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Operator $\Box^r$ on a submanifold of Riemannian manifold and its applications

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Abstract. The paper generalizes the self-adjoint differential operator $\Box$ on hypersurfaces of a constant curvature manifold to submanifolds, introduced by Cheng-Yau. Using the series of such operators, a new way to prove Minkowski-Hsiung integral formula is given and some integral formulas for compact submanifolds is derived. An application to a hypersurface of a Riemannian manifold with harmonic Riemannian curvature is presented.

Key Words: Newton tensor, operator, submanifold, hypersurface, Codazzi tensor
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Introduction

Denote by V and W an n-dimensional and a p-dimensional vector spaces, respectively, V^* the dual space of V, $\{e_i\}(i = 1, \ldots, n)$ and $\{e_\alpha\}(\alpha = 1, \ldots, p)$ bases of V and of W, respectively. Let the tensor $D = \sum\limits_{\alpha,i,j} D^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha \in V^* \bigotimes V^* \bigotimes W$ be symmetric which means that $D^\alpha_{ij} = D^\alpha_{ji}$, where $\{\omega_i\}$ is the dual basis of $\{e_i\}$. In this paper we first define the r-th Newton tensor $T_{(r)}(D)\,(r = 0, 1, \ldots, n)$, determined by the tensor D of type $(1, 2)$ which will be called the generalized Newton tensor. When V is the tangent space to a submanifold at some point, and D is the second fundamental form of the submanifold (associated with the metric), the r-th elementary symmetric functions are called the modified mean curvatures. Following this, we define in the paper the r-th modified mean curvatures of D^α_{ij} and call them Q_r. We also study some algebraic properties of the r-th Newton tensor associated

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with r-th modified mean curvatures and the properties of them for a submanifold of a space with constant sectional curvature. We note that these definitions and properties are natural generalizations of the classical Newton tensor and the r-th elementary symmetric polynomial's definitions and properties (see [17]). Then, following the operator introduced by Cheng-Yau in [6] and using the Newton tensor we induce a series of differential operators $\Box^r$ which are adjoint relative to the L^2-inner product. In the study of those properties, we find a new way to prove Minkowski-Hsiung integral formula and derive some integral formulas for compact submanifolds, which are analogous to the usual Minkowski-Hsiung integral formula. Considering the case $\Box^r$ acts on Q_r, we obtain two general conclusions. Finally, we focus on the $\Box^2$ operator for a hypersurface of a Riemannian manifold with harmonic Riemanian curvature to study and obtain a result of [20].

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1 The generalized Newton tensor and the higher order mean curvatures

We begin with an algebra, first recalling some fundamental formulas. Let V be a (real) n-dimensional vector space, and $D : V \longrightarrow V$ be a diagonalizable linear transformation. We fix a basis $\{v_i, i = 1, \ldots, n\}$ of V, and denote the matrix of D relative to this basis by (D_{ij}), and the eigenvalues of D relative to this basis by $k_1, \ldots, k_n$.
The r-th elementary symmetric function is

$$Q_r = \sum_{1\le i_1\le\cdots\le i_r\le n} k_{i_1}\cdots k_{i_r} = \frac{1}{r!}\sum_{i_1,\cdots,i_r} k_{i_1}\cdots k_{i_r}.$$

The r-th Newton tensor is

$$T_{(r)}(D) = Q_r I - Q_{r-1}D + \cdots + (-1)^r D_r,$$

where D_r denotes the r-times linear transformation on the vector space V by D. Relative to $\{v_i\}$, the matrix of $T_r(D)$ is

$$T_{(r)ij} = Q_r\delta_{ij} - Q_{r-1}D_{ij} + \cdots + (-1)^r D_{ii_1}\cdots D_{i_r j}.$$

R. C. Reilly gave the following properties (see [17]):

1). $T_{(r+1)}(D) = Q_{r+1}I - DT_{(r)}$, $r = 0, 1, \ldots, n$, where I is the identity transformation.

2). $T_{(r)}(D) = DT_{(r)}$.

3). $(r+1)Q_{r+1} = \text{Trace}(DT_{(r)})$.

4). Let $D = D(t)$ be a smooth one-parameter family of diagonalizable transformations of V. Then for $r = 0, 1, \ldots, n$ we have

$$\frac{\partial Q_{r+1}}{\partial t} = \mathrm{Trace}(\frac{\partial D}{\partial t} T_{(r)}).$$

We recall the definition of **the generalized Kronecker symbols** (see [4]):

$$\varepsilon^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} = \begin{cases} +1, & \text{if } (j_1,\ldots,j_r) \text{ are distinct, and } (j_1,\ldots,j_r) \\ & \text{is an even permutation of } (i_1,\ldots,i_r); \\ -1, & \text{if } (j_1,\ldots,j_r) \text{ are distinct, and } (j_1,\ldots,j_r) \\ & \text{is an odd permutation of } (i_1,\ldots,i_r); \\ 0, & \text{other case .} \end{cases}$$

Remark 1.1 *Moreover, the generalized Kronecker symbol can be expressed in terms of the matrix*

$$\varepsilon^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} = \begin{vmatrix} \delta_{i_1 j_1} & \cdots & \delta_{i_1 j_r} \\ \delta_{i_2 j_1} & \cdots & \delta_{i_2 j_r} \\ \vdots & \ddots & \vdots \\ \delta_{i_r j_1} & \ldots & \delta_{i_r j_r} \end{vmatrix}, \tag{1.1}$$

where δ_{ij} is the standard Kronecker delta, which means:

$$\delta_{ij} = \begin{cases} +1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Lemma 1.1

$$Q_r = \frac{1}{r!} \sum \varepsilon^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} D_{i_1 j_1} \cdots D_{i_r j_r}, \tag{1.2}$$

$$T_{(r)ij} = \frac{1}{r!} \sum \varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,j} D_{i_1 j_1} \cdots D_{i_r j_r}. \tag{1.3}$$

Proof. For $D_{i_1 j_1} = k_{i_1}\delta_{i_1 j_1}, \ldots, D_{i_r j_r} = k_{i_r}\delta_{i_r j_r}$, we have

$$\begin{aligned} \frac{1}{r!} \sum \varepsilon^{i_1,\ldots,i_r}_{j_1,\ldots,j_r} D_{i_1 j_1} \cdots D_{i_r j_r} &= \frac{1}{r!} \sum \varepsilon^{i_1,\ldots,i_r}_{i_1\ldots,i_r} k_{i_1} \cdots k_{i_r} \\ &= \frac{1}{r!} \sum k_{i_1} \cdots k_{i_r} \\ &= Q_r. \end{aligned}$$

¿From the right part of (1.3), we know that the generalized Kronecker symbol can be expressed in terms of (1.1), then if we express the matrix of $\varepsilon^{i_1,\ldots,i_r,i}_{i_1,\ldots,i_r,j}$ by unfolding the matrix along its last line, we obtain

$$\begin{aligned} \frac{1}{r!} \sum \varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,j} D_{i_1 j_1} \cdots D_{i_r j_r} &= \frac{1}{r!} \sum \varepsilon^{i_1,\ldots,i_r,i}_{i_1,\ldots,i_r,j} k_{i_1} \cdots k_{i_r} \\ &= Q_r \delta_{ij} - T_{(r-1)il} D_{lj}. \end{aligned}$$

Using the property 1 given by R. C. Reilly, we obtain that (1.3) is true. □

Remark 1.2 *These can be viewed as the second expression of the r-th elementary symmetric function and the r-th Newton tensor (the papers [11, 16] make use of this kind of expression).*

Let V and W denote an n-dimensional and a p-dimensional vector spaces, respectively, V^* denotes the dual space of V, $\{e_i\}\,(i=1,\ldots,n)$ and $\{e_\alpha\}\,(\alpha=1,\ldots,p)$ denote bases of V and of W, respectively.
Let $D=\sum\limits_{\alpha,i,j} D^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha \in V^*\bigotimes V^*\bigotimes W$ be symmetric which means $D^\alpha_{ij}=D^\alpha_{ji}$, where $\{\omega_i\}$ is the dual basis to $\{e_i\}$. In this paper we begin with defining the r-th Newton tensor $T_{(r)}(D)(r=0,1,\ldots,n)$. Closely following the second exposition for the Newton tensor, and imitating the definition of the mean curvature in [17, 11], we define the generalized Newton tensor as follows:

Definition 1.1 *1) If r is an odd integer, $r=2k+1$ $(k=0,1,\ldots)$, then $T_{(r)}(D)$ is a mapping $T_{(r)}(D):V^*\bigotimes V^*\bigotimes W\longrightarrow V^*\bigotimes V^*$ such that for $Z=Z^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha$*

$$T_{(r)}(D)Z=\frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,l}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}}D^{\alpha_k}_{i_{r-1}j_{r-1}})(D^\alpha_{i_rj_r}Z^\alpha_{lj})\omega_i\otimes\omega_j.$$

Denoting $T^\alpha_{(r)il}(D)=\frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,l}(D^{\alpha_1}_{i_1j_1},D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}},D^{\alpha_k}_{i_{r-1}j_{r-1}})D^\alpha_{i_rj_r}$, we have

$$T_{(r)}(D)Z=T^\alpha_{(r)il}Z^\alpha_{lj}\omega_i\otimes\omega_j,\quad (T_{(r)}(D)Z)_{ij}=T^\alpha_{(r)il}Z^\alpha_{lj}.$$

2) If r is an even integer, $r=2k$ $(k=0,1,\ldots)$, then $T_{(r)}(D)$ is detemined as a map $T_{(r)}(D):V^*\bigotimes V^*\bigotimes W\longrightarrow V^*\bigotimes V^*\bigotimes W$ such that

$$T_{(r)}(D)Z=\frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,l}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})Z^\alpha_{lj}\omega_i\otimes\omega_j\otimes e_\alpha.$$

Denoting $T_{(r)il}(D)=\frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,l}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})$, we obtain

$$T_{(r)}(D)Z=T_{(r)il}Z^\alpha_{lj}\omega_i\otimes\omega_j\otimes e_\alpha,\quad (T_{(r)}(D)Z)^\alpha_{ij}=T_{(r)il}Z^\alpha_{lj}.$$

The map $T_{(r)}(D)$ is called **the generalized Newton transformation (or tensor)** *of D.*

Remark 1.3 *For convenience to compute, in this Section, we shall agree that repeated indices are summed, and $T_{(r)}(D)$ is viewed as $T_{(r)}$ if $r=0$, $T_{(0)ij}=\delta_{ij}$, if $r=n$, $T_{(n)ij}=0$. Also, we suppose $T_{(r)}=T_{(r)}(D)$.*

We are really interested only in the situation where V is the tangent space to a submanifold, and D is the second fundamental form of the submanifold (associated with the metric), the r-th elementary symmetric functions calling the r-th modified mean curvatures. Then, following this we define **the r-th modified mean curvatures** of D^α_{ij} and call them Q_r.

Definition 1.2 *1) If r is an odd integer, $r = 2k+1$, then define*

$$\mathbf{Q_r} := \frac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}}D^{\alpha_k}_{i_{r-1}j_{r-1}})D^{\alpha}_{i_rj_r}e_\alpha,$$

$$Q^\alpha_r := \frac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}}D^{\alpha_k}_{i_{r-1}j_{r-1}})D^{\alpha}_{i_rj_r}.$$

2) If r is an even integer, $r = 2k$, then define

$$Q_r := \frac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r}).$$

Remark 1.4 *If σ_r is a formal* **r-th mean curvature** *of D, then it is not difficult to know that $Q_r = \binom{n}{r}\sigma_r$, where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. Suppose $Q_0 = 1$, and if r is 1, then $\mathbf{Q_1} = n\sigma_1 = \sum_{i,\alpha} D^\alpha_{ii}e_\alpha$.*

We are going to prove some algebraic properties of the r-th Newton tensor associated with the r-th modified mean curvatures. Those properties are natural generalizations of the algebraic properties of classical Newton tensor and the r-th elementary symmetric polynomial.

Lemma 1.2

$$(r+1)Q_{r+1} = \mathrm{Trace}(T_{(r)}D). \tag{1.4}$$

Proof. If r is an odd integer,

$$\begin{aligned}
\mathrm{Trace}(T_{(r)}D) &= T^\alpha_{(r)il}D^\alpha_{li}\\
&= \frac{1}{r!}\varepsilon^{i_1,\dots,i_r,i}_{j_1,\dots,j_r,l}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}}D^{\alpha_k}_{i_{r-1}j_{r-1}})D^{\alpha}_{i_rj_r}D^\alpha_{li}\\
&= \frac{(r+1)!}{r!}\frac{1}{(r+1)!}\varepsilon^{i_1,\dots,i_r,i_{r+1}}_{j_1,\dots,j_r,j_{r+1}}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}}D^{\alpha_k}_{i_{r-1}j_{r-1}})\\
&\quad\cdot(D^{\alpha_{k+1}}_{i_rj_r}D^{\alpha_{k+1}}_{i_{r+1}j_{r+1}})\\
&= (r+1)Q_{r+1}.
\end{aligned}$$

If r is an even integer,

$$\begin{aligned}
\mathrm{Trace}(T_{(r)}D) &= T_{(r)il}D^\alpha_{li}e_\alpha\\
&= \frac{1}{r!}\varepsilon^{i_1,\dots,i_r,i}_{j_1,\dots,j_r,l}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})D^\alpha_{li}e_\alpha\\
&= \frac{(r+1)!}{r!}\frac{1}{(r+1)!}\varepsilon^{i_1,\dots,i_r,i_{r+1}}_{j_1,\dots,j_r,j_{r+1}}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})\\
&\quad\cdot D^\alpha_{i_{r+1}j_{r+1}}e_\alpha\\
&= (r+1)\mathbf{Q_{r+1}}.
\end{aligned}$$

□

Lemma 1.3 *If r is an odd integer,*

$$T^{\alpha}_{(r)ij} = T^{\alpha}_{(r)ji} \tag{1.5}$$

If r is an even integer,

$$T_{(r)ij} = T_{(r)ji}. \tag{1.6}$$

Proof. Using the symmetry of D, if r is an odd integer, set $r = 2k+1$,

$$\begin{aligned}
T^{\alpha}_{(r)ij} &= \frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,j}(D^{\alpha_1}_{i_1j_1}, D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}}, D^{\alpha_k}_{i_{r-1}j_{r-1}})D^{\alpha}_{i_rj_r}\\
&= \frac{1}{r!}\varepsilon^{j_1,\ldots,j_r,j}_{i_1,\ldots,i_r,i}(D^{\alpha_1}_{j_1i_1} D^{\alpha_1}_{j_2i_2})\cdots(D^{\alpha_k}_{j_{r-2}i_{r-2}} D^{\alpha_k}_{j_{r-1}i_{r-1}})D^{\alpha}_{j_ri_r}\\
&= \frac{1}{r!}\varepsilon^{j_1,\ldots,j_r,j}_{i_1,\ldots,i_r,i}(D^{\alpha_1}_{i_1j_1} D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}} D^{\alpha_k}_{i_{r-1}j_{r-1}})D^{\alpha}_{i_rj_r}\\
&= \frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,j}_{j_1,\ldots,j_r,i}(D^{\alpha_1}_{i_1j_1} D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-2}j_{r-2}} D^{\alpha_k}_{i_{r-1}j_{r-1}})D^{\alpha}_{i_rj_r}\\
&= T^{\alpha}_{(r)ji}.
\end{aligned}$$

If r is an even integer, set $r = 2k$,

$$\begin{aligned}
T_{(r)ij} &= \frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j_r,j}(D^{\alpha_1}_{i_1j_1}, D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}, D^{\alpha_k}_{i_rj_r})\\
&= \frac{1}{r!}\varepsilon^{j_1,\ldots,j_r,j}_{i_1,\ldots,i_r,i}(D^{\alpha_1}_{j_1i_1} D^{\alpha_1}_{j_2i_2})\cdots(D^{\alpha_k}_{j_{r-1}i_{r-1}} D^{\alpha_k}_{j_ri_r})\\
&= \frac{1}{r!}\varepsilon^{j_1,\ldots,j_r,j}_{i_1,\ldots,i_r,i}(D^{\alpha_1}_{i_1j_1} D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}} D^{\alpha_k}_{i_rj_r})\\
&= \frac{1}{r!}\varepsilon^{i_1,\ldots,i_r,j}_{j_1,\ldots,j_r,i}(D^{\alpha_1}_{i_1j_1} D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}} D^{\alpha_k}_{i_rj_r})\\
&= T_{(r)ji}.
\end{aligned}$$

□

Lemma 1.4 *If r is an even integer,*

$$T_{(r)}(D) = Q_r I - T^{\alpha}_{(r-1)}D^{\alpha}. \tag{1.7}$$

If $r = 1$,

$$T^{\alpha}_{(1)}(D) = Q^{\alpha}_1 I - T_{(0)}D^{\alpha}. \tag{1.8}$$

Proof. If r is an even integer,

$$\begin{aligned}
T_{(r)ij} &= \frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(D^{\alpha_1}_{i_1j_1} D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}} D^{\alpha_k}_{i_rj_r})\\
&= \frac{1}{r!}(\delta^{i_1}_j\varepsilon^{i_2,i_3,\ldots,i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r} - \delta^{i_2}_j\varepsilon^{i_1,i_3,\ldots,i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r} + \ldots + \delta^{i}_j\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r}_{j_1,j_2,\ldots,j_{r-1},j_r})\\
&\quad \cdot(D^{\alpha_1}_{i_1j_1} D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}} D^{\alpha_k}_{i_rj_r})
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{r!}\varepsilon^{i_2,i_3,\ldots,i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r}(D^{\alpha_1}_{jj_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})\\
&\quad -\frac{1}{r!}\varepsilon^{i_1,i_3,\ldots,i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{jj_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})\\
&\quad +\cdots \quad \cdots \quad \cdots\\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r}_{j_1,j_2,\ldots,j_{r-1},j_r}\cdot(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})\delta^i_j\\
&= -\frac{1}{r!}\varepsilon^{i_2,i_3,\ldots,i_r,i}_{j_2,j_3\ldots,j_r,j_1}(D^{\alpha_2}_{i_3j_3}D^{\alpha_2}_{i_4j_4})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})(D^{\alpha_1}_{i_2j_2}D^{\alpha_1}_{jj_1})\\
&\quad -\frac{1}{r!}\varepsilon^{i_1,i_3,\ldots,i_r,i}_{j_1,j_3\ldots,j_r,j_2}(D^{\alpha_2}_{i_3j_3}D^{\alpha_2}_{i_4j_4})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{jj_2})\\
&\quad -\cdots \quad \cdots \quad \cdots\\
&\quad +Q_r\delta^i_j\\
&= -\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i}_{j_1,j_2\ldots,j_{r-1},l}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-3}j_{r-3}}D^{\alpha_k}_{i_{r-2}j_{r-2}})(D^{\alpha}_{i_{r-1}j_{r-1}}D^{\alpha}_{lj})\\
&\quad -\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i}_{j_1,j_2\ldots,j_{r-1},l}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-3}j_{r-3}}D^{\alpha_k}_{i_{r-2}j_{r-2}})(D^{\alpha}_{i_{r-1}j_{r-1}}D^{\alpha}_{lj})\\
&\quad -\cdots \quad \cdots \quad \cdots\\
&\quad +Q_r\delta^i_j\\
&= -\frac{1}{r!}T^{\alpha}_{(r-1)il}D^{\alpha}_{lj}-\cdots-\frac{1}{r!}T^{\alpha}_{(r-1)il}D^{\alpha}_{lj}+Q_r\delta^i_j\\
&= Q_r\delta^i_j - T^{\alpha}_{(r-1)il}D^{\alpha}_{lj}.
\end{aligned}$$

If $r = 1$,

$$\begin{aligned}
T^{\alpha}_{(1)}(D) &= \varepsilon^{i_1 i}_{j_1 j}D^{\alpha}_{i_1j_1}\\
&= \delta^{i_1}_{j_1}\delta^i_jD^{\alpha}_{i_1j_1} - \delta^{i_1}_{j}\delta^i_{j_1}D^{\alpha}_{i_1j_1}\\
&= Q^{\alpha}_1\delta^i_j - \delta^i_{j_1}D^{\alpha}_{jj_1}\\
&= Q^{\alpha}_1\delta^i_j - T_{(0)ij_1}D^{\alpha}_{jj_1}.
\end{aligned}$$

□

Lemma 1.5 *Let $D = D(t)$ be a smooth one-parameter family of D, then for $r = 1, \ldots, n+1$ we have*
If r is even,

$$\frac{\partial Q_r}{\partial t} = \mathrm{Trace}(T^{\alpha}_{(r-1)}\frac{\partial D^{\alpha}}{\partial t}). \qquad (1.9)$$

If $r = 1$,

$$\frac{\partial Q^{\alpha}_1}{\partial t} = \mathrm{Trace}(T_{(0)}\frac{\partial D^{\alpha}}{\partial t}). \qquad (1.10)$$

Proof. If r is even, from the equation

$$(r)Q_r = \mathrm{Trace}(T_{(r-1)}D)$$

and

$$Q_r = \frac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r}),$$

we have

$$\begin{aligned}
\frac{\partial Q_r}{\partial t} &= \tfrac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}\Big(\frac{\partial D^{\alpha_1}_{i_1j_1}}{\partial t}D^{\alpha_1}_{i_2j_2}+D^{\alpha_1}_{i_1j_1}\frac{\partial D^{\alpha_1}_{i_2j_2}}{\partial t}\Big)\cdots(D^{\alpha_k}_{i_{r-1}j_{r-1}}D^{\alpha_k}_{i_rj_r})\\
&\quad+\cdots\quad\cdots\quad\cdots\\
&\quad+\tfrac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_{k-1}}_{i_{r-3}j_{r-3}}D^{\alpha_{k-1}}_{i_{r-2}j_{r-2}})\\
&\quad\Big(\frac{\partial D^{\alpha_k}_{i_{r-1}j_{r-1}}}{\partial t}D^{\alpha_k}_{i_rj_r}+D^{\alpha_k}_{i_{r-1}j_{r-1}}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\Big)\\
&= \tfrac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_{k-1}}_{i_{r-3}j_{r-3}}D^{\alpha_{k-1}}_{i_{r-2}j_{r-2}})\\
&\quad\Big(\frac{\partial D^{\alpha_k}_{i_{r-1}j_{r-1}}}{\partial t}D^{\alpha_k}_{i_rj_r}+D^{\alpha_k}_{i_{r-1}j_{r-1}}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\Big)\\
&\quad+\cdots\quad\cdots\quad\cdots\\
&\quad+\tfrac{1}{r!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_{k-1}}_{i_{r-3}j_{r-3}}D^{\alpha_{k-1}}_{i_{r-2}j_{r-2}})\\
&\quad\Big(\frac{\partial D^{\alpha_k}_{i_{r-1}j_{r-1}}}{\partial t}D^{\alpha_k}_{i_rj_r}+D^{\alpha_k}_{i_{r-1}j_{r-1}}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\Big)\\
&= \tfrac{(r-1)!}{r!}\tfrac{1}{(r-1)!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_{k-1}}_{i_{r-3}j_{r-3}}D^{\alpha_{k-1}}_{i_{r-2}j_{r-2}})\\
&\quad\Big(\frac{\partial D^{\alpha_k}_{i_{r-1}j_{r-1}}}{\partial t}D^{\alpha_k}_{i_rj_r}+D^{\alpha_k}_{i_{r-1}j_{r-1}}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\Big)\\
&\quad+\cdots\quad\cdots\quad\cdots\\
&\quad+\tfrac{(r-1)!}{r!}\tfrac{1}{(r-1!)}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_{k-1}}_{i_{r-3}j_{r-3}}D^{\alpha_{k-1}}_{i_{r-2}j_{r-2}})\\
&\quad\Big(\frac{\partial D^{\alpha_k}_{i_{r-1}j_{r-1}}}{\partial t}D^{\alpha_k}_{i_rj_r}+D^{\alpha_k}_{i_{r-1}j_{r-1}}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\Big)\\
&= \tfrac{2}{r}\tfrac{1}{(r-1)!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_{k-1}}_{i_{r-3}j_{r-3}}D^{\alpha_{k-1}}_{i_{r-2}j_{r-2}})D^{\alpha_k}_{i_{r-1}j_{r-1}}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\\
&\quad+\cdots\quad\cdots\quad\cdots\\
&\quad+\tfrac{2}{r}\tfrac{1}{(r-1)!}\varepsilon^{i_1,\dots,i_r}_{j_1,\dots,j_r}(D^{\alpha_1}_{i_1j_1}D^{\alpha_1}_{i_2j_2})\cdots(D^{\alpha_{k-1}}_{i_{r-3}j_{r-3}}D^{\alpha_{k-1}}_{i_{r-2}j_{r-2}})D^{\alpha_k}_{i_{r-1}j_{r-1}}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\\
&= \tfrac{2}{r}T^{\alpha_k}_{(r-1)i_rj_r}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\\
&\quad+\cdots\quad\cdots\quad\cdots\\
&\quad+\tfrac{2}{r}T^{\alpha_k}_{(r-1)i_rj_r}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\\
&= T^{\alpha_k}_{(r-1)i_rj_r}\frac{\partial D^{\alpha_k}_{i_rj_r}}{\partial t}\\
&= \mathrm{Trace}\Big(T^{\alpha}_{(r-1)}\frac{\partial D^{\alpha}}{\partial t}\Big).
\end{aligned}$$

If $r=1$,

$$\begin{aligned}
\frac{\partial Q^{\alpha}_1}{\partial t} &= \frac{\partial(\varepsilon^{i_1}_{j_1}D^{\alpha_1}_{i_1j_1})}{\partial t}\\
&= \varepsilon^{i_1}_{j_1}\frac{\partial D^{\alpha_1}_{i_1j_1}}{\partial t}\\
&= \mathrm{Trace}\Big(T_{(0)}\frac{\partial D^{\alpha}}{\partial t}\Big).
\end{aligned}$$

□

2 Operator $\Box^r$ on a submanifold of a space with constant sectional curvatures and it's applications

In this Section, we follow closely the exposition of the moving frame in [3, 21], and we agree that Q_r is a vector, formal in a submanifold like as in the above Section, however being the modified mean curvature function in a hypersurface. Let $x : M^n \to N^{n+p}$ be an isometric immersion of n-dimensional Riemannian M^n as a submanifold in $(n+p)$-dimensional space N. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ of N^{n+p} such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M. We shall make use of the following convention on the ranges of indices

$$1 \le A, B, C, \ldots \le n+p, \quad 1 \le i, j, k, \ldots \le n,$$

$$n+1 \le \alpha, \beta, \gamma, \ldots \le n+p,$$

and we shall agree that repeated indices are summed over the respective ranges. With respect to the frame field of N chosen above, let $\omega_1, \ldots, \omega_{n+p}$ be the field of the dual frame.
Then the structure equations of N are given by

$$d\omega_A = \sum \omega_B \wedge \omega_{BA}, \ \omega_{BA} + \omega_{AB} = 0, \tag{2.1}$$

$$d\omega_{AB} = \sum \omega_{AC} \wedge \omega_{CB} + \Phi_{AB}, \quad \Phi_{AB} = -\frac{1}{2} \sum \overline{R}_{ABCD} \omega_C \wedge \omega_D, \tag{2.2}$$

where ω_{AB} is the Levi-civita connection of N with respect to e_A and $\overline{R}_{ABCD}$ is the Riemannian curvature tensor of N. We know that $\overline{R}_{ABCD}$ satisfies the following identities

$$\overline{R}_{ABCD} = -\overline{R}_{ABDC} = -\overline{R}_{BADC}\,, \quad \overline{R}_{ABCD} = \overline{R}_{CDAB}, \tag{2.3}$$

$$\overline{R}_{ABCD} + \overline{R}_{ACDB} + \overline{R}_{ADBC} = 0. \tag{2.4}$$

We restrict these forms to M by the same letters. Then

$$\omega_\alpha = 0. \tag{2.5}$$

The structure equations of M are

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \quad \omega_{ji} + \omega_{ij} = 0, \tag{2.6}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Phi_{ij}, \quad \Phi_{ij} = -\frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l. \tag{2.7}$$

Since $0 = d\omega_\alpha = \sum \omega_j \wedge \omega_{j\alpha}$, by Cartan's lemma we may write

$$\omega_{i\alpha} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.8}$$

¿From these formulas we obtain

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.9}$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Phi_{\alpha\beta}, \quad \Phi_{\alpha\beta} = -\frac{1}{2}\sum R_{\alpha\beta kl}\omega_k \wedge \omega_l. \tag{2.10}$$

$$R_{\alpha\beta kl} = \overline{R}_{\alpha\beta kl} + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \tag{2.11}$$

Here (ω_{ij}) defines a connection of M, and $(\omega_{\alpha\beta})$ a connection in the normal bundle of M. We call $B = \sum h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha$ the second fundamental form of the immersed manifold M. We take exterior differentiation of (2.8) and use $h^\alpha_{ij,k}$ to denote the covariant derivatives by

$$\sum h^\alpha_{ij,k}\omega_k = dh^\alpha_{ij} + \sum h^\alpha_{ik}\omega_{kj} + \sum h^\alpha_{kj}\omega_{ki} - \sum h^\beta_{ij}\omega_{\alpha\beta}. \tag{2.12}$$

Then

$$h^\alpha_{ij,k} - h^\alpha_{ik,j} = \overline{R}_{\alpha ijk}. \tag{2.13}$$

Now we introduce the operator $\Box^r$.
For a C^∞ function f defined on M, we define its gradient and Hessian by the following formulas

$$df := \sum f_{,i}\omega_i, \quad \sum f_{,ij}\omega_j := df_{,i} + \sum f_{,j}\omega_j \quad (f_{,ij} = f_{,ji}). \tag{2.14}$$

For a section $\xi = \xi^\alpha e_\alpha$ of the normal bundle $T^\perp(M)$ we define the covariant derivative of ξ^α by

$$\sum \xi^\alpha_{,i}\omega_i = d\xi^\alpha + \sum \xi^\beta \omega_{\beta\alpha} \tag{2.15}$$

and the convariant derivative of $\xi^\alpha_{,i}$ by

$$\sum \xi^\alpha_{,ij}\omega_j = d\xi^\alpha_{,i} + \sum \xi^\alpha_{,j}\omega_{ji} - \sum \xi^\beta_{,i}\omega_{\alpha\beta}. \tag{2.16}$$

When $p > 1$ and r is odd, we can define the differential operator $\Box^r$.

Definition 2.1 *For a section $\xi = \xi^\alpha e_\alpha$ of the normal bundle $T^\perp(M)$ we denote the differential operator*

$$\Box^{r*} : C^\infty(T^\perp(M)) \longrightarrow C^\infty(M)$$

by

$$\Box^{r*}\xi = \sum T^{\alpha}_{(r)ij}\xi^{\alpha}_{,ij}. \qquad (2.17)$$

For a C^∞ function f of M we define the differential operator

$$\Box^r : C^\infty(M) \longrightarrow C^\infty(T^\perp(M))$$

by

$$\Box^r f = \sum T^{\alpha}_{(r)ij} f_{,ij} e_\alpha. \qquad (2.18)$$

If $p > 1$ and r is odd, we define differential operator $\Box^r$, and in the case $p = 1$ we also define differential operator $\Box^r$ as well as the above definitions.

Definition 2.2 *For a C^∞ function f of M we can define the differential operator*

$$\Box^r : C^\infty(M) \longrightarrow C^\infty(M)$$

$$\Box^r f = \sum T_{(r)ij} f_{,ij}. \qquad (2.19)$$

Remark 2.1 *If $r = 0$, then $\Box^r f = \sum\limits_i f_{,ii} = \Delta f$.*

Now we suppose that N is of constant curvature c, then

$$\overline{R}_{\alpha jkl} = 0, \quad \overline{R}_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

So we have the Gauss equation

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}) \qquad (2.20)$$

and

$$h^\alpha_{ij,k} - h^\alpha_{ik,j} = 0. \qquad (2.21)$$

So the fundamental form B must be a Codazzi tensor. We have a lemma as follows:

Lemma 2.1 *Let $x : M^n \to N^{n+p}(c)$ be an immersion of a compact orientable n-dimensional Riemannian manifold M^n as a submanifold in the $(n+p)$-dimensional Riemannian N^{n+p} with constant sectional curvature c, and let B be the second fundamental form of M^n.*
i) If $p > 1$ and r is an even integer, then the r-th Newton tensor of B is divergence-free, i.e.,

$$\sum_j T_{(r)ij,j} = 0.$$

If $p > 1$ and r is an odd integer, then

$$\sum_j T^{\alpha}_{(r)ij,j} = 0.$$

ii) If $p = 1$ and r is any integer, then the r-th Newton tensor of B is divergence-free, i.e.,

$$\sum_j T_{(r)ij,j} = 0.$$

Proof. Since ii) is proved in [17], we are going to do only i).
If r is an even integer, set $r = 2k$,

$$T_{(r)ij} = \frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-1}j_{r-1}}h^{\alpha_k}_{i_rj_r}).$$

We have

$$\begin{aligned}
T_{(r)ij,j} &= \frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_1}_{i_1j_1,j}h^{\alpha_1}_{i_2j_2} + h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2,j})\cdots(h^{\alpha_k}_{i_{r-1}j_{r-1}}h^{\alpha_k}_{i_rj_r})\\
&\quad +\cdots \quad \cdots \quad \cdots\\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-1}j_{r-1},j}h^{\alpha_k}_{i_rj_r} + h^{\alpha_k}_{i_{r-1}j_{r-1},j}h^{\alpha_k}_{i_rj_r,j})\\
&= \frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_k}_{i_1j_1}h^{\alpha_k}_{i_2j_2})\cdots(h^{\alpha_1}_{i_{r-1}j_{r-1},j}h^{\alpha_1}_{i_rj_r} + h^{\alpha_1}_{i_{r-1}j_{r-1},j}h^{\alpha_1}_{i_rj_r,j})\\
&\quad +\cdots \quad \cdots \quad \cdots\\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-1}j_{r-1},j}h^{\alpha_k}_{i_rj_r} + h^{\alpha_k}_{i_{r-1}j_{r-1},j}h^{\alpha_k}_{i_rj_r,j})\\
&= \frac{k}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-1}j_{r-1},j}h^{\alpha_k}_{i_rj_r} + h^{\alpha_k}_{i_{r-1}j_{r-1},j}h^{\alpha_k}_{i_rj_r,j})\\
&= \frac{1}{(r-1)!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-1}j_{r-1},j}h^{\alpha_k}_{i_rj_r,j}).
\end{aligned}$$

and we know that

$$\begin{cases} \varepsilon^{i_1,\ldots,i_ri}_{j_1,\ldots,j_rj} + \varepsilon^{i_1,\ldots,i_r,i}_{j_1,\ldots,j,j_r} = 0 \\ h^{\alpha_k}_{i_rj_r,j} = h^{\alpha_k}_{i_rj,j_r} \end{cases},$$

so we have

$$\sum_j T_{(r)ij,j} = 0.$$

If r is an odd integer, set $r = 2k + 1$,

$$\begin{aligned}
\sum_j T^{\alpha}_{(r)ij,j} &= \frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1,j}h^{\alpha_1}_{i_2j_2}+h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2,j})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1}})h^{\alpha}_{i_rj_r} \\
&\quad \cdot h^{\alpha}_{i_rj_r}+\cdots \quad \cdots \quad \cdots \\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1},h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2},j}h^{\alpha_k}_{i_{r-1}j_{r-1}}+h^{\alpha_k}_{i_{r-2}j_{r-2},j}h^{\alpha_k}_{i_{r-1}j_{r-1},j}) \\
&\quad \cdot h^{\alpha}_{i_rj_r} \\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1}})h^{\alpha}_{i_rj_r,j} \\
&= \frac{2}{r!}\varepsilon^{i_1,i_2,\ldots,i_r,i}_{j_1,j_2,\ldots,j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2,j})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r} \\
&\quad +\cdots \quad \cdots \quad \cdots \\
&\quad +\frac{2}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r} \\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r,j}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2k}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r} \\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r,j} \\
&\downarrow \quad \text{(by exchanging the second integers)} \\
&= \frac{2k}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j,j_r,j_{r-1}}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j,j_{r-1}})h^{\alpha}_{i_rj_r} \\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j,j_r}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj,j_r} \\
&\downarrow \quad (B \text{ is a Codazzi tensor}) \\
&= \frac{2k}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j,j_r,j_{r-1}}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r} \\
&\quad +\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j,j_r}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r,j} \\
&\downarrow \quad \text{(the generalized Kronecker sign is anti-symmetric)} \\
&= -\frac{2k}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r} \\
&\quad -\frac{1}{r!}\varepsilon^{i_1,i_2,\ldots,i_{r-1},i_r,i}_{j_1,j_2,\ldots,j_{r-1},j_r,j}(h^{\alpha_1}_{i_1j_1}h^{\alpha_1}_{i_2j_2})\cdots(h^{\alpha_k}_{i_{r-2}j_{r-2}}h^{\alpha_k}_{i_{r-1}j_{r-1},j})h^{\alpha}_{i_rj_r,j}.
\end{aligned}$$

So we have

$$\sum_j T^{\alpha}_{(r)ij,j} = 0.$$

This completes the proof of the lemma 2.4. □

For any point $q \in M$, the Riemannian metric $\langle , \rangle_q$ induces inner products in the normal bundles over M, also denoted by $\langle , \rangle_q$. Then, we can define for any $\xi, \eta \in C^\infty(T^\perp(M))$ a function $\langle \xi, \eta \rangle_q = \langle \xi_q, \eta_q \rangle_q$, since $\xi_q, \eta_q \in T^\perp(M)_q$. It also induces the operator $*$ by requiring

$$\langle \xi, \eta \rangle_q dM = \eta \wedge *\xi.$$

So we can define the L^2-inner product on $C^\infty(T^\perp(M))$ as

$$(,) : (\xi, \eta) = \int_M \langle \xi, \eta \rangle dM = \int_M \eta \wedge *\xi,$$

denoting the volume form by dM, see [5], [7]. We recall Theorem 1.1 of [7].

Lemma 2.2 *Let M be a compact oriented submanifold. If ϕ satisfies the conditions*

$$(i)\ \phi^\alpha_{ij} = \phi^\alpha_{ji}, \quad (ii) \sum \phi^\alpha_{ij,j} = 0,$$

then $\Box^*_\phi$ and $\Box_\phi$ are adjoint, which means

$$(f, \Box^*_\phi \xi) = (\xi, \Box_\phi f), \tag{2.22}$$

in particular

$$\int_M \Box^*_\phi \xi dM = 0. \tag{2.23}$$

We have the following key Theorem.

Theorem 2.1 *Let M be a compact oriented submanifold of a space with constant sectional curvature, the definitions of $\Box^r$ and $\Box^{r*}$ being as above. Then*

i) if $p > 1$ and r is an odd integer, $1 \le r \le n$, $\Box^r$ and $\Box^{r*}$ are adjoint relative to the L^2-inner product of M, i.e.,

$$(\Box^{r*}\xi, f) = (\xi, \Box^r f), \tag{2.24}$$

in particular,

$$\int_M \Box^{r*}\xi dM = 0. \tag{2.25}$$

ii) if $p > 1$ and r is an even integer or $p = 1$, $\Box^r$ is self-adjoint relative to the L^2-inner product of M, i.e.,

$$(g, \Box^r f) = (f, \Box^r g), \tag{2.26}$$

in particular

$$\int_M \Box^r f dM = 0. \tag{2.27}$$

Suppose ξ is a vector of the tangent bundle $T^{\top}(M)$. We have

$$\int_M \Box^r \xi dM = 0,$$

where we denote the volume form by dM.

Proof. If $p > 1$ and r is an even integer, or $p = 1$,

$$\begin{aligned} g\,\Box^r f : &= g\sum_{i,j} T_{(r)ij} f_{,ij} = \sum_j (g \sum_i T_{(r)ij} f_{,i})_{,j} \\ &\quad - \sum_{i,j} g_{,j} T_{(r)ij} f_{,i} - g \sum_{i,j} T_{(r)ij,j} f_{,i} \\ &= \sum_j (g \sum_i T_{(r)ij} f_{,i})_{,j} - \sum_{i,j} (g_{,j} T_{(r)ij} f)_{,i} + \sum_{ij} g_{,ji} T_{(r)ij} f \\ &\quad + \sum_{i,j} g_{,j} T_{(r)ij,i} f_{,i} - g \sum_{i,j} T_{(r)ij,j} f_{,i}. \end{aligned}$$

If r is even, we have from Lemma 2.1

$$\sum_j T_{(r)ij,j} = 0, \quad \sum_i T_{(r)ij,i} = 0.$$

Meanwhile, set $\xi_j = g \sum_i T_{(r)ij} f_{,i}$, $\eta_i = f \sum_j T_{(r)ij} g_{,i}$. Making use of Green's theorem we get

$$(g, \Box^r f) = (f, \Box^r g).$$

If $g = 1$ we have $\int_M \Box^r f dM = 0$.
If $p > 1$ and r is an odd integer, making use of lemma 2.1 and Lemma 2.2 we obtain

$$(\Box^{r*}\xi, f) = (\xi, \Box^r f).$$

This completes the proof of the theorem. □

Remark 2.2 *When $p = 1$, $\Box^1$ is the same as the operator $\Box$ derived by Cheng-Yau, [6].*

Now we choose a vector $a \in R^{n+p+1}$ of N^{n+p} with the constant curvatures. For any point $p \in M^n$, let $X(p) \in M^n$ be the position vector. We define the height function by $\varphi := \langle a, X \rangle$, where $\langle , \rangle$ is the inner product in M^n.
We are going to calculate $\Box^r \varphi$.
The moving equation of M^n in N^{n+p} is

$$\begin{cases} dX &= \omega_i e_i, \\ de_i &= \omega_{ij} e_j + \omega_{i\alpha} e_\alpha - cX\omega_i, \\ de_\alpha &= -\omega_{i\alpha} e_i + \omega_{\alpha\beta} e_\beta. \end{cases} \tag{2.28}$$

Hence

$$d\varphi = \langle dX, a\rangle = \langle \omega_i e_i, a\rangle.$$

If we set $\varphi_{,i} = \langle e_i, a\rangle$ from the definition (2.14) of the covariant derivatives of $\varphi_{,i}$ we obtain

$$\varphi_{,ij}\omega_j = d\varphi_{,i} + \varphi_{,j}\omega_{ji}. \tag{2.29}$$

Taking moving equations (2.28) into the above equation, we obtain

$$\begin{aligned}\varphi_{,ij}\omega_j &= \omega_{ij}\langle e_j, a\rangle + h^\alpha_{ij}\langle e_\alpha, a\rangle\omega_j \\ &\quad -c\langle X, a\rangle\omega_i + \langle e_j, a\rangle\omega_{ji} \\ &= [h^\alpha_{ij}\langle e_\alpha, a\rangle - c\langle X, a\rangle\delta_{ij}]\omega_j.\end{aligned} \tag{2.30}$$

So

$$\varphi_{,ij} = h^\alpha_{ij}\langle e_\alpha, a\rangle - c\langle X, a\rangle\delta_{ij}. \tag{2.31}$$

If r is even and $p > 1$, then

$$\begin{aligned}\Box^r\varphi &= \sum_{i,j} T_{(r)ij}\varphi_{,ij} \\ &= \sum_{\alpha,i,j} T_{(r)ij}h^\alpha_{ij}\langle e_\alpha, a\rangle - c\sum_{i,j} T_{(r)ij}\langle X, a\rangle\delta_{ij} \\ &= \langle \mathrm{Trace}(T_{(r)}B), a\rangle - c\langle X, a\rangle \mathrm{Trace}(T_{(r)}).\end{aligned} \tag{2.32}$$

From (1.4), we know $\mathrm{Trace}(T_{(r)}B) = (r+1)\mathbf{Q_{r+1}}$,
and from (1.7), $T_{(r)}(B) = Q_r I - T^\alpha_{(r-1)}B^\alpha$ we have

$$\mathrm{Trace}(T_{(r)}) = nQ_r - rQ_r = (n-r)Q_r. \tag{2.33}$$

Then

$$\Box^r\varphi = (r+1)\langle \mathbf{Q_{r+1}}, a\rangle - c\langle X, a\rangle(n-r)Q_r. \tag{2.34}$$

The same proof works for the case where $p = 1$ and r is any integer, so

$$\Box^r\varphi = (r+1)Q_{r+1}\langle e_{n+1}, a\rangle - c\langle X, a\rangle(n-r)Q_r, \tag{2.35}$$

and

$$Q_r = \binom{n}{r}\sigma_r.$$

Using Theorem 2.1, we obtain the following theorem.

Theorem 2.2 *Let $x : M^n \to N^{n+p}(c)$ be an immersion of the compact orientable n-dimensional Riemannian manifold M^n as a submanifold in the $(n+p)$-dimensional Riemannian $N^{n+p} \subset R^{n+p+1}$ with constant sectional curvature, let a be any fixed element of N^{n+p}, e_α $(\alpha = 1, \dots, p)$ be the unit normal vector field of M^n, σ_r $(r = 0, 1, \dots, (n-1))$ be the mean curvature*

on M^n, and X be the position vector of M^n.
i) If $p > 1$ and r is an even integer, then

$$\int_M (\langle \sigma_{\mathbf{r+1}}, a\rangle - c\langle X, a\rangle \sigma_r) dM = 0. \tag{2.36}$$

ii) If $p = 1$ and r is any integer, then

$$\int_M (\sigma_{r+1}\langle e_{n+1}, a\rangle - c\langle X, a\rangle \sigma_r) dM = 0. \tag{2.37}$$

Remark 2.3 *For hypersurfaces in the unit sphere, from the theorem we can deduce Theorem A of R. C. Reilly, in [15], so our Theorem is a generalization of the mentioned result.*

Corollary 2.1 (Reilly [15]) *Let $x : M^n \to S^{n+1}$ be an immersion of the compact orientable n-dimensional Riemannian manifold M^n as a hypersurface in the $(n+1)$-dimensional unit sphere $S^{n+1} \subset R^{n+2}$. Let a be any fixed element of S^{n+1}, e_{n+1} be the unit normal vector field of M^n, σ_r $(r = 0, 1, \ldots, (n-1))$ be the mean curvature functions on M^n, and X be the position vector of M^n. Then*

$$\int_M (\sigma_{r+1}\langle e_{n+1}, a\rangle - \langle X, a\rangle \sigma_r) dM = 0. \tag{2.38}$$

If $p = 1$, and e_{n+1} is the unit normal vector field on $M^n \subset N^{n+1}(c)$, using the moving equations (2.28) we are going to compute $\Box^r e_{n+1}$. By

$$de_{n+1} = -\sum \omega_{i(n+1)} e_i = \sum -h_{ij} e_i \omega_j,$$

we know

$$e_{n+1,j} = -\sum_i h_{ij} e_i, \quad e_{n+1,i} = -\sum_j h_{ij} e_j, \tag{2.39}$$

and we have

$$\begin{aligned}\sum_j e_{n+1,ij}\omega_j &= de_{n+1,i}+\sum_j e_{n+1,j}\omega_{ji}\\ &= -\sum_j[(dh_{ij})e_j+h_{ij}de_j+\sum_k h_{kj}\omega_{ji}e_k]\\ &= -(\sum_{k,j}h_{ij,k}\omega_k e_j+\sum_{k,j}h_{ik}\omega_{jk}e_j+\sum_{k,j}h_{kj}\omega_{ik}e_j\\ &\quad -cX\sum_j h_{ij}\omega_j+\sum_{k,j}h_{ij}\omega_{jk}e_k+\sum_{k,j}h_{kj}\omega_{ji}e_k\\ &\quad +\sum_{k,j}h_{kj}h_{ij}\omega_k e_{n+1})\\ &= -(\sum_{k,j}h_{ij,k}\omega_k e_j+\sum_{k,j}h_{kj}\omega_k e_{n+1})+cX\sum_j h_{ij}\omega_j\\ &= -\sum_j(\sum_k h_{ij,k}e_k+\sum_k h_{jk}h_{ki}e_{n+1}-cXh_{ij})\omega_j.\end{aligned}$$

So we can obtain

$$e_{n+1,ij}=-\sum_k h_{ij,k}e_k-\sum_k h_{jk}h_{ki}e_{n+1}+cXh_{ij}. \tag{2.40}$$

On the other hand,

$$\begin{aligned}\sum_{i,j,k}T_{(r)ij}h_{ik}h_{kj} &= \sum_{j,k}(Q_{r+1}\delta_{jk}-T_{(r)jk})h_{kj}\\ &= Q_{r+1}Q_1-(r+2)Q_{r+2}.\end{aligned} \tag{2.41}$$

Making use of this identity we obtain

$$\begin{aligned}\Box^r e_{n+1} &= \sum_{i,j}T_{(r)ij}e_{n+1,ij}\\ &= -\sum_{i,j,k}T_{(r)ij}h_{ij,k}e_k-\sum_{i,j,k}T_{(r)ij}h_{jk}h_{ki}e_{n+1}+cX\sum_{i,j}T_{(r)ij}h_{ij}\\ &= -\sum_k(r+1)Q_{r+1,k}e_k-(Q_{r+1}Q_1-(r+2)Q_{r+2})e_{n+1}\\ &\quad +c(r+1)Q_{r+1}X.\end{aligned} \tag{2.42}$$

Theorem 2.3 *Let $x: M^n\to N^{n+1}(c)$ be an immersion of a compact orientable n-dimensional Riemannian manifold M^n as a hypersurface in $(n+1)$-dimensional Riemannian manifold $N^{n+1}\subset R^{n+2}$. Let e_{n+1} be the unit normal vector field of M^n, Q_r $(r=0,1,\ldots,(n-1))$ be the r-th modified mean curvature functions on M^n, and X be the position vector of M^n. Then*

$$\int_M(\sum_k(r+1)Q_{r+1,k}e_k+(Q_{r+1}Q_1-(r+2)Q_{r+2})e_{n+1}-c(r+1)Q_{r+1}X)dM=0. \tag{2.43}$$

Remark 2.4 *If $Q_{r+1} = const$, then*

$$\int_M ((Q_{r+1}Q_1 - (r+2)Q_{r+2})e_{n+1} - c(r+1)Q_{r+1}X)dM = 0. \tag{2.44}$$

If $Q_{r+1} = 0$, then

$$\int_M Q_{r+2}e_{n+1}dM = 0. \tag{2.45}$$

Furthermore, if a is a fixed vector of $N^{n+1}(c)$, it is easy to know

$$\langle e_{n+1}, a\rangle_{,ij} = \langle e_{n+1,ij}, a\rangle. \tag{2.46}$$

¿From above computing we obtain

$$\begin{aligned} &\Box^r\langle e_{n+1}, a\rangle = -\sum_k (r+1)Q_{r+1,k}\langle e_k, a\rangle \\ &-(Q_{r+1}Q_1 - (r+2)Q_{r+2})\langle e_{n+1}, a\rangle + c(r+1)Q_{r+1}\langle X, a\rangle. \end{aligned} \tag{2.47}$$

Combining this equation with the equation (2.34), we get

$$\begin{aligned} \Box^r\langle e_{n+1}, a\rangle - \Box^{r+1}\langle X, a\rangle &= -\sum_k (r+1)Q_{r+1,k}\langle e_k, a\rangle \\ &\quad -Q_{r+1}Q_1\langle e_{n+1}, a\rangle + cnQ_{r+1}\langle X, a\rangle. \end{aligned} \tag{2.48}$$

Corollary 2.2 *Let $x : M^n \to N^{n+1}(c)$ be an immersion of a compact orientable n-dimensional Riemannian manifold M^n as a hypersurface in (n+1)-dimensional Riemannian manifolds $N^{n+1} \subset R^{n+2}$. Let e_{n+1} be the unit normal vector field of M^n, Q_r $(r = 0, 1, \ldots, (n-1))$ be the r-th modified mean curvature functions on M^n, a be any fixed element of N^{n+1}, and X be the position vector of M^n. Then*

$$\int_M (\sum_k (r+1)Q_{r+1,k}\langle e_k, a\rangle + Q_{r+1}Q_1\langle e_{n+1}, a\rangle - cnQ_{r+1}\langle X, a\rangle)dM = 0. \tag{2.49}$$

If N^{n+p} is the Euclidean space R^{n+p}, for any point $p \in M^n$ let $X^\top$ be the projection of the position vector X on the tangent space T_pM at the point p, $X^\perp$ be the projection of the position vector X on the normal space $T_p^\perp M$ at the point p, so

$$\begin{aligned} X^\top &= \langle X^\top, e_i\rangle e_i = u_{,i}e_i, \\ u_{,i} :&= \langle X, e_i\rangle, \\ X^\perp &= \langle X, e_\alpha\rangle e_\alpha = p^\alpha e_\alpha, \\ p^\alpha :&= \langle X, e_\alpha\rangle. \end{aligned} \tag{2.50}$$

We are going to compute $\Box^r X^\top$.
¿From (2.14) we denote the covariant derivative of $u_{,i}$ by

$$u_{,ij}\omega_j = du_{,i} + u_{,j}\omega_j. \tag{2.51}$$

Taking (2.28) and $c = 0$ into the above equation, we obtain by a direct calculation

$$u_{,ij} = h^{\alpha}_{ij} p^{\alpha} + \delta_{ij}. \tag{2.52}$$

So we have

$$\begin{aligned} \Box^r X^{\top} &= \sum_{i,j} T_{(r)ij} u_{,ij} \\ &= \text{Trace} T_{(r)}(B) + \text{Trace}(T_{(r)} h^{\alpha}) p^{\alpha} \\ &= (n-r) Q_r + (r+1) \langle \mathbf{Q_{r+1}}, X \rangle. \end{aligned} \tag{2.53}$$

Using Theorem 2.1 we get the following theorem

Theorem 2.4 (Reilly [16]). *Let $x : M^n \to R^{n+p}$ be an immersion of a compact orientable n-dimensional Riemannian manifold M^n as a submanifold in the $(n+p)$-dimensional Euclidean space R^{n+p}, $e_\alpha (\alpha = 1, \dots, p)$ be the unit normal vector field of M^n, $\sigma_r (r = 0, 1, \dots, (n-1))$ be the mean curvature on M^n, and X be the position vector of M^n.*
i) If $p > 1$ and r is an even integer, then

$$\int_M (\langle \sigma_{\mathbf{r+1}}, X \rangle + \sigma_r) dM = 0. \tag{2.54}$$

ii) If $p = 1$ and r is any integer, then

$$\int_M (\langle \sigma_{r+1} e_{n+1}, X \rangle + \sigma_r) dM = 0. \tag{2.55}$$

Remark 2.5 *The theorem is Lemma A in Reilly, [16]. Here ii) is the one of classical Minkowski-Hsiung integral formulas, [8, 10].*

3 Related results to $\Box^r$

Let $e_i (i = 1, \dots, n)$ be a local orthonormal frame field on an n-dimensional Riemannian manifold M^n, ω_i be its dual coframe field. The structure equations of M are equations (2.6) and (2.7) in Section 2. Let $\varphi = \sum_{ij} \varphi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, $T_{(r)}$ the r-th Newton transformation of φ, Q_r the r-th modified curvature,

$$\Box^r Q_r := \sum T_{(r)ij} Q_{r,ij}. \tag{3.1}$$

Then

$$\begin{aligned} \Box^r Q_r &= \sum T_{(r)ij} Q_{r,ij} \\ &= \sum (Q_r \delta_{ij} - \sum_l T_{(r-1)il} \varphi_{lj}) Q_{r,ij}. \end{aligned}$$

We suppose that $C_{(r)ij} = T_{(r)il}\varphi_{lj}$. It is clear that $C_{(r)}$ is a symmetric tensor here and $rQ_r =$ Trace $C_{(r-1)}$, hence we have

$$\Box^r Q_r = \frac{1}{r}\sum(\frac{1}{r}trC_{(r-1)}\delta_{ij} - C_{(r-1)ij})(trC_{(r-1)})_{,ij}. \tag{3.2}$$

The convariant derivative of $C_{(r)ij}$ is defined by the following formula

$$\sum_k C_{(r)ij,k}\omega_k = dC_{(r)ij} + \sum_k C_{(r)kj}\omega_{ki} + \sum_k C_{(r)ik}\omega_{kj}, \tag{3.3}$$

and the convariant derivative of $C_{(r)ij,k}$ is defined by

$$\sum_l C_{(r)ij,kl}\omega_l = dC_{(r)ij,k} + \sum_m C_{(r)mj,k}\omega_{mi} + \sum_m C_{(r)im,k}\omega_{mj} + \sum_m C_{(r)ij,m}\omega_{mk}. \tag{3.4}$$

Taking exterior differentiation of the equation (3.3), we obtain

$$\sum_{lk} C_{(r)ij,kl}\omega_l \wedge \omega_k = \sum_m C_{(r)mj}\Phi_{mi} + \sum_m C_{(r)im}\Phi_{mj}. \tag{3.5}$$

Therefore

$$\sum_{lk}(C_{(r)ij,kl} - C_{(r)ij,lk})\omega_l \wedge \omega_k = 2\sum_m(C_{(r)mj}\Phi_{mi} + \sum_m C_{(r)im}\Phi_{mj}). \tag{3.6}$$

Taking (2.7) into (3.6), we have the Ricci identity

$$C_{(r)ij,kl} - C_{(r)ij,lk} = \sum_m C_{(r)mj}R_{mikl} + \sum_m C_{(r)im}R_{mjkl}. \tag{3.7}$$

Let $C_{(r-1)}$ be a Codazzi tensor, which satisfies

$$C_{(r-1)ij,k} - C_{(r-1)ik,j} = 0. \tag{3.8}$$

¿From the results of Theorem 2.1, it is easy to obtain the following lemma:

Lemma 3.1 *Let M^n be a compact orientable n-dimensional Riemannian manifold, φ be a symmetric tensor on M, $C_{(r)} = T_{(r)}\varphi$. If $C_{(r-1)}$ is a Codazzi tensor, then for* $(0 \le r \le n)$,

$$\int_M \Box^r Q_r dM = 0.$$

$\sum_k C_{(r-1)ij,kk}$ being defined as the Laplacian of $C_{(r-1)ij}$. Using equation (3.7), and following the methods of Calabi, Simons, Chern, Cheng-Yau, [1, 2, 18, 3, 6, 18], one can compute the Laplacian of the tensor,

$$\begin{aligned}\Delta C_{(r-1)ij} &= \sum_k C_{(r-1)ij,kk}\\ &= \sum_k (C_{(r-1)ij,kk} - C_{(r-1)ik,jk}) + \sum_k (C_{(r-1)ik,jk} - C_{(r-1)ik,kj})\\ &\quad + \sum_k (C_{(r-1)ik,kj} - C_{(r-1)kk,ij}) + \sum_k C_{(r-1)kk,ij}\\ &= \sum_{m,k} C_{(r-1)mk} R_{mijk} + \sum_{m,k} C_{(r-1)im} R_{mkjk} + \sum_k (C_{(r-1)ij,kk} - C_{(r-1)ik,jk})\\ &\quad + \sum_k (C_{(r-1)ik,kj} - C_{(r-1)kk,ij}) + \sum_k C_{(r-1)kk,ij}.\end{aligned} \tag{3.9}$$

Using (3.8), we obtain

$$\Delta C_{(r-1)ij} = \sum_{m,k} C_{(r-1)mk} R_{mijk} + \sum_{m,k} C_{(r-1)im} R_{mkjk} + (trC_{(r-1)})_{,ij}. \tag{3.10}$$

Let us set $|C_{(r-1)}|^2 = \sum_{i,j} C^2_{(r-1)ij}$, $|\nabla C_{(r-1)}|^2 = \sum_{i,j,k} C^2_{(r-1)ij,k}$. Making use of the equation (3.10), we obtain

$$\begin{aligned}\frac{1}{2}\Delta |C_{(r-1)}|^2 &= |\nabla C_{(r-1)}|^2 + \sum_{i,j,m,k} C_{(r-1)ij} C_{(r-1)mk} R_{mijk}\\ &\quad + \sum_{i,j,m,k} C_{(r-1)ij} C_{(r-1)im} R_{mkjk} + \sum_{i,j} C_{(r-1)ij} (trC_{(r-1)})_{,ij}.\end{aligned} \tag{3.11}$$

Near a point $p \in M^n$ we choose an orthonormal frame fields e_i $(i = 1, \ldots, n)$ such that $C_{(r-1)ij} = C_{(r-1)ii}\delta_{ij}$ at p. Then (3.11) is simplified to

$$\begin{aligned}\frac{1}{2}\nabla |C_{(r-1)}|^2 &= |\nabla C_{(r-1)}|^2 + \tfrac{1}{2}\sum_{i,j} R_{ijij}(C_{(r-1)ii} - C(r-1)_{jj})^2\\ &\quad + \sum_i C_{ii}(trC_{(r-1)})_{,ii}.\end{aligned} \tag{3.12}$$

From equations (3.2) and (3.12), we have

$$\begin{aligned}\Box^r Q_r &= \frac{1}{r}\sum (\frac{1}{r} trC_{(r-1)}\delta_{ij} - C_{(r-1)ij})(trC_{(r-1)})_{,ij}\\ &= \frac{1}{r}(\frac{1}{2r}\Delta |trC_{(r-1)}|^2 - \frac{1}{r}|\nabla trC_{(r-1)}|^2 - \frac{1}{2}\Delta |C_{(r-1)}|^2\\ &\quad + |\nabla C_{(r-1)}|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(C_{(r-1)ii} - C_{(r-1)jj})^2).\end{aligned} \tag{3.13}$$

¿From Lemma 3.1 we obtain immediately the following theorem :

Theorem 3.1 *Let M^n be a compact orientable n-dimensional Riemannian manifold, φ be a symmetric tensor. For $(1 \le r \le n)$ set $C_{(r-1)} = T_{(r-1)}\varphi$. Near a point $P \in M$ we choose orthomormal frame fields $\{e_i\}(i = 0, 1, \ldots, n)$*

such that $C_{(r-1)ij} = C_{(r-1)ii}\delta_{ij}$. *Suppose* $C_{(r-1)}$ *is a Codazzi tensor. Then*

$$\int_M (|\nabla C_{(r-1)}|^2 - \frac{1}{r}|\nabla trC_{(r-1)}|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(C_{(r-1)ii} - C_{(r-1)jj})^2)dM = 0.$$

If $M^n \subset N^{n+1}(c)$ and $r = 1$, the operator derived by Cheng-Yau (see [6]) identifies our operator

$$\Box^1 f = \sum_{i,j}((Tr\varphi)\delta_{ij} - \varphi_{ij})f_{,ij}.$$

Supposing M a compact hypersurface, the integral formula in the Theorem 3.1 proves to be

$$\int_M (|\nabla\varphi|^2 - |\nabla tr\varphi|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\varphi_{ii} - \varphi_{jj})^2)dM = 0. \tag{3.14}$$

Now we set $\varphi_{ij} = h_{ij}$. When M is of nonnegative sectional curvature, it is obvious from the above integral formula that

$$|\nabla B|^2 - n^2|\nabla H|^2 \geq 0, \tag{3.15}$$

where $H = \frac{1}{n}\sum_i h_{ii}$. When H is constant, the condition above is naturally true. A lot of works have been done for this case, see [1], [14], [18] and [19]. If $R - C = \text{const} \geq 0$ where R is the normalized scalar curvature, the condition is also true. Cheng-Yau ([6]), Yau ([22]), Li ([12], and [13]) have discussed the geometric meaning of the case.
Let (R_{ij}) be the matrix of the Ricci curvature tensor on M, r be the scalar curvature,

$$R_{ij} := \sum_k R_{kikj}, \quad r := \sum_k R_{kk}. \tag{3.16}$$

Schouten tensor $S = \sum_{ij} S_{ij}\omega_i \otimes \omega_j$, where

$$S_{ij} := R_{ij} - \frac{1}{2(n-1)} r\delta_{ij}. \tag{3.17}$$

It is well known that Schouten tensor is a Codazzi tensor on a local conformal symmetric space. In this situation we set $\varphi_{ij} = S_{ij}$, and then the integral formula (3.14) exists. The geometric meaning of the case is discussed in [9].

When $r = 2$,

$$\Box^2 Q_2 = \sum_{i,j}(TrC_{(1)}\delta_{ij} - C_{(1)ij})(\frac{TrC_{(1)}}{2})_{,ij}.$$

Suppose $M^n \subset N^{n+1}(c)$ and M^n is of harmonic Riemannian curvatures, however from the definitions of the convariant derivatives of R_{ij} and R_{ijkl}

$$\sum R_{ij,k}\omega_k := dR_{ij} + \sum R_{ik}\omega_{kj} + \sum R_{kj}\omega_{ki}, \tag{3.18}$$

$$\begin{aligned} \sum R_{ijkl,m}\omega_m : &= dR_{ijkl} + \sum R_{mjkl}\omega_{mi} + \sum R_{imkl}\omega_{mj}, \\ &+ \sum R_{ijml}\omega_{mk} + \sum R_{ijkm}\omega_{ml}. \end{aligned} \tag{3.19}$$

Taking exterior differention of equation (2.7) we obtain the following Bianchi identity

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0. \tag{3.20}$$

Combining (2.3) (2.4) with (3.19) we obtain

$$\begin{aligned} R_{ij,k} - R_{ik,j} &= \sum R_{lijk,l} \\ &\downarrow \ \text{(Riemanian curvature is harmonic)} \\ &= 0. \end{aligned} \tag{3.21}$$

We set $\varphi_{ij} = h_{ij}$. By

$$R_{ij} = C_{(1)ij} + (n-1)c\delta_{ij},$$

we know that $C_{(1)}$ is a Codazzi tensor. So the integral formula of Theorem 3.1 proves to be

$$\int_M \left(\sum_{i,j,k} R^2_{ij,k} - \frac{1}{2}\sum_k r^2_{,k} + \frac{1}{2}\sum_{i,j} R_{ijij}(R_{ii} - R_{jj})^2 \right) dM = 0. \tag{3.22}$$

However

$$\begin{aligned} \sum_j R_{ij,j} &= \sum_{k,j} R_{ikjk,j} \\ &= \sum_{k,j} R_{jkik,j} = 0. \end{aligned}$$

So we have

$$\begin{aligned} r_{,i} &= \sum_j R_{jj,i} \\ &= \sum_j R_{ij,j} = 0. \end{aligned}$$

Hence the integral formula (3.22) is

$$\int_M (\sum_{i,j,k} R^2_{ij,k} + \frac{1}{2}\sum_{i,j} R_{ijij}(R_{ii} - R_{jj})^2) dM = 0. \tag{3.23}$$

Xia in [20] discussed the geometric meaning of this situation.

Corollary 3.1 *Let M^n be a compact Riemannian manifold with harmonic curvature tensor and nonnegative sectional curvature. If M^n can be immersed into S^{n+1} as a hypersurface, then M^n is isometric with either $S^k(a) \times S^{n-k}(b)(a^2 + b^2 = 1)$ or S^n.*

References

[1] H. Alenlar and M. Do. Carmo, *Hypersurfaces with constant mean curvature in sphere*, Proc. Amer. Math. Soc., **120** (1994), 1223–1229.

[2] E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem*, K. Jörgens. Mich. Math. J., **5** (1958), 105.

[3] S. Y. Cheng, M. Do. Carmo, and S. Kobayashi, *Minimal submanifold of a sphere with second fundamental form of constant length*, in Fuctional Analysis and Related Fields (F. E. Browder, ed.), Springer-Verlag, New York, 1970, 59–75.

[4] S. S. Chern, W. H. Chen and K. S. Lam, *Lectures on Differential Geometry*, Peking University Press, 1983, in Chinese.

[5] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic press, INC, 1984.

[6] S. Y. Cheng and S. T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann., **225** (1977) 195–204.

[7] Z. Guo, *Willmore submanifolds in the unit sphere*, Coll. Math., **55** (3) (2004), 279–287.

[8] C. C. Hsiung, *Some integral formulas for hypersurfaces*, Math. Scand, **2** (1954), 286–294.

[9] N. Ji, *Schouten tensor on the Riemannian manifold with harmonic conformal curvature and its applications*, master degree dissertation, Yunnan Normal University, 2005.

[10] Y. Katsurada, *Generalized Minkowski formulas for closed hypersurfaces in Riemann space*, Ann. Mat. Pur. Appl., **57** (1962), 283–293.

[11] A. M. Li, *A kind of Variational problem and integral formula on Riemannian manifolds*, Acta Math. Sinica, **28** (2) (1985), 145–153.

[12] H. Li, *Hypersurface with constant scalar curvature in space forms*, Math. Ann., **305** (1996), 665–672.

[13] H. Li, *Global Rigidity theorems of hypersurfaces*, Ark. Mat., **35** (1997), 327–351.

[14] K. Nomizu, and B. Smyth, *A formula of Simon's type and hypersurfaces*, J. Differential Geom., **3** (1969), 367–377.

[15] R. C. Reilly, *Extrinsic rigidity theorems for compact submanifolds of the Sphere*, J. Differential Geom., **3** (1970), 487–497.

[16] R. C. Reilly, *On the first eigenvalue of the Laplacian for compact submanifolds of euclidean space*, Comment. Math. Helvetici, **52** (1977), 465–477.

[17] R. C. Reilly, *Variational properties of fuctions of the mean curvatres for hypersurfaces in space forms*, J. Differential Geom., **8** (1977), 525–533.

[18] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math., **88** (1968), 62–105.

[19] Z. Q. Sun, *Compact hypersurfaces with constant scalar curvature in S^4*, Chinese Ann. Math. Ser. A, **8: 3** (1987), 273–286.

[20] Y. W. Xia, *Rigidity on a hypersurface on the unit sphere with harmonic Ricci tensor curvature*, master degree dissertation, Yunnan Normal University, 2005.

[21] S. T. Yau, *Submanifolds with constant mean curvature*, I, Amer. J. Math., **96**(2) (1974), 346–366.

[22] S. T. Yau, *Submanifolds with constant mean curvature*, II, Amer. J. Math., **97**(1) (1975), 76–100.

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