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A Note on Bi-Periodic Leonardo Sequence

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Abstract. In this work, we define a new generalization of the Leonardo sequence by the recurrence relation $GLe_n = aGLe_{n-1} + GLe_{n-2} + a$ (for even n) and $GLe_n = bGLe_{n-1} + GLe_{n-2} + b$ (for odd n) with the initial conditions $GLe_0 = 2a - 1$ and $GLe_1 = 2ab - 1$, where a and b are real nonzero numbers. Some algebraic properties of the sequence $\{GLe_n\}_{n\geq 0}$ are studied and several identities, including the generating function and Binet's formula, are established.

Key Words: Leonardo Sequence, Bi-Periodic Fibonacci Sequence, Binet's Formula, Generating Function, Catalan's Identity, Cassini's Identity, d'Ocgane's Identity
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Introduction

One of the well-known sequence is the Fibonacci sequence $\{F_n\}_{n\geq 0}$, which consists of integer numbers defined by a recurrence relation of order two, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with the initial conditions $F_0 = 0$ and $F_1 = 1$. The results concerning this sequence motivated the study of many other numerical sequences, some of which are closely related to it as Lucas, Pell, and Jacobsthal sequences (see [9, 10] and [11] for their applications). The sequence of Leonardo, introduced by Catarino and Borges in [5], is one sequence motivated by Fibonacci numbers. The Leonardo sequence $\{Le_n\}_{n\geq 0}$ is defined by the recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \qquad n \geq 2, \tag{1}$$

or, equivalently,

$$Le_{n+1} = 2Le_n - Le_{n-2}, \qquad n \geq 2, \tag{2}$$

with initial conditions $Le_0 = Le_1 = 1$. According to Proposition 2.2 in [5], Leonardo and Fibonacci numbers are related as follows

$$Le_n = 2F_{n+1} - 1, \qquad n \geq 0. \tag{3}$$

Leonardo sequence served as the basis for a number of publications, including the work of Alp and Koçer [1], Alves and Vieira [2], Catarino and Borges [4], Kara and Yilmaz [8], Kuhapatanakul and Chobsorn [12], Tan and Leung [14], and Gokbas [7], among others. Various generalizations of known number sequences have also been considered. For example, for any real nonzero numbers a and b, Edson and Yayenie [6] introduced the generalization of the Fibonacci sequence $\{F_n^{(a,b)}\}_{n\geq 0}$ defined as

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & \text{if } n \text{ is even}, \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & \text{if } n \text{ is odd}, \end{cases} \qquad n \geq 2, \tag{4}$$

with the initial conditions $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$. When $a = b = 1$, we have the classical Fibonacci sequence, and for $a = b = 2$, we get the Pell numbers. If we set $a = b = k$ for some positive integer k, we come to the k-Fibonacci numbers. Theorem 5 in [6] established the following Binet formula

$$F_n^{(a,b)} = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{5}$$

where $\alpha = (ab + \sqrt{a^2b^2 + 4ab})/2$ and $\beta = (ab - \sqrt{a^2b^2 + 4ab})/2$ are the roots of the characteristic equation $x^2 - abx - ab = 0$, and $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$ is the parity function, that is,

$$\xi(n) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases} \tag{6}$$

In [6], it was shown that the following relations hold true:

$$\alpha + \beta = ab,\ \alpha - \beta = \sqrt{a^2b^2 + 4ab},\ \alpha\beta = -ab,\ \frac{\alpha}{\beta} = -\frac{\alpha^2}{ab},\ \frac{\beta}{\alpha} = -\frac{\beta^2}{ab}. \tag{7}$$

In the same way, Bilgici [3] introduced a new generalization of the Lucas sequence, denoted by $\{l_n\}_{n\geq 0}$ and called the bi-periodic Lucas sequence which is defined recursively

$$l_n = \begin{cases} al_{n-1} + l_{n-2} & \text{if } n \text{ is even}, \\ bl_{n-1} + l_{n-2} & \text{if } n \text{ is odd}, \end{cases} \qquad n \geq 2, \tag{8}$$

with the initial conditions $l_0 = 2$ and $l_1 = a$. When $a = b = 1$, we have the classical Lucas sequence. If we set $a = b = k$, for some positive integer k, we get the k-Lucas numbers.

Uygun and Owusu [16] introduced and studied a generalization of the Jacobsthal numbers, which is called the bi-periodic Jacobsthal sequence and defined recursively by

$$\hat{j}_n = \begin{cases} b\hat{j}_{n-1} + 2\hat{j}_{n-2} & \text{if } n \text{ is even}, \\ a\hat{j}_{n-1} + 2\hat{j}_{n-2} & \text{if } n \text{ is odd}, \end{cases} \qquad n \geq 2,$$

with the initial conditions $\hat{j}_0 = 0$ and $\hat{j}_1 = 1$. When $a = b = 1$, we have the classical Jacobsthal sequence, and for $a = b = k$, with some positive integer k, we get the k-Jacobsthal numbers.

In [15], the same authors study a new generalization of the Jacobsthal–Lucas numbers, which is called the bi-periodic Jacobsthal–Lucas sequence and defined as follows

$$\hat{c}_n = \begin{cases} b\hat{c}_{n-1} + 2\hat{c}_{n-2} & \text{if } n \text{ is even,} \\ a\hat{c}_{n-1} + 2\hat{c}_{n-2} & \text{if } n \text{ is odd,} \end{cases} \qquad n \geq 2,$$

with the initial conditions $\hat{c}_0 = 2$ and $\hat{c}_1 = a$. When $a = b = 1$, we have the classical Jacobsthal Lucas sequence, and for $a = b = k$, with some positive integer k, we get the k-Jacobsthal–Lucas numbers.

The main goal of this work is to define a new generalization for the Leonardo sequence. Such sequence, which we shall call the bi-periodic Leonardo sequence, is introduced in the next section. The Binet formula is stated in Section 2, and the generating function is provided in Section 3. Catalan's, Cassini's, and several other identities are established in Section 4.

1 The bi-periodic Leonardo sequence

In this section, we define a generalization of the Leonardo sequence similar to the generalized Fibonacci sequence given by Edson and Yayenie [6] and to the two generalizations of the Lucas sequence given by Bilgici [3].

Definition 1 *For any real nonzero numbers a and b, the bi-periodic Leonardo sequence is defined recursively by*

$$GLe_n = \begin{cases} aGLe_{n-1} + GLe_{n-2} + a & \text{if } n \text{ is even,} \\ bGLe_{n-1} + GLe_{n-2} + b & \text{if } n \text{ is odd,} \end{cases} \qquad n \geq 2, \tag{9}$$

with the initial conditions $GLe_0 = 2a - 1$ and $GLe_1 = 2ab - 1$.

The first few elements of the bi-periodic Leonardo sequence are presented in Table 1.

When $a = b = 1$, expression (9) defines the Leonardo sequence (1). If $a = b = k$ for some positive integer k, we get the k-Leonardo numbers studied, for example, by Mangueira, Alves, and Catarino [13].

Note that expression (9) is equivalent to

$$GLe_{n+2} - a^{1-\xi[n]}b^{\xi[n]}GLe_{n+1} - GLe_n = a^{1-\xi[n]}b^{\xi[n]}, \tag{10}$$

where $\xi[\cdot]$ is the parity function defined by (6). For n even, by subtracting $GLe_{n+1} = bGLe_n + GLe_{n-1} + b$ from $GLe_n = aGLe_{n-1} + GLe_{n-2} + a$, we obtain

$$GLe_{n+1} = (b + 1)GLe_n + (1 - a)GLe_{n-1} - GLe_{n-2} + (b - a), \tag{11}$$

Table 1: The first six elements of $\{GLe_n\}_{n\geq 0}$

n	GLe_n
0	$2a-1$
1	$2ab-1$
2	$2a\left(ab+1\right)-1$
3	$2ab\left(ab+2\right)-1$
4	$2a\left(ab\left(ab+3\right)+1\right)-1$
5	$2ab\left(ab\left(ab+4\right)+3\right)-1$

while subtracting $GLe_{n+1}=aGLe_n+GLe_{n-1}+a$ from $GLe_n=bGLe_{n-1}+GLe_{n-2}+b$ for odd n, we obtain

$$GLe_{n+1}=(a+1)GLe_n+(1-b)GLe_{n-1}-GLe_{n-2}+(a-b). \tag{12}$$

Expressions (11) and (12) permit us to establish the following property.

Lemma 1 *The bi-periodic Leonardo sequence $\{GLe_n\}_{n=0}^{\infty}$ satisfies the following property*

$$\begin{aligned} GLe_{n+1} &= (a^{\xi[n]}b^{1-\xi[n]}+1)GLe_n+(1-a^{1-\xi[n]}b^{\xi[n]})GLe_{n-1} \\ &\quad -GLe_{n-2}+(-1)^n(b-a), \end{aligned} \tag{13}$$

for $n\geq 2$ and any real nonzero numbers a and b.

Putting $a=b=1$ in (13), we obtain the well-know recurrence (2) for Leonardo numbers (see [5]).

The next result gives us the relation between any bi-periodic Leonardo number with the two previous terms of this sequence with the same parity type of index.

Lemma 2 *The bi-periodic Leonardo sequence $\{GLe_n\}_{n=0}^{\infty}$ satisfies the following property: for any real nonzero numbers a and b and $n\geq 4$,*

$$GLe_n=\left(ab+2\right)GLe_{n-2}-GLe_{n-4}+ab. \tag{14}$$

Proof. For even n, the statement is obtained from the following chain of equalities:

$$\begin{aligned} GLe_{2n} &= aGLe_{2n-1}+GLe_{2n-2}+a \\ &= a\left(bGLe_{2n-2}+GLe_{2n-3}+b\right)+GLe_{2n-2}+a \\ &= \left(ab+1\right)GLe_{2n-2}+aGLe_{2n-3}+ab+a \\ &= \left(ab+1\right)GLe_{2n-2}+GLe_{2n-2}-GLe_{2n-4}+ab \\ &= \left(ab+2\right)GLe_{2n-2}-GLe_{2n-4}+ab. \end{aligned}$$

For odd n, the proof proceeds similarly. □

From identity (4), we get the bi-periodic Leonardo sequence of order n in terms of the bi-periodic Fibonacci number of order $n+1$.

Theorem 1 *For $n \geq 0$,*

$$GLe_n = 2a^{1-\xi[n]}b^{\xi[n]}F_{n+1}^{(a,b)} - 1. \tag{15}$$

Proof. First note that expression (15) is equivalent to

$$GLe_n = \begin{cases} 2a\left(F_{n+1}^{(a,b)}\right) - 1 & \text{if } n \text{ is even,} \\ 2b\left(F_{n+1}^{(a,b)}\right) - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{16}$$

The proof is carried out by induction on n. For $n = 0$, we have $GLe_0 = 2a - 1 = 2aF_1^{(a,b)} - 1$ and the statement is true, and for $n = 1$, we obtain $GLe_1 = 2ab - 1 = 2bF_2^{(a,b)} - 1$. Now suppose that expression (15) is true for all $2 \leq k \leq n$. We want to prove that the statement remains valid for $k = n+1$. If n is even then $n+1$ is odd, and taking into account the induction hypothesis and identity (4), we can write

$$\begin{aligned} GLe_{n+1} &= bGLe_n + GLe_{n-1} + b \\ &= b\left(2aF_{n+1}^{(a,b)} - 1\right) + \left(2bF_n^{(a,b)} - 1\right) + b \\ &= 2b\left(aF_{n+1}^{(a,b)} + F_n^{(a,b)}\right) - 1 \\ &= 2bF_{n+2}^{(a,b)} - 1. \end{aligned}$$

Further, if n is odd, then $n+1$ is even. Thus,

$$\begin{aligned} GLe_{n+1} &= aGLe_n + GLe_{n-1} + a \\ &= a\left(2bF_{n+1}^{(a,b)} - 1\right) + \left(2aF_n^{(a,b)} - 1\right) + a \\ &= 2a\left(bF_{n+1}^{(a,b)} + F_n^{(a,b)}\right) - 1 \\ &= 2aF_{n+2}^{(a,b)} - 1. \end{aligned}$$

□

According to Theorem 1, we can establish the next result, which can be easily proved.

Proposition 1 *For any nonzero integers a and b, and $n \geq 0$, GLe_n is an odd number.*

Moreover, Theorem 3 in [3] provides that for every integer n,

$$l_n = F_{n+1}^{(a,b)} + F_{n-1}^{(a,b)}.$$

Hence, Theorem 1 permits us to establish a connection between the bi-periodic Leonardo numbers and the bi-periodic Lucas numbers, given in the next proposition.

Proposition 2 *For the bi-periodic Leonardo numbers GLe_n and a, b nonzero real numbers, the following identity holds:*

$$GLe_{n+1} + GLe_{n-1} = 2a^{1-\xi[n]}b^{\xi[n]}l_{n+1} - 2, \tag{17}$$

where l_n is the n-th bi-periodic Lucas number.

Proof. By combining identities

$$l_n = F_{n+1}^{(a,b)} + F_{n-1}^{(a,b)}$$

and

$$GLe_n = 2a^{1-\xi[n]}b^{\xi[n]}F_{n+1}^{(a,b)} - 1,$$

we get

$$\begin{aligned} GLe_{n+1} + GLe_{n-1} &= 2a^{1-\xi[n]}b^{\xi[n]}(F_{n+2}^{(a,b)} + F_{n}^{(a,b)}) - 2 \\ &= 2a^{1-\xi[n]}b^{\xi[n]}l_{n+1} - 2. \end{aligned}$$

□

Proposition 3 *For the bi-periodic Leonardo numbers GLe_n and a, b nonzero real numbers, the following identity holds, whenever $ab + 4 \neq 0$,*

$$GLe_n = 2a^{1-\xi[n]}b^{\xi[n]}\left(\frac{l_n + l_{n+2}}{ab+4}\right) - 1,$$

where l_n is the n-th bi-periodic Lucas number.

2 The Binet formula and some other identities

In this section, we provide the Binet formula for the bi-periodic Leonardo sequence. We obtain the following result taking into account the Binet formula for the bi-periodic Fibonacci sequence given by expression (5) and Theorem 1.

Theorem 2 *For $n \geq 0$, the n-th bi-periodic Leonardo number is given by*

$$GLe_n = \frac{1}{\alpha - \beta}\left[2a^{1-\xi[n]}b^{\xi[n]}\left(\frac{a^{1-\xi(n+1)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\right)(\alpha^{n+1} - \beta^{n+1}) - (\alpha - \beta)\right], \tag{18}$$

where $\alpha = (ab + \sqrt{a^2b^2 + 4ab})/2$ and $\beta = (ab - \sqrt{a^2b^2 + 4ab})/2$ are the roots of the characteristic equation $x^2 - abx - ab = 0$, and $\xi[\cdot]$ is the parity function.

The proof of Theorem 2 is given by substituting the Binet formula (5) for bi-periodic Fibonacci number of order n in (15).

Corollary 1 in [3] established a negative integers extension by using the Binet formula for the bi-periodic Fibonacci and Lucas numbers, namely,

$$F_{-n}^{(a,b)} = (-1)^{n+1} F_n^{(a,b)}, \tag{19}$$

$$l_{-n}^{(a,b)} = (-1)^{n+1} l_n^{(a,b)}.$$

Similarly, the next result establishes a negative integers extension to the bi-periodic Leonardo numbers by using the Binet formula (18), and identities (15) and (19).

Corollary 1 *For any integer n, we have*

$$GLe_{-n} = (-1)^n (GLe_{n-2} + 2(1 - \xi[n])).$$

Proof. Recall that

$$GLe_n = 2a^{1-\xi[n]} b^{\xi[n]} F_{n+1}^{(a,b)} - 1.$$

By replacing n for $-n$ in (15), we get

$$GLe_{-n} = 2a^{1-\xi[-n]} b^{\xi[-n]} F_{-n+1}^{(a,b)} - 1.$$

Since $\xi[-n] = \xi[n]$ and $\xi[n-2] = \xi[n]$

$$\begin{aligned}
GLe_{-n} &= 2a^{1-\xi[-n]} b^{\xi[-n]} F_{-(n-1)}^{(a,b)} - 1, \\
&= 2a^{1-\xi[n]} b^{\xi[n]} (-1)^n F_{n-1}^{(a,b)} - 1, \\
&= (-1)^n (2a^{1-\xi[n]} b^{\xi[n]} (-1)^n F_{n-1}^{(a,b)} + (-1)^{n+1}), \\
&= (-1)^n (2a^{1-\xi[n-2]} b^{\xi[n-2]} F_{n-1}^{(a,b)} + (-1)^{n+1}), \\
&= (-1)^n ((2a^{1-\xi[n-2]} b^{\xi[n-2]} F_{n-1}^{(a,b)} - 1) + 1 + (-1)^n), \\
&= (-1)^n (GLe_{n-2} + 2(1 - \xi[n])).
\end{aligned}$$

□

3 The generating function for the bi-periodic Leonardo sequence

Generating functions give a technique for solving linear homogeneous recurrence relations. In this section, we establish the generating function for the bi-periodic Leonardo sequence.

Theorem 3 *The generating function for bi-periodic Leonardo numbers is given by*

$$\begin{aligned}\sum_{n=0}^{\infty} GLe_n x^n &= \frac{(4a-2-ab)-abx-(2a-1)x^2+(2ab-1)(ab+2)x^3}{(1-(ab+2)x^2+x^4)} \\ &+ \frac{ab}{(1-x)((1-(ab+2)x^2+x^4))}. \qquad (20)\end{aligned}$$

Proof. Consider the generating functions on parity:

$$h(x)=\sum_{j=0}^{\infty} GLe_{2j}x^{2j},$$

$$g(x)=\sum_{j=0}^{\infty} GLe_{2j+1}x^{2j+1}.$$

Note that

$$h(x)=(2a-1)+(2a(ab+1)-1)x^2+\sum_{j=2}^{\infty} GLe_{2j}x^{2j},$$

$$\begin{aligned}(ab+2)x^2h(x) &= \sum_{j=0}^{\infty}(ab+2)GLe_{2j}x^{2j+2} \\ &= (2a-1)(ab+2)x^2+\sum_{m=2}^{\infty}(ab+2)GLe_{2m-2}x^{2m},\end{aligned}$$

or

$$x^4h(x)=\sum_{j=0}^{\infty} GLe_{2j}x^{2j+4}=\sum_{m=2}^{\infty} GLe_{2m-4}x^{2m}.$$

Thus,

$$\begin{aligned}(1-(ab+2)x^2+x^4)h(x) &= (2a-1)+(2a(ab+1)-1)x^2 \\ &\quad -(2a-1)(ab+2)x^2+ab\sum_{m=2}^{\infty} x^{2m}.\end{aligned}$$

Similarly, we have

$$g(x)=(2ab-1)x+\sum_{m=1}^{\infty} GLe_{2m+1}x^{2m+1},$$

$$\begin{aligned}(ab+2)x^2g(x) &= \sum_{j=0}^{\infty}(ab+2)GLe_{2j+1}x^{2j+3} \\ &= (2ab-1)(ab+2)x^3+\sum_{m=2}^{\infty}(ab+2)GLe_{2m-1}x^{2m+1}\end{aligned}$$

and

$$x^4 g(x) = \sum_{j=0}^{\infty} GLe_{2j-1} x^{2j+5} = \sum_{m=2}^{\infty} GLe_{2m-3} x^{2m+1}.$$

Thus,

$$(1-(ab+2)x^2+x^4)h(x) = (2a-1)+(2ab-1)(ab+2)x^3 + ab\sum_{m=2}^{\infty} x^{2m-1}.$$

Therefore,

$$\begin{aligned} F(x) &= \frac{(4a-2-ab)-abx-(2a-1)x^2+(2ab-1)(ab+2)x^3}{(1-(ab+2)x^2+x^4)} \\ &+ \frac{ab}{(1-x)((1-(ab+2)x^2+x^4))}. \end{aligned}$$

□

We can obtain the generating function in terms of the parity function as follows.

Theorem 4 *The generating function for bi-periodic Leonardo numbers is given by*

$$\begin{aligned} \sum_{n=0}^{\infty} GLe_n x^n = & \frac{(2a-1)(1-a^{1-\xi[n]}b^{\xi[n]}x)+(2ab-1)}{(1-a^{1-\xi[n]}b^{\xi[n]}x-x^2)} \\ & + \frac{a^{1-\xi[n]}b^{\xi[n]}x^2}{(1-x)(1-a^{1-\xi[n]}b^{\xi[n]}x-x^2)}. \end{aligned} \tag{21}$$

Proof. Expression (10) multiplied by x^{n+2}, gives us

$$GLe_{n+2}x^{n+2} - a^{1-\xi[n]}b^{\xi[n]}GLe_{n+1}x^{n+2} - GLe_n x^{n+2} = a^{1-\xi[n]}b^{\xi[n]}x^{n+2}.$$

This implies

$$\begin{aligned} & \sum_{n=0}^{\infty} GLe_{n+2}x^{n+2} - a^{1-\xi[n]}b^{\xi[n]}\sum_{n=0}^{\infty} GLe_{n+1}x^{n+2} - \sum_{n=0}^{\infty} GLe_n x^{n+2} \\ = \ & a^{1-\xi[n]}b^{\xi[n]}\sum_{n=0}^{\infty} x^{n+2}, \end{aligned}$$

or, by considering the expression $F(x) = \sum_{j=0}^{\infty} GLe_j x^j$,

$$(F(x)-GLe_0-GLe_1)-a^{1-\xi[n]}b^{\xi[n]}x(F(x)-GLe_0)-x^2F(x) = \frac{a^{1-\xi[n]}b^{\xi[n]}x^2}{1-x}.$$

Thus,

$$F(x) = \frac{(2a-1)(1-a^{1-\xi[n]}b^{\xi[n]}x)+(2ab-1)}{(1-a^{1-\xi[n]}b^{\xi[n]}x-x^2)} + \frac{a^{1-\xi[n]}b^{\xi[n]}x^2}{(1-x)(1-a^{1-\xi[n]}b^{\xi[n]}x-x^2)}.$$

□

4 Catalan's, Cassini's and d'Ocgane's identities

In this section, we provide some identities involving the bi-periodic Leonardo numbers. In particular, Catalan's and Cassini's identities for bi-periodic Leonardo numbers are established.

First of all, recall a result established in [Theorem 5, [3]]: for all integers m and n, it holds

$$F_{m+n}^{(a,b)} = \frac{1}{2}\left[\left(\frac{b}{a}\right)^{\xi[m+1]\xi[n]} F_m^{(a,b)} l_n + \left(\frac{b}{a}\right)^{\xi[m]\xi[n+1]} F_n^{(a,b)} l_m\right]. \tag{22}$$

We prove the following new relation between the Fibonacci and Lucas bi-periodic numbers.

Lemma 3 *For any nonnegative integer n, we have*

$$F_{n+1+r}^{(a,b)} + F_{n+1-r}^{(a,b)} = \begin{cases} F_{n+1}^{(a,b)} l_r & \text{if } n \text{ is even,} \\ F_r^{(a,b)} l_{n+1} & \text{if } n \text{ is odd,} \end{cases} \tag{23}$$

Proof. By (22), we obtain

$$F_{(n+1)+r}^{(a,b)} = \frac{1}{2}\left[\left(\frac{b}{a}\right)^{\xi[n+2]\xi[r]} F_{n+1}^{(a,b)} l_r + \left(\frac{b}{a}\right)^{\xi[n+1]\xi[r+1]} F_r^{(a,b)} l_{n+1}\right]. \tag{24}$$

On the other side, since $F_{-n}^{(a,b)} = (-1)^{n+1}F_{-n}^{(a,b)}$ and $l_{-n} = (-1)^{n+1}l_{-n}$ by [Corollary 1, [3]], we get

$$F_{(n+1)-r}^{(a,b)} = \frac{(-1)^r}{2}\left[\left(\frac{b}{a}\right)^{\xi[n+2]\xi[r]} F_{n+1}^{(a,b)} l_r - \left(\frac{b}{a}\right)^{\xi[n+1]\xi[r+1]} F_r^{(a,b)} l_{n+1}\right]. \tag{25}$$

Thus, for r even, the sum of expressions (24) and (25) gives

$$F_{n+1+r}^{(a,b)} + F_{n+1-r}^{(a,b)} = F_{n+1}^{(a,b)} l_r.$$

Hence, for r odd we verify the equality

$$F_{n+1+r}^{(a,b)} + F_{n+1-r}^{(a,b)} = F_r^{(a,b)} l_{n+1}.$$

□

Note also that Lemma 3 can be proved by direct application of the Binet formulas for $F_n^{(a,b)}$ and l_n, namely,

$$F_n^{(a,b)} = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right) \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$l_n = \left(\frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}}\right) (\alpha^n - \beta^n),$$

by considering the parity of integers n and r.

The next result establishes Catalan's identity for the bi-periodic Leonardo numbers.

Theorem 5 *For two positive integers n and r, the following equation holds true:*

$$\begin{aligned} & a^{\xi[n-r]} b^{1-\xi[n-r]} GLe_{n-r} GLe_{n+r} - a^{\xi[n]} b^{1-\xi[n]} GLe_n^2 \\ = \; & ab(a^{\xi[n+1]+2\xi[r-1]-2} b^{1-\xi[n+1]-2\xi[r-1]} (-1)^{n+2-r} (GLe_{r-1}+1)^2 \\ & +2a^{\xi[n]} b^{1-\xi[n]} (GLe_n + 1) - a^{\xi[n-r]} b^{1-\xi[n-r]} (GLe_{n-r} + GLe_{n+r} + 2) \\ & +a^{\xi[n-r]} b^{1-\xi[n-r]} - a^{\xi[n]} b^{1-\xi[n]}. \end{aligned} \tag{26}$$

Proof. By (15), we get

$$\begin{aligned} & a^{\xi[n-r]} b^{1-\xi[n-r]} GLe_{n-r} GLe_{n+r} - a^{\xi[n]} b^{1-\xi[n]} GLe_n^2 \\ = \; & a^{\xi[n-r]} b^{1-\xi[n-r]} (2a^{1-\xi[n-r]} b^{\xi[n-r]} F_{n-r+1}^{(a,b)} - 1)(2a^{1-\xi[n+r]} b^{\xi[n+r]} F_{n+r+1}^{(a,b)} - 1) \\ & -a^{\xi[n]} b^{1-\xi[n]} (2a^{1-\xi[n]} b^{\xi[n]} F_{n+1}^{(a,b)} - 1)^2, \\ = \; & 4a^{2-\xi[n+r]} b^{1+\xi[n+r]} F_{n-r+1}^{(a,b)} F_{n+r+1}^{(a,b)} - 4a^{2-\xi[n]} b^{1+\xi[n]} (F_{n+1}^{(a,b)})^2 \\ & +4abF_{n+1}^{(a,b)} - 2ab(F_{n-r+1}^{(a,b)} + bF_{n+r+1}^{(a,b)}) + a^{\xi[n-r]} b^{1-\xi[n-r]} - a^{\xi[n]} b^{1-\xi[n]}. \end{aligned} \tag{27}$$

Since $\xi[n-r+1] = 1-\xi[n-r] = 1-\xi[n+r]$ and $\xi[n+1] = 1-\xi[n]$, further we can write

$$\begin{aligned} & 4ab(a^{1-\xi[n+r]} b^{\xi[n+r]} F_{n-r+1}^{(a,b)} F_{n+r+1}^{(a,b)} - a^{1-\xi[n]} b^{\xi[n]} (F_{n+1}^{(a,b)})^2) \\ = \; & 4ab(a^{\xi[n-r+1]} b^{1-\xi[n-r+1]} F_{n-r+1}^{(a,b)} F_{n+r+1}^{(a,b)} - a^{\xi[n+1]} b^{1-\xi[n+1]} (F_{n+r}^{(a,b)})^2) \\ = \; & 4ab(a^{\xi[n+1]} b^{1-\xi[n+1]} (-1)^{n+2-r} (F_r^{(a,b)})^2). \end{aligned} \tag{28}$$

Hence, by substituting (28) in (27), we obtain

$$\begin{aligned} & a^{\xi[n-r]}b^{1-\xi[n-r]}GLe_{n-r}GLe_{n+r} - a^{\xi[n]}b^{1-\xi[n]}GLe_n^2 \\ = \; & 4ab(a^{\xi[n+1]}b^{1-\xi[n+1]}(-1)^{n+2-r}(F_r^{(a,b)})^2) + 4abF_{n+1}^{(a,b)} \\ & -2ab(F_{n-r+1}^{(a,b)} + F_{n+r+1}^{(a,b)}) + a^{\xi[n-r]}b^{1-\xi[n-r]} - a^{\xi[n]}b^{1-\xi[n]}. \end{aligned} \tag{29}$$

Therefore, putting

$$\frac{GLe_{r-1}+1}{2a^{1-\xi[r-1]}b^{\xi[r-1]}} = F_r^{(a,b)}, \qquad \frac{GLe_n+1}{2a^{1-\xi[n]}b^{\xi[n]}} = F_{n+1}^{(a,b)},$$

$$\frac{GLe_{n-r}+1}{2a^{1-\xi[n-r]}b^{\xi[n-r]}} = F_{n-r+1}^{(a,b)}, \qquad \frac{GLe_{n+r}+1}{2a^{1-\xi[n+r]}b^{\xi[n+r]}} = F_{n+r+1}^{(a,b)}$$

in (29), we get

$$\begin{aligned} & a^{\xi[n-r]}b^{1-\xi[n-r]}GLe_{n-r}GLe_{n+r} - a^{\xi[n]}b^{1-\xi[n]}GLe_n^2 \\ = \; & ab(a^{\xi[n+1]+2\xi[r-1]-2}b^{1-\xi[n+1]-2\xi[r-1]}(-1)^{n+2-r}(GLe_{r-1}+1)^2 \\ & +2a^{\xi[n]}b^{1-\xi[n]}(GLe_n+1) - a^{\xi[n-r]}b^{1-\xi[n-r]}(GLe_{n-r}+GLe_{n+r}+2) \\ & +a^{\xi[n-r]}b^{1-\xi[n-r]} - a^{\xi[n]}b^{1-\xi[n]}. \end{aligned}$$

□

The next result provides Cassini's identity for the bi-periodic Leonardo numbers, established in [Proposition 4.1, [5]], it also can be verified by setting $r = 1$ in Theorem 5.

Proposition 4 *For any integer n, it holds*

$$\begin{aligned} & a^{1-\xi[n]}b^{\xi[n]}GLe_{n-1}GLe_{n+1} - a^{\xi[n]}b^{1-\xi[n]}GLe_n^2 \\ = \; & 4(-1)^{n+1}(ab)a^{1-\xi[n]}b^{\xi[n]} - a^{\xi[n]}b^{1-\xi[n]}(2GLe_{n-1} \\ & +(a^{1-\xi[n]}b^{\xi[n]}-2)GLe_n + ab + 4a^{\xi[n]}b^{1-\xi[n]}) + (-1)^n(a-b), \end{aligned} \tag{30}$$

Proof. By (15), we get

$$\begin{aligned} & a^{\xi[n-1]}b^{1-\xi[n-1]}GLe_{n-1}GLe_{n+1} - a^{\xi[n]}b^{1-\xi[n]}GLe_n^2 \\ = \; & 4a^{(2-\xi[n+1])}b^{1+\xi[n+1]}F_n^{(a,b)}F_{n+2}^{(a,b)} - 2abF_n^{(a,b)} - 2abF_{n+2}^{(a,b)} + a^{\xi[n-1]}b^{1-\xi[n-1]} \\ & -4a^{2-\xi[n]}b^{1+\xi[n]}(F_{n+1}^{(a,b)})^2 + 4abF_{n+1}^{(a,b)} - a^{\xi[n]}b^{1-\xi[n]}. \end{aligned} \tag{31}$$

Now, according to (31) for n even, we can write

$$\begin{aligned} & aGLe_{n-1}GLe_{n+1} - bGLe_n^2 \\ = \; & 4ab^2F_n^{(a,b)}F_{n+2}^{(a,b)} - 2ab(F_n^{(a,b)} + F_{n+2}^{(a,b)}) + a - 4a^2b(F_{n+1}^{(a,b)})^2 + 4abF_{n+1}^{(a,b)} - b \\ = \; & 4ab(bF_n^{(a,b)}F_{n+2}^{(a,b)} - a(F_{n+1}^{(a,b)})^2) - 2ab(F_n^{(a,b)} + F_{n+2}^{(a,b)}) + 4abF_{n+1}^{(a,b)} + a - b. \end{aligned}$$

Since

$$bF_n^{(a,b)}F_{n+2}^{(a,b)} - a(F_{n+1}^{(a,b)})^2 = a(-1)^{n+1}(F_1^{(a,b)})^2,$$

for n even,

$$\begin{aligned} aGLe_{n-1}GLe_{n+1} - bGLe_n^2 &= 4a(-1)^{n+1}ab - b(2GLe_{n-1}) \\ &\quad -b((a-2)GLe_n + ab + 4b) + a - b. \end{aligned}$$

Similarly, according to (31) for n odd, we have

$$\begin{aligned} bGLe_{n-1}GLe_{n+1} - aGLe_n^2 &= 4a^2b^1F_n^{(a,b)}F_{n+2}^{(a,b)} - 2abF_n^{(a,b)} - 2abF_{n+2}^{(a,b)} \\ &\quad +b - 4ab^2(F_{n+1}^{(a,b)})^2 + 4abF_{n+1}^{(a,b)} - a. \end{aligned}$$

Since

$$aF_n^{(a,b)}F_{n+2}^{(a,b)} - b(F_{n+1}^{(a,b)})^2 = b(-1)^{n+1}(F_1^{(a,b)})^2,$$

for n odd, we obtain

$$\begin{aligned} bGLe_{n-1}GLe_{n+1} - aGLe_n^2 &= 4(-1)^{n+1}ab^2 - a(2GLe_{n-1}) \\ &\quad -a(b-2)GLe_n + ab + 4a) + (b-a). \end{aligned}$$

□

As a corollary of Proposition 4, we have Cassini's identity for the Leonardo numbers established in also [Proposition 4.1, [5]] and [Corollary 2.16,[1]]).

Corollary 2 *For positive integer n and $a = b = 1$, the following identity holds*

$$GLe_{n-1}GLe_{n+1} - GLe_n^2 = 4(-1)^{n+1} - (GLe_{n-1} - GLe_{n-2}). \tag{32}$$

The next result establishes d'Ocgane's identity for the bi-periodic Leonardo numbers.

Theorem 6 *For m, n positive integers, the following expressions hold true:*

$$\begin{aligned} & a^{\xi[mn+m]}b^{\xi[mn+n]}GLe_{m-1}GLe_n - a^{\xi[mn+n]}b^{\xi[mn+m]}GLe_mGLe_{n-1} \\ = \; & GLe_{m-1}GLe_n - GLe_mGLe_{n-1} \qquad (33) \\ = \; & (-1)^n 2a^{2-\xi[m-1]-\xi[n]}b^{\xi[m-1]+\xi[n]-1}(GLe_{m-n-1}+1) \\ & -GLe_{m-1} + GLe_{n-1} - GLe_n + GLe_m \end{aligned}$$

if m and n have the same parity, and, otherwise,

$$\begin{aligned} & a^{\xi[mn+m]}b^{\xi[mn+n]}GLe_{m-1}GLe_n - a^{\xi[mn+n]}b^{\xi[mn+m]}GLe_mGLe_{n-1} \\ = \; & a^{5\xi[n]-2}b^{3\xi[m]-2\xi[n]}(GLe_{m-1}+1)(GLe_n+1) \\ & -a^{5\xi[m]-2}b^{3\xi[n]-2\xi[m]}(GLe_m+1)(GLe_{n-1}+1) \qquad (34) \\ & -a^{2\xi[m]+\xi[n]-1}b^{3\xi[n]}(GLe_{m-1}+GLe_n+2) \\ & +a^{2\xi[n]+\xi[m]-1}b^{3\xi[m]}(GLe_m+GLe_m+2). \end{aligned}$$

Proof. By (15), we get

$$\begin{aligned}
& a^{\xi[mn+m]}b^{\xi[mn+n]}GLe_{m-1}GLe_n - a^{\xi[mn+n]}b^{\xi[mn+m]}GLe_mGLe_{n-1} \\
= \; & a^{\xi[mn+m]}b^{\xi[mn+n]}4a^{2-\xi[m-1]-\xi[n]}b^{\xi[m-1]+\xi[n]}F_m^{(a,b)}F_{n+1}^{(a,b)} \\
& -a^{\xi[mn+n]}b^{\xi[mn+m]}4a^{2-\xi[m]-\xi[n-1]}b^{\xi[m]+\xi[n-1]}F_{m+1}^{(a,b)}F_n^{(a,b)} \\
& +a^{\xi[mn+m]}b^{\xi[mn+n]}(-2a^{1-\xi[m-1]}b^{\xi[m-1]}F_m^{(a,b)} - 2a^{1-\xi[n]}b^{\xi[n]}F_{n+1}^{(a,b)}) \\
& -a^{\xi[mn+n]}b^{\xi[mn+m]}(-2a^{1-\xi[m]}b^{\xi[m]}F_{m+1}^{(a,b)} - 2a^{1-\xi[n-1]}b^{\xi[n-1]}F_n^{(a,b)}).
\end{aligned} \tag{35}$$

Considering the same parity $\xi[n] = \xi[m]$ for n and m in (35), we have

$$2-\xi[m-1]-\xi[n] = 2-\xi[n-1]-\xi[m], \quad \xi[m-1]+\xi[n] = \xi[n-1]+\xi[m].$$

Then

$$\begin{aligned}
& a^{\xi[mn+m]}b^{\xi[mn+n]}GLe_{m-1}GLe_n - a^{\xi[mn+n]}b^{\xi[mn+m]}GLe_mGLe_{n-1} \\
= \; & 4a^{2-\xi[m-1]-\xi[n]}b^{\xi[m-1]+\xi[n]}(a^{\xi[mn+m]}b^{\xi[mn+n]}F_m^{(a,b)}F_{n+1}^{(a,b)} \\
& -a^{\xi[mn+n]}b^{\xi[mn+m]}F_{m+1}^{(a,b)}F_n^{(a,b)}) \\
= \; & a^{\xi[mn+m]}b^{\xi[mn+n]}(-2a^{1-\xi[m-1]}b^{\xi[m-1]}F_m^{(a,b)} - 2a^{1-\xi[n]}b^{\xi[n]}F_{n+1}^{(a,b)}) \\
& -a^{\xi[mn+n]}b^{\xi[mn+m]}(-2a^{1-\xi[m]}b^{\xi[m]}F_{m+1}^{(a,b)} - 2a^{1-\xi[n-1]}b^{\xi[n-1]}F_n^{(a,b)}).
\end{aligned}$$

According to the identities

$$\begin{aligned}
& a^{\xi[mn+m]}b^{\xi[mn+n]}F_m^{(a,b)}F_{n+1}^{(a,b)} - a^{\xi[mn+n]}b^{\xi[mn+m]}F_{m+1}^{(a,b)}F_n^{(a,b)} \\
= \; & (-1)^n a^{\xi[m-n]}F_{m-n}^{(a,b)},
\end{aligned}$$

$$\xi[nm+n] = \xi[mn+n] = 0,$$

and (15), we obtain

$$\begin{aligned}
& a^{\xi[mn+m]}b^{\xi[mn+n]}GLe_{m-1}GLe_n - a^{\xi[mn+n]}b^{\xi[mn+m]}GLe_mGLe_{n-1} \\
= \; & GLe_{m-1}GLe_n - GLe_mGLe_{n-1} \\
= \; & (-1)^n 2a^{2-\xi[m-1]-\xi[n]}b^{\xi[m-1]+\xi[n]-1}(GLe_{m-n-1}+1) \\
& -GLe_{m-1} + GLe_{n-1} - GLe_n + GLe_m,
\end{aligned}$$

for m, n with the same parity.

Now suppose that m and n have opposite parity. This implies that $\xi[mn+m] = \xi[m]$ and $\xi[mn+n] = \xi[n]$. Then, we can rewrite (35) in the following form:

$$\begin{aligned}
& a^{\xi[mn+m]}b^{\xi[mn+n]}GLe_{m-1}GLe_n - a^{\xi[mn+n]}b^{\xi[mn+m]}GLe_mGLe_{n-1} \\
= \; & a^{\xi[m]}b^{\xi[n]}GLe_{m-1}GLe_n - a^{\xi[n]}b^{\xi[m]}GLe_mGLe_{n-1} \\
= \; & 4(a^{3\xi[n]}b^{3\xi[m]}F_m^{(a,b)}F_{n+1}^{(a,b)} - a^{3\xi[m]}b^{3\xi[n]}F_{m+1}^{(a,b)}F_n^{(a,b)}) \\
& -2a^{2\xi[m]}b^{2\xi[n]}(F_m^{(a,b)} + F_{n+1}^{(a,b)}) + 2a^{2\xi[n]}b^{2\xi[m]}(F_{m+1}^{(a,b)} + F_n^{(a,b)}).
\end{aligned}$$

By (15), we get

$$\begin{aligned}
& a^{\xi[mn+m]}b^{\xi[mn+n]}GLe_{m-1}GLe_n - a^{\xi[mn+n]}b^{\xi[mn+m]}GLe_mGLe_{n-1} \\
= \ & a^{\xi[m]}b^{\xi[n]}GLe_{m-1}GLe_n - a^{\xi[n]}b^{\xi[m]}GLe_mGLe_{m-1} \\
= \ & a^{5\xi[n]-2}b^{3\xi[m]-2\xi[n]}(GLe_{m-1}+1)(GLe_n+1) \\
& -a^{5\xi[m]-2}b^{3\xi[n]-2\xi[m]}(GLe_m+1)(GLe_{n-1}+1) \\
& -a^{2\xi[m]+\xi[n]-1}b^{3\xi[n]}(GLe_{m-1}+GLe_n+2) \\
& +a^{2\xi[n]+\xi[m]-1}b^{3\xi[m]}(GLe_m+GLe_m+2).
\end{aligned}$$

□

5 Conclusions

In this paper, we defined a new generalization of the Leonardo sequence of numbers, the bi-periodic Leonardo sequence $\{GLe_n\}_{n\geq 0}$, and established algebraic properties, the Binet formula and generating function for this sequence. In addition, identities involving the bi-periodic Leonardo numbers, the bi-periodic Fibonacci numbers, and the bi-periodic Lucas numbers are provided. Moreover, we established the analogous of classical identities, such as Catalan's, Cassini's, and d'Ocgane's identities for the bi-periodic Leonardo numbers.

We believe that the introduced sequence of numbers can serve as an object for study in several aspects such as combinatorial, analytical, and matrix perspectives.

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