n(j)$ — discrete Fourier coefficients of Bernoulli polynomial $B_j(x)$

$C[-1,1]$ — Continuous functions

$C^k[-1,1]$ — functions with k-th Continuous Derivative

$C_{q,m}(f)$ — see Lemma 1.7

c_q — see Theorem 1.3

$c_{q,m}(f)$ — see Theorem 1.8

$c_{0,m}^*$ — see (1.221)

$D_{N,m}(f,x)$ — see Theorem 1.6

$D^p_{N,m}(f,x)$ — see Theorem 2.4

$D^{p,*}_{N,m}(x)$ — see (2.115)

$D_{N,m,q}(f,x)$ — see Theorem 3.1

$D^p_{N,m,q}(f,x)$ — see Theorem 3.2

$\delta_{s,k}$ — the Kronecker symbol

$\delta^p_n\left(\{y_s\}^{\infty}_{s=-\infty}\right) = \delta^p_n\left(\{y_s\}\right)$ — finite differences defined by (1.52)

$\delta^p_n\left(\lambda, \{y_s\}^{\infty}_{s=-\infty}\right) = \delta^p_n\left(\lambda, \{y_s\}\right)$ — generalized finite differences defined by (2.1)

$\Delta^k_n\left(\mu, \{y_s\}^{\infty}_{s=-\infty}\right) = \Delta^k_n\left(\mu, \{y_s\}\right)$ — generalized finite differences defined by (2.39)

$\Delta^p_n\left(\{y_s\}^{\infty}_{s=-\infty}\right) = \Delta^p_n\left(\{y_s\}\right)$ — finite differences defined by (1.53)

$E_k(x)$ — see (3.35)

f_n — Fourier coefficient of function f, see (1.77)

$\check{f}_n$ — discrete Fourier coefficient defined by (1.3)

$\check{f}_{n,m}$ — discrete Fourier coefficients defined by (1.39)

$f^*(x)$ — see (1.75)

f^*_n — Fourier coefficients of $f^*(x)$

$F_{n,m}$— coefficients of the QP interpolation, see (1.38)

$\Phi_{k,m}(x)$ — see (1.87)

$\gamma_k(\tau)$ — see (2.38)

$h_p(\beta,\tau)$ — see (2.62)

$I_N(f,x)$ — Classical trigonometric interpolation, see (1.2)

$I^p_N(f,x)$ — Classical RT interpolation, see (2.3)

$I_{N,m}(f,x)$ — QP interpolation, see (1.37)

$I^p_{N,m}(f,x)$ — QPR approximation, see (2.34)

$I_{N,m,q}(f,x)$ — QPP interpolation, see (3.18)

$I^p_{N,m,q}(f,x)$ — QPRP approximation, see (3.23)

$\widetilde{I}_{N,m,q}(f,x)$ — QPP approximation with approximated jumps, see (3.40)

$\widetilde{I}^p_{N,m,q}(f,x)$ — QPRP approximation with approximated jumps, see (3.42)

$\ell_{x\to\pm1,q}(h)$ — see Theorem 1.4

ℓ^*_q — see (1.18)

$\ell_{x\to\pm1,q,m}(f,h)$ — see Theorem 1.10

$\ell_{q,m}(f)$ — see (1.230)

$\ell^*_{x\to\pm1,0,m}$ — see (1.233)

$\ell^*_{0,m}$ — see (1.232)

$\ell^p_{x\to\pm1,q}(h)$ — see (2.3)

$\ell^p_{x\to\pm1,q,m}(f,h)$ — see Theorem 2.7

$\ell^{p,*}_{x\to\pm1,q,m}(h)$ — see (2.159)

$\ell^{p,*}_{0,m}$ — see (2.162)

$\mu_{q,m}(f,x)$ — see Lemma 1.9

$\mu^*_{0,m}(f,x)$ — see (1.218)

$\mu^p_{q,m}(f,x)$ — see Theorem 2.7

$\mu^{p,*}_m(f,x)$ — see (2.160)

$\nu_{q,m}(f,x)$ — see Lemma 1.9

$\nu^*_{0,m}(f,x)$ — see (1.219)

$\nu^p_{q,m}(f,x)$ — see Theorem 2.7

$\nu^{p;*}_m(f,x)$ — see (2.161)

$\omega_{p,t}$ — see (1.61)

$R_N(f,x)$ — error of Classical trigonometric interpolation, see (1.4)

$R^p_N(f,x)$ — error of Classical RT interpolation, see (2.4)

$R_{N,m}(f,x)$ — error of QP interpolation, see (1.127)

$R^p_{N,m}(f,x)$ — error of QPR approximation, see (2.35)

$R_{N,m,q}(f,x)$ — error of QPP interpolation, see (3.19)

$R^p_{N,m,q}(f,x)$ — error of QPRP approximation, see (3.24)

$\widetilde{R}_{N,m,q}(f,x)$ — error of QPP approximation with approximated jumps, see (3.41)

$\widetilde{R}^p_{N,m,q}(f,x)$ — error of QPRP approximation with approximated jumps, see (3.43)

$L_2[-1,1]$ — Square-Integrable functions

$\|\cdot\|_{L_2[-1,1]}$ – standard L_2-norm, see (0.16)

$S(k,j)$ — the Stirling numbers of the second kind

σ — $\sigma = \dfrac{2N}{2N+m+1}$, see (1.20)

$\theta_{n,\ell}$ — see (1.40)

τ_k — see (2.5)

$v_{s,\ell} = \alpha_\ell^{s-1}$ — elements of the Vandermonde matrix, see (1.32)

x_k — grid of interpolation. For the QP interpolation $x_k = \frac{k}{N}$, $|k| \le N$. For the classical interpolations $x_k = \frac{2k}{2N+1}$, $|k| \le N$

Abbreviations

RT — Rational Trigonometric

QP — Quasi-Periodic

QPR — Quasi-Periodic Rational

QPP — Quasi-Periodic Polynomial

QPRP — Quasi-Periodic Rational Polynomial

FP — Fourier-Pade

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