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An Elementary Proof of the Transformation Formula for the Dedekind Eta Function

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Abstract. In this work, we give an elementary proof of the transformation formula for the Dedekind eta function under the action of the modular group $\mathrm{PSL}\,(2,\mathbb{Z})$. We start by giving a proof of the transformation formula $\eta(\tau)$ under the transformation $\tau \to -1/\tau$, using the Jacobi triple product identity and the Poisson summation formula. After we establish some identities for the Dedekind sum, the transformation formula for $\eta(\tau)$ under the transformation induced by a general element of the modular group $\mathrm{PSL}\,(2,\mathbb{Z})$ is derived by induction.

Key Words: Dedekind Eta Function, Transformation Formula, Modular Group, Functional Equation
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1 Introduction

The Dedekind eta function is introduced by Dedekind in 1877 and is defined in the upper half plane $\mathbb{H} = \{\tau \,|\, \mathrm{Im}\,\tau > 0\}$ by the equation

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right).$$

It is closely related to the theory of modular forms [2]. In this note, we are going to derive the following formula which describes the transformation of $\eta(\tau)$ under a linear fractional transformation defined by an element of the modular group $\Gamma = \mathrm{PSL}\,(2,\mathbb{Z})$:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \exp\left\{\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right\}\{-i(c\tau+d)\}^{1/2}\,\eta(\tau). \quad (1)$$

Here $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an element of Γ with $c > 0$, and $s(-d, c)$ is a Dedekind sum. Equation (1) is known as the Dedekind functional equation for the Dedekind eta function. One can establish that the number

$$\omega(a, b, c, d) = \frac{a+d}{c} + 12s(-d, c)$$

is an integer. Therefore, the function $f(\tau) = \eta(\tau)^{24}$ satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right)\frac{1}{(c\tau + d)^{12}} = f(\tau) \quad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

In other words, $\eta(\tau)^{24}$ is a modular form of weight 12 for the modular group Γ.

The Dedekind functional equation (1) was proved using a more general transformation formula of Iseki [4] in the book [2]. In the special case $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $s(0, 1) = 0$, and formula (1) reduces to

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{-1/2}\eta(\tau). \tag{2}$$

This formula has been proved using various methods such as the contour integral method by Siegel [7] (see also [2]). A slight drawback of Siegel's method is that it involves a limiting process which needs to be justified using advanced theories.

In this note, we present a proof of (2) using elementary methods. We first present the proof of the Jacobi triple product formula

$$\prod_{n=1}^{\infty}\left(1 - w^{2n}\right)\left(1 + w^{2n-1}z^2\right)\left(1 + w^{2n-1}z^{-2}\right) = \sum_{n=-\infty}^{\infty} w^{n^2}z^{2n}, \;\; |w| < 1, z \neq 0,$$

following the approach in [1]. From here we derive the Euler pentagonal number formula

$$\prod_{n=1}^{\infty}(1 - w^n) = \sum_{n=-\infty}^{\infty}(-1)^n w^{\frac{3n^2-n}{2}}, \quad |w| < 1.$$

The Poisson summation formula is then employed to prove the transformation formula (2).

It is well known that the modular group Γ is generated by the two elements $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The transformation of η under T is given by

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau),$$

which is obvious from its definition. The fact that Γ is generated by T and S can be proved by induction on c (see, for example, [2].) Using this idea, we prove the Dedekind functional equation (1) for general transformation using induction. This proof is completely elementary.

The purpose of this work is to give a self-contained elementary proof for the Dedekind functional equation. Therefore, we present in detail the proofs of all the results we need.

2 Fractional Linear Transformations and the Modular Group

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. It is well known that a mapping $w : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is analytic and bijective if and only if w is a linear fractional transformation, namely,

$$w(z) = \frac{az+b}{cz+d}$$

for some 4-tuple (a, b, c, d) with $ad-bc \neq 0$. For any nonzero complex number k, the 4-tuples (a, b, c, d) and (ka, kb, kc, kd) define the same fractional linear transformation. Therefore, we can normalize a, b, c, d by

$$ad - bc = 1,$$

and associate this linear fractional transformation with the two-by-two matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{3}$$

The set of two-by-two matrices of the form (3) with $ad - bc = 1$ is denoted by $\mathrm{SL}\,(2, \mathbb{C})$. This is a group under matrix multiplication.

Since (a, b, c, d) and $(-a, -b, -c, -d)$ define the same fractional linear transformation, we can define an equivalence relation on $\mathrm{SL}\,(2, \mathbb{C})$ in the following way. If A and B are in $\mathrm{SL}\,(2, \mathbb{C})$, then $A \sim B$ if and only if

$$A = \pm B.$$

The quotient of $\mathrm{SL}\,(2, \mathbb{C})$ by this equivalence relation is denoted by $\mathrm{PSL}\,(2, \mathbb{C})$. Let I be the two-by-two identity matrix. Then $H = \{I, -I\}$ is a normal subgroup of $\mathrm{SL}\,(2, \mathbb{C})$. One can easily see that

$$\mathrm{PSL}\,(2, \mathbb{C}) = \mathrm{SL}\,(2, \mathbb{C})/H.$$

Therefore, $\mathrm{PSL}\,(2,\mathbb{C})$ is also a group, which we call the group of fractional linear transformations. The group operation is precisely composition of transformations.

The modular group $\Gamma = \mathrm{PSL}\,(2,\mathbb{Z})$ is the subgroup of $\mathrm{PSL}\,(2,\mathbb{C})$ consisting of elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with integers a, b, c, d satisfying $ad - bc = 1$. It is well known that it is generated by the two elements

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{4}$$

which describe, respectively, the linear transformations

$$z \mapsto z + 1 \qquad \text{and} \qquad z \mapsto -\frac{1}{z}.$$

3 Jacobi Triple Product Identity

In this section, we derive the Jacobi triple product identity following the approach in [1].

Theorem 1 *Let w and z be complex numbers with $|w| < 1$ and $z \neq 0$. Then*

$$\prod_{n=1}^{\infty} \left(1 - w^{2n}\right)\left(1 + w^{2n-1}z^2\right)\left(1 + w^{2n-1}z^{-2}\right) = \sum_{n=-\infty}^{\infty} w^{n^2} z^{2n}. \tag{5}$$

Proof. When $|w| < 1$ and $z \neq 0$, the triple product on the left-hand side of (5) converges absolutely. The sum on the right-hand side of (5) also converges absolutely.

For $|w| < 1$ and $z \neq 0$, define the function $F(w, z)$ by

$$F(w, z) = \prod_{n=1}^{\infty} \left(1 - w^{2n}\right)\left(1 + w^{2n-1}z^2\right)\left(1 + w^{2n-1}z^{-2}\right).$$

For a fixed w, $F(w, z)$ can be expanded into a power series in z. Since $F(w, z)$ is even in z and $F(w, z) = F(w, z^{-1})$, the corresponding power series has the form

$$F(w, z) = \sum_{n=-\infty}^{\infty} a_n(w) z^{2n}$$

with

$$a_{-n}(w) = a_n(w).$$

Since $F(0, z) = 1$, we find that $a_0(0) = 1$ and $a_n(0) = 0$ if $n \neq 0$.

Now, note that

$$\begin{aligned}F(w,wz) &= \prod_{n=1}^{\infty}\left(1-w^{2n}\right)\left(1+w^{2n+1}z^2\right)\left(1+w^{2n-3}z^{-2}\right)\\&= \frac{1+w^{-1}z^{-2}}{1+wz^2}F(w,z)\\&= w^{-1}z^{-2}F(w,z).\end{aligned}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} a_n(w)w^{2n}z^{2n} = w^{-1}z^{-2}\sum_{n=-\infty}^{\infty} a_n(w)z^{2n} = \sum_{n=-\infty}^{\infty} a_{n+1}(w)w^{-1}z^{2n}.$$

This implies that for any integer n,

$$a_{n+1}(w) = w^{2n+1}a_n(w).$$

By induction, we find that for $n \geq 1$,

$$a_{-n}(w) = a_n(w) = w^{n^2}a_0(w).$$

Therefore,

$$F(w,z) = a_0(w)\sum_{n=-\infty}^{\infty} w^{n^2}z^{2n}. \tag{6}$$

To prove the theorem, we need to show that $a_0(w) = 1$ for all $|w| < 1$. Setting $z = e^{\frac{\pi i}{4}}$ in (6), we have

$$\frac{F\left(w, e^{\frac{\pi i}{4}}\right)}{a_0(w)} = \sum_{n=-\infty}^{\infty} w^{n^2}i^n. \tag{7}$$

Since $i^{2n} = i^{-2n} = (-1)^n$ and $i^{-(2n+1)} = -i^{2n+1}$, we find that the odd terms in the right hand side of (7) cancel, and only the even terms left. This gives

$$\frac{F\left(w, e^{\frac{\pi i}{4}}\right)}{a_0(w)} = \sum_{n=-\infty}^{\infty} (-1)^n w^{4n^2}. \tag{8}$$

Setting $z = i$ and replacing w with w^4 in (6), we have

$$\frac{F\left(w^4, i\right)}{a_0(w^4)} = \sum_{n=-\infty}^{\infty} (-1)^n w^{4n^2}. \tag{9}$$

A comparison of (8) and (9) gives

$$\frac{a_0(w^4)}{a_0(w)} = \frac{F(w^4, i)}{F\left(w, e^{\frac{\pi i}{4}}\right)}.$$

This implies that

$$\begin{aligned}\frac{a_0(w^4)}{a_0(w)} &= \prod_{n=1}^{\infty} \frac{(1-w^{8n})(1-w^{8n-4})^2}{(1-w^{2n})(1+iw^{2n-1})(1-iw^{2n-1})}\\ &= \prod_{n=1}^{\infty} \frac{(1-w^{8n})(1-w^{8n-4})^2}{(1-w^{2n})(1+w^{4n-2})}.\end{aligned}$$

Since every positive integer of the form $4n$ is either of the form $8n$ or of the form $8n-4$, we find that

$$\prod_{n=1}^{\infty}(1-w^{8n})(1-w^{8n-4}) = \prod_{n=1}^{\infty}(1-w^{4n}).$$

On the other hand,

$$(1-w^{8n-4}) = (1-w^{4n-2})(1+w^{4n-2}).$$

Therefore,

$$\frac{a_0(w^4)}{a_0(w)} = \prod_{n=1}^{\infty} \frac{(1-w^{4n})(1-w^{4n-2})}{1-w^{2n}}.$$

Since every positive integer of the form $2n$ is either of the form $4n$ or of the form $4n-2$, we find that

$$\prod_{n=1}^{\infty}(1-w^{4n})(1-w^{4n-2}) = \prod_{n=1}^{\infty}(1-w^{2n}).$$

This implies

$$a_0(w^4) = a_0(w).$$

For any w with $|w| < 1$ and any positive integer k,

$$a_0(w) = a_0(w^4) = \cdots = a_0(w^{4k}).$$

Since $w^{4k} \to 0$ when $k \to \infty$, we obtain

$$a_0(w) = a_0(0) = 1.$$

Hence,

$$F(w, z) = \sum_{n=-\infty}^{\infty} w^{n^2} z^{2n},$$

which completes the proof. □

Note that $w = e^{\pi i\tau}$ maps the upper half plane $\mathbb{H} = \{\operatorname{Im}\tau > 0\}$ to the unit disc $\mathbb{D} = \{|w| < 1\}$. Replacing w by $e^{\pi i\tau}$ and z by $e^{\pi iz}$, the Jacobi triple product identity takes the following form.

Corollary 1 *For any complex numbers τ and z with $Im\,\tau > 0$, it holds*

$$\begin{aligned}&\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)\left(1+e^{\pi i(2n-1)\tau}e^{2\pi iz}\right)\left(1+e^{\pi i(2n-1)\tau}e^{-2\pi iz}\right)\\&=\sum_{n=-\infty}^{\infty}e^{\pi in^2\tau}e^{2\pi inz}.\end{aligned}\tag{10}$$

4 Poisson Summation Formula

Poisson summation formula, which is useful in the study of number theory, is a consequence of the theory of Fourier series. In this section, we present the Poisson summation formula and apply it to the Gaussian function.

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that*

$$\sum_{n=-\infty}^{\infty} f(x+n) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} f'(x+n)$$

converge uniformly on the closed interval $[0, 1]$. Then for any $x \in \mathbb{R}$,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx} \tag{11}$$

where

$$\widehat{f}(n) = \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx.$$

Proof. Define

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) \quad \text{and} \quad G(x) = \sum_{n=-\infty}^{\infty} f'(x+n). \tag{12}$$

Due to the assumption of uniform convergence on $[0, 1]$ and the fact that f and f' are continuous, F and G are continuous functions on $[0, 1]$. It is easy to verify that the series for $F(x)$ and $G(x)$ converge uniformly on any closed and bounded interval,

$$F(x+1) = F(x), \qquad G(x+1) = G(x),$$

and

$$F'(x) = G(x).$$

In particular, F is also continuously differentiable. Now, since F is a periodic function with period 1, Dirichlet theorem for Fourier series implies that the Fourier series of $F(x)$ converges to $F(x)$. Namely,

$$\sum_{n=-\infty}^{\infty} f(x+n) = F(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

where

$$c_n = \int_0^1 F(x) e^{-2\pi i n x} dx.$$

Let us compute c_n in terms of f. We have

$$c_n = \int_0^1 \sum_{k=-\infty}^{\infty} f(x+k) e^{-2\pi i n x} dx.$$

Since the first series in (12) converges uniformly, we can interchange summation and integration to obtain

$$c_n = \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k) e^{-2\pi i n x} dx.$$

Making the change of variables $x \mapsto x - k$, we have

$$\begin{aligned} c_n &= \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(x) e^{-2\pi i n (x-k)} dx \\ &= \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(x) e^{-2\pi i n x} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx. \end{aligned}$$

This completes the proof of the theorem. □

Before applying the Poisson summation formula to a Gaussian function, let us verify the uniform convergence of the corresponding series.

Lemma 1 *Let u be a positive number and let b be any real number. Define the function $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = e^{-2\pi i b x} e^{-\pi u x^2}.$$

Then the two series

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n) \quad \text{and} \quad G(x) = \sum_{n=-\infty}^{\infty} f'(x+n)$$

converge uniformly on $[0, 1]$.

Proof. It suffices to consider the case $b = 0$. We prove the uniform convergence for the series $G(x)$ by applying the Weierstrass M-test. The proof for the series $F(x)$ is similar.

Note that when $b = 0$,

$$f'(x) = -2\pi u x e^{-\pi u x^2}.$$

For $x \in [0, 1]$ and $n \geq 1$, we can write

$$|f'(x+n)| \leq 2\pi u(n+1)e^{-\pi u n^2} \leq 2\pi u(n+1)e^{-\pi u n}.$$

When $n \geq 2$,

$$|f'(x-n)| \leq 2\pi u n e^{-\pi u(n-1)^2} \leq 2\pi u n e^{-\pi u(n-1)}.$$

It remains to note that the two series

$$\sum_{n=1}^{\infty} 2\pi u(n+1)e^{-\pi u n} \qquad \text{and} \qquad \sum_{n=2}^{\infty} 2\pi u n e^{-\pi u(n-1)}$$

are both convergent. □

Theorem 3 *Let u be a positive number. For any real numbers a and b, one has*

$$\sum_{n=-\infty}^{\infty} e^{-2\pi i(n+a)b} e^{-\pi u(n+a)^2} = \frac{1}{\sqrt{u}} \sum_{n=-\infty}^{\infty} e^{2\pi i n a} e^{-\pi(n+b)^2/u}. \tag{13}$$

Proof. Let $f(x)$ be the function defined in Lemma 1. By the Poisson summation formula (11), we have

$$\sum_{n=-\infty}^{\infty} e^{-2\pi i(n+a)b} e^{-\pi u(n+a)^2} = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n a}.$$

Now we only need to compute $\widehat{f}(n)$. We can write

$$\begin{aligned}\widehat{f}(n) &= \int_{-\infty}^{\infty} e^{-2\pi i b x} e^{-\pi u x^2} e^{-2\pi i n x} dx \\ &= e^{-\pi(n+b)^2/u} \int_{-\infty}^{\infty} e^{-\pi u(x+i(n+b)/u)^2} dx.\end{aligned}$$

Here the function $e^{-\pi u z^2}$ is integrated over the closed contour $\operatorname{Im} z = (n + b)/u$. Since $e^{-\pi u z^2}$ is analytic, we can shift the contour of integration to the real line $\operatorname{Im} z = 0$. This gives

$$\int_{-\infty}^{\infty} e^{-\pi u(x+i(n+b)/u)^2} dx = \int_{-\infty}^{\infty} e^{-\pi u x^2} dx = \frac{1}{\sqrt{u}}.$$

Therefore,

$$\widehat{f}(n) = \frac{1}{\sqrt{u}} e^{-\pi(n+b)^2/u},$$

and the proof is completed. □

Let

$$D = \left\{(\tau, z, w) \in \mathbb{C}^3 \,|\, \mathrm{Im}\,\tau > 0\right\}.$$

Note that both the series

$$H_1(\tau, z, w) = \sum_{n=-\infty}^{\infty} e^{-2\pi i(n+z)w} e^{\pi i \tau (n+z)^2}$$

and

$$H_2(\tau, z) = (-i\tau)^{-1/2} \sum_{n=-\infty}^{\infty} e^{2\pi i n z} e^{-\pi i (n+w)^2/\tau}$$

converge absolutely and uniformly on any compact subsets of D. Hence, both of them define analytic functions on D. When $\tau = iu$, $u > 0$ and $z = a$, $w = b$, $a, b \in \mathbb{R}$, it follows from (13) that

$$H_1(iu, a, b) = H_2(iu, a, b).$$

By analytic continuation, we obtain

$$H_1(\tau, z, w) = H_2(\tau, z, w) \qquad \text{for all } (\tau, z, w) \in D.$$

Corollary 2 *For any complex numbers τ, z and w with $Im\,\tau > 0$, it holds*

$$\sum_{n=-\infty}^{\infty} e^{-2\pi i(n+z)w} e^{\pi i \tau (n+z)^2} = (-i\tau)^{-1/2} \sum_{n=-\infty}^{\infty} e^{2\pi i n z} e^{-\pi i (n+w)^2/\tau}. \tag{14}$$

5 Transformation Defined by the Generators of the Modular Group

In this section, we consider the transformation of the Dedekind eta function $\eta(\tau)$ under the action of the two generators T and S (4) of the modular group $\mathrm{PSL}\,(2, \mathbb{Z})$.

For the generator T, its power T^m defines the transformation $\tau \mapsto \tau + m$. The transformation of $\eta(\tau)$ under the action of T^m is easily deduced.

Proposition 4 *If $\tau \in \mathbb{H}$ and m is an integer, we have*

$$\eta(\tau + m) = \exp\left(\frac{\pi i m}{12}\right)\eta(\tau).$$

For the transformation of $\eta(\tau)$ under the generator S, we have the following result.

Theorem 5 *When $Im\tau > 0$, it holds*

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau). \tag{15}$$

There are various methods that can be used to prove this transformation formula. In [2], (15) was proved using Siegel's method which employs residue calculus. In [5], (15) was derived as a consequence of the corresponding transformation formula for the Eisenstein series $E_2(z)$. In [8], (15) was proved using the Jacobi triple product formula as well as the Poisson summation formula. However, it was first proved that

$$\eta\left(-\frac{1}{\tau}\right)^3 = (-i\tau)^{3/2}\eta(\tau)^3.$$

Here we are going to give an alternative proof of (15) by first deriving the Euler pentagonal number theorem.

Theorem 6 *When $Im\tau > 0$, one has*

$$\prod_{n=1}^{\infty}\left(1 - e^{2\pi in\tau}\right) = \sum_{n=-\infty}^{\infty}(-1)^n e^{\pi i(3n^2-n)\tau} = \sum_{n=-\infty}^{\infty}(-1)^n e^{\pi i(3n^2+n)\tau}.$$

Proof. Replacing τ by 3τ, and z by $(\tau+1)/2$ in the Jacobi triple product identity (10), we obtain

$$\prod_{n=1}^{\infty}\left(1 - e^{2\pi i(3n)\tau}\right)\left(1 - e^{2\pi i(3n-1)\tau}\right)\left(1 - e^{2\pi i(3n-2)\tau}\right) = \sum_{n=-\infty}^{\infty}(-1)^n e^{\pi i(3n^2+n)\tau}.$$

When n runs through positive integers, $3n$, $3n-1$ and $3n-2$ also runs through positive integers. Hence,

$$\prod_{n=1}^{\infty}\left(1 - e^{2\pi i(3n)\tau}\right)\left(1 - e^{2\pi i(3n-1)\tau}\right)\left(1 - e^{2\pi i(3n-2)\tau}\right) = \prod_{n=1}^{\infty}\left(1 - e^{2\pi in\tau}\right).$$

□

Using the Euler pentagonal number theorem, we can express the Dedekind eta function as a sum.

Corollary 3 *When $Im\tau > 0$,*

$$\eta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi in} e^{3\pi i\left(n+\frac{1}{6}\right)^2\tau}. \tag{16}$$

Using the identity (14), we can now prove the transformation formula (15).

Proof of Theorem 5 Replacing τ by $\tau/3$ and setting $z = 1/2$, $w = 1/6$ in (14), we obtain

$$\sum_{n=-\infty}^{\infty} e^{\pi i n} e^{-3\pi i(n+1/6)^2/\tau} = \frac{(-i\tau)^{1/2}}{\sqrt{3}} \sum_{n=-\infty}^{\infty} e^{-\pi i n/3} e^{-\pi i/6} e^{\pi i\tau(n+1/2)^2/3}. \tag{17}$$

By (16), the left-hand side of (17) is $\eta(-1/\tau)$. For the right-hand side, note that when n runs through all integers, $3n$, $3n-1$ and $3n+1$ together also run through all integers. Therefore,

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} e^{-\pi i n/3} e^{-\pi i/6} e^{\pi i\tau(n+1/2)^2/3}\\ =& \sum_{n=-\infty}^{\infty} e^{-\pi i n} e^{-\pi i/6} e^{\pi i\tau(3n+1/2)^2/3} + \sum_{n=-\infty}^{\infty} e^{-\pi i n} e^{\pi i/6} e^{\pi i\tau(3n-1/2)^2/3}\\ &+ \sum_{n=-\infty}^{\infty} e^{-\pi i n} e^{-\pi i/2} e^{\pi i\tau(3n+3/2)^2/3}.\end{aligned} \tag{18}$$

The first two terms on the right-hand side of (18) give

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} e^{-\pi i n} e^{-\pi i/6} e^{\pi i\tau(3n+1/2)^2/3} + \sum_{n=-\infty}^{\infty} e^{-\pi i n} e^{\pi i/6} e^{\pi i\tau(3n-1/2)^2/3}\\ &= 2\cos\frac{\pi}{6} \sum_{n=-\infty}^{\infty} (-1)^n e^{3\pi i\tau(n+1/6)^2}\\ &= \sqrt{3}\eta(\tau).\end{aligned} \tag{19}$$

The last term on the right-hand side of (18) is

$$I = -i \sum_{n=-\infty}^{\infty} (-1)^n e^{3\pi i\tau(n+1/2)^2}.$$

When n runs through all integers, $-1-n$ also runs through all integers. We then find that

$$I = -i \sum_{n=-\infty}^{\infty} (-1)^{n+1} e^{3\pi i\tau(-n-1/2)^2} = i \sum_{n=-\infty}^{\infty} (-1)^n e^{3\pi i\tau(n+1/2)^2} = -I.$$

Thus, $I = 0$, and we conclude from (17), (18) and (19) that

$$\eta\left(1/\tau\right) = (-i\tau)^{1/2}\eta(\tau).$$

□

Remark 1 *Define the function* $\chi : \mathbb{Z} \to \mathbb{C}$ *by*

$$\chi(1) = \chi(11) = 1, \quad \chi(5) = \chi(7) = -1,$$

$$\chi(2) = \chi(3) = \chi(4) = \chi(6) = \chi(8) = \chi(9) = \chi(10) = \chi(12) = 0,$$

and

$$\chi(n+12) = \chi(n) \qquad \text{for all}\, n \in \mathbb{Z}.$$

Then $\chi(n)$ *is a Dirichlet character modulo 12. It satisfies the multiplicativity property:*

$$\chi(mn) = \chi(m)\chi(n) \qquad \text{for all}\, m, n \in \mathbb{Z}.$$

As n *runs through all integers,* $2n$ *and* $2n+1$ *together also runs through all integers. The formula for* $\eta(\tau)$ (16) *shows that*

$$\begin{aligned}
\eta(\tau) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau (12n+1)^2/12} - \sum_{n=-\infty}^{\infty} e^{\pi i \tau (12n+7)^2/12} \\
&= \frac{1}{2}\left\{ \sum_{n=-\infty}^{\infty} \chi(12n+1) e^{\pi i \tau (12n+1)^2/12} + \sum_{n=-\infty}^{\infty} \chi(12n-1) e^{\pi i \tau (12n-1)^2/12} \right. \\
&\quad \left. + \sum_{n=-\infty}^{\infty} \chi(12n+7) e^{\pi i \tau (12n+7)^2/12} + \sum_{n=-\infty}^{\infty} \chi(12n-7) e^{\pi i \tau (12n-7)^2/12} \right\} \\
&= \frac{1}{2} \sum_{m=1}^{12} \chi(m) \sum_{n=-\infty}^{\infty} e^{\pi i \tau (12n+m)^2/12} \\
&= \frac{1}{2} \sum_{n=-\infty}^{\infty} \chi(n) e^{\pi i \tau n^2/12}.
\end{aligned}$$

As in [3], this formula can be used to give another proof of Theorem 5.

Second proof of Theorem 5 One can easily verify that for all integer n,

$$\chi(n) = \frac{1}{\sqrt{12}} \sum_{m=1}^{12} \chi(m) e^{\frac{2\pi i m n}{12}}.$$

Therefore,

$$\eta(-1/\tau) = \frac{1}{2\sqrt{12}} \sum_{m=1}^{12} \chi(m) \sum_{n=-\infty}^{\infty} e^{\frac{2\pi i m n}{12}} e^{-\pi i n^2/(12\tau)}.$$

For each $1 \leq m \leq 12$, using (14) with τ replaced by 12τ, $z = m/12$ and $w = 0$, we find that

$$\begin{aligned}\eta(-1/\tau) &= \frac{(-i\tau)^{1/2}}{2}\sum_{m=1}^{12}\chi(m)\sum_{n=-\infty}^{\infty} e^{12\pi i\tau(n+m/12)^2}\\ &= \frac{(-i\tau)^{1/2}}{2}\sum_{m=1}^{12}\chi(m)\sum_{n=-\infty}^{\infty} e^{\pi i\tau(12n+m)^2/12}\\ &= (-i\tau)^{1/2}\eta(\tau).\end{aligned}$$

Thus, the transformation formula (15) is a special case of a more general transformation formula for the theta function associated with Dirichlet characters [6]. $\square$

6 The Dedekind Sums

For the transformation formula for η under a general element of the modular group, we first define the Dedekind sum.

If h is an integer and k is a positive integer larger than 1, the Dedekind sum $s(h,k)$ is defined as

$$s(h,k) = \sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right). \tag{20}$$

When $k = 1$, we put $s(h,1) = 0$ for any integer h.

The Dedekind sums have the following properties.

Lemma 2 *Let k be a positive integer and let h and h' be integers relatively prime to k. If $h \equiv h' \mod k$, then*

$$s(h,k) = s(h',k).$$

Proof. The statement is obvious if $k = 1$. If $k > 1$, there is an integer m such that

$$h' = km + h.$$

Then for any integer r,

$$\begin{aligned}\frac{h'r}{k} - \left\lfloor\frac{h'r}{k}\right\rfloor &= \frac{(km+h)r}{k} - \left\lfloor\frac{(km+h)r}{k}\right\rfloor\\ &= mr + \frac{hr}{k} - \left\lfloor mr + \frac{hr}{k}\right\rfloor\\ &= mr + \frac{hr}{k} - mr - \left\lfloor\frac{hr}{k}\right\rfloor\\ &= \frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor.\end{aligned}$$

It follows from the definition (20) that $s(h',k) = s(h,k)$. □

There is a simple relation between $s(h,k)$ and $s(-h,k)$.

Lemma 3 *If k is a positive integer and h is an integer relatively prime to k, then*

$$s(-h,k) = -s(h,k).$$

Proof. For each $1 \le r \le k-1$, there is an r' such that $1 \le r' \le k-1$ and

$$hr \equiv r' \mod k.$$

This implies that

$$\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor = \frac{r'}{k},$$

Since

$$-hr \equiv k - r' \mod k$$

and $1 \le k - r' \le k-1$, we have

$$\frac{-hr}{k} - \left\lfloor \frac{-hr}{r} \right\rfloor = \frac{k-r'}{k} = 1 - \frac{r'}{k}.$$

Therefore,

$$\begin{aligned} s(-h,k) &= \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{-hr}{k} - \left\lfloor \frac{-hr}{k} \right\rfloor - \frac{1}{2}\right) \\ &= \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{1}{2} - \frac{r'}{k}\right) \\ &= -\sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right) \\ &= -s(h,k). \end{aligned}$$

□

Next we establish the reciprocity relation between $s(h,k)$ and $s(k,h)$ when h and k are positive integers. The main idea is the same as in [2], where one evaluates a sum in two different ways. To make it easier to understand, we extract some identities as lemmas. Instead of using purely number theoretic argument as in [2], we give an interpretation in terms of counting lattice points, an idea that has been used in one of the proofs of the law of quadratic reciprocity.

Lemma 4 *If k is a positive integer and h is an integer relative prime to k, then*

$$\sum_{r=1}^{k-1}\left\lfloor\frac{hr}{k}\right\rfloor=\frac{(h-1)(k-1)}{2}.$$

Proof. As in the proof of Lemma 3, for each $1\leq r\leq k-1$, there is an integer r' such that $1\leq r'\leq k-1$ and

$$hr\equiv r'\mod k.$$

This implies that

$$\frac{hr}{k}-\left\lfloor\frac{hr}{k}\right\rfloor=\frac{r'}{k}.$$

As r runs through the integers from 1 to $k-1$, r' also runs through the integers from 1 to $k-1$. Therefore,

$$\sum_{r=1}^{k-1}\left\lfloor\frac{hr}{k}\right\rfloor=\sum_{r=1}^{k-1}\frac{hr}{k}-\sum_{r'=1}^{k-1}\frac{r'}{k}=\frac{(h-1)(k-1)}{2}.$$

□

Lemma 5 *If h and k are positive integers with $(h,k)=1$, then*

$$\sum_{r=1}^{k-1}\left(\left\lfloor\frac{hr}{k}\right\rfloor\right)^2=2hs(k,h)+\frac{(2hk-3h-k+3)(h-1)}{6}.$$

Proof. Using Lemma 4, we find that

$$\begin{aligned}\sum_{r=1}^{k-1}\left(\left\lfloor\frac{hr}{k}\right\rfloor\right)^2&=\sum_{r=1}^{k-1}\left\lfloor\frac{hr}{k}\right\rfloor\left(\left\lfloor\frac{hr}{k}\right\rfloor+1\right)-\sum_{r=1}^{k-1}\left\lfloor\frac{hr}{k}\right\rfloor\\&=2\sum_{r=1}^{k-1}\sum_{s=1}^{\left\lfloor\frac{hr}{k}\right\rfloor}s-\frac{(h-1)(k-1)}{2}.\end{aligned}$$

Consider the lattice points (r,s) with $1\leq r\leq k-1$ and $1\leq s\leq h-1$. Since h and k are relatively prime, none of these points lie on the line $hx=ky$. Hence,

$$\begin{aligned}\sum_{r=1}^{k-1}\sum_{s=1}^{\left\lfloor\frac{hr}{k}\right\rfloor}s&=\sum_{\substack{1\leq r\leq k-1,\,1\leq s\leq h-1\\ks\leq hr}}s\\&=\sum_{1\leq r\leq k-1,\,1\leq s\leq h-1}s-\sum_{\substack{1\leq r\leq k-1,\,1\leq s\leq h-1\\hr\leq ks}}s.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{r=1}^{k-1}\left(\left\lfloor\frac{hr}{k}\right\rfloor\right)^2 &= (k-1)(h-1)h - 2\sum_{s=1}^{h-1}\sum_{r=1}^{\left\lfloor\frac{ks}{h}\right\rfloor} s - \frac{(h-1)(k-1)}{2}\\ &= \frac{(k-1)(h-1)(2h-1)}{2} - 2\sum_{s=1}^{h-1} s\left\lfloor\frac{ks}{h}\right\rfloor\\ &= 2h\sum_{s=1}^{h-1}\frac{s}{h}\left(\frac{ks}{h}-\left\lfloor\frac{ks}{h}\right\rfloor-\frac{1}{2}\right) - 2h\sum_{s=1}^{h-1}\frac{s}{h}\left(\frac{ks}{h}-\frac{1}{2}\right)\\ &\quad + \frac{(k-1)(h-1)(2h-1)}{2}.\end{aligned}$$

A straightforward computation gives

$$\sum_{r=1}^{k-1}\left(\left\lfloor\frac{hr}{k}\right\rfloor\right)^2 = 2hs(k,h) + \frac{(2hk-3h-k+3)(h-1)}{6}.$$

□

Now, we can establish the reciprocity law for Dedekind sums.

Theorem 7 *If h and k are positive integers with $(h,k)=1$, then*

$$s(h,k)+s(k,h) = \frac{h^2+k^2-3hk+1}{12hk}.$$

Proof. Since there is a symmetry in h and k, we can assume that $h \geq k$. It is easy to check that the formula is true when $h=k=1$. When $k=1$ and $h>1$,

$$\begin{aligned}s(h,k)+s(k,h) &= s(1,h)\\ &= \sum_{r=1}^{h-1}\frac{r}{h}\left(\frac{r}{h}-\frac{1}{2}\right)\\ &= \frac{h^2-3h+2}{12h}\\ &= \frac{h^2+k^2-3hk+1}{12hk}.\end{aligned}$$

Let $h \geq k > 1$. Since $(h,k)=1$, we must have $h>k$. As in the proof of Lemma 3, for each integer $1 \leq r \leq k-1$, there is a unique r' such that $1 \leq r' \leq k-1$ and

$$hr \equiv r' \mod k,$$

which implies that

$$\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor = \frac{r'}{k}.$$

Hence,

$$\begin{aligned}\sum_{r'=1}^{k-1}\left(\frac{r'}{k}\right)^2 &= \sum_{r=1}^{k-1}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor\right)^2 \\ &= 2\sum_{r=1}^{k-1}\frac{hr}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right) - \sum_{r=1}^{k-1}\frac{h^2r^2}{k^2} \\ &+ \sum_{r=1}^{k-1}\left(\left\lfloor \frac{hr}{k} \right\rfloor\right)^2 + \sum_{r=1}^{k-1}\frac{hr}{k}.\end{aligned}$$

Using Lemma 5, we find that

$$\begin{aligned}2hs(h,k) = &\sum_{r'=1}^{k-1}\left(\frac{r'}{k}\right)^2 + \sum_{r=1}^{k-1}\frac{h^2r^2}{k^2} - \sum_{r=1}^{k-1}\frac{hr}{k} - 2hs(k,h) \\ &- \frac{(2hk-3h-k+3)(h-1)}{6}\end{aligned}$$

Hence,

$$\begin{aligned}&s(h,k)+s(k,h) \\ &= \frac{1}{2h}\left\{\sum_{r'=1}^{k-1}\left(\frac{r'}{k}\right)^2 + \sum_{r=1}^{k-1}\frac{h^2r^2}{k^2} - \sum_{r=1}^{k-1}\frac{hr}{k} - \frac{(2hk-3h-k+3)(h-1)}{6}\right\} \\ &= \frac{h^2+k^2-3hk+1}{12hk}.\end{aligned}$$

□

7 Dedekind's Functional Equation

The main result of this section is the induction proof of the Dedekind's functional equation presented in Theorem 8.

Theorem 8 *If* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ *and* $c > 0$*, then*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \exp\left(\frac{\pi i \omega(a,b,c,d)}{12}\right)\{-i(c\tau+d)\}^{1/2}\eta(\tau)$$

where

$$\omega(a,b,c,d) = \frac{a+d}{c} + 12s(-d,c) \tag{21}$$

is an integer.

Proof. We use induction on c. When $c = 1$, $b = ad - 1$. Thus,

$$\frac{a\tau + b}{c\tau + d} = \frac{a(\tau + d) - 1}{\tau + d} = a - \frac{1}{\tau + d}.$$

It follows from Proposition 4 and Theorem 6 that

$$\begin{aligned}\eta\left(\frac{a\tau + b}{c\tau + d}\right) &= \eta\left(a - \frac{1}{\tau + d}\right)\\ &= \exp\left(\frac{\pi i a}{12}\right)\eta\left(-\frac{1}{\tau + d}\right)\\ &= \exp\left(\frac{\pi i a}{12}\right)\{-i(\tau + d)\}^{1/2}\eta(\tau + d)\\ &= \exp\left(\frac{\pi i(a + d)}{12}\right)\{-i(\tau + d)\}^{1/2}\eta(\tau)\\ &= \exp\left(\frac{\pi i\omega(a, b, c, d)}{12}\right)\{-i(c\tau + d)\}^{1/2}\eta(\tau),\end{aligned}$$

where

$$\omega(a, b, c, d) = a + d$$

is an integer. Since $s(-d, c) = s(-d, 1) = 0$, this proves the statement of the theorem when $c = 1$.

Now we will use principle of strong induction. Let c be an integer larger than or equal to 2. Suppose that for all $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \Gamma$ with $1 \le c' \le c - 1$, the statement of the theorem is proved. Consider $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 1$. Since c and d are relatively prime, there is a unique positive integer r less than c such that $-d \equiv r \bmod c$. In other words, there is an integer q such that

$$d = cq - r.$$

Let

$$u = aq - b.$$

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} u & a \\ r & c \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}.$$

Let γ_1, γ_2, γ_3 be linear fractional transformations defined by

$$\gamma_1(\tau) = \frac{u\tau + a}{r\tau + c}, \quad \gamma_2(\tau) = S(\tau) = -\frac{1}{\tau}, \quad \gamma_3(\tau) = T^q(\tau) = \tau + q.$$

Then

$$\frac{a\tau + b}{c\tau + d} = \gamma_1(\tau') = \frac{u\tau' + a}{r\tau' + c}, \quad \tau' = \gamma_2(\gamma_3(\tau)) = -\frac{1}{\tau + q}.$$

Since $0 < r < c$, we can apply induction hypothesis and obtain

$$\begin{aligned}\eta\left(\frac{a\tau+b}{c\tau+d}\right) &= \eta\left(\frac{u\tau'+a}{r\tau'+c}\right)\\ &= \exp\left(\frac{\pi i\omega(u,a,r,c)}{12}\right)\{-i(r\tau'+c)\}^{1/2}\eta(\tau'),\end{aligned}$$

where

$$\omega(u,a,r,c) = \frac{u+c}{r} + 12s(-c,r)$$

is an integer. From the case $c = 1$, we have

$$\eta(\tau') = \eta\left(-\frac{1}{\tau+q}\right) = \exp\left(\frac{\pi i q}{12}\right)\{-i(\tau+q)\}^{1/2}\eta(\tau).$$

Since

$$(r\tau'+c)(\tau+q) = c(\tau+q) - r = c\tau + d$$

and

$$(-i)^{1/2} = \exp\left(-\frac{\pi i}{4}\right),$$

we find that

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \exp\left(\frac{\pi i\omega(a,b,c,d)}{12}\right)\{-i(c\tau+d)\}^{1/2}\eta(\tau),$$

where

$$\omega(a,b,c,d) = \omega(u,a,r,c) + q - 3 = \frac{u+c+qr}{r} + 12s(-c,r) - 3.$$

From the first equality, we conclude by the inductive hypothesis that $\omega(a,b,c,d)$ is an integer. Now we need to prove that $\omega(a,b,c,d)$ is given by (21). By Lemma 3,

$$s(-c,r) = -s(c,r).$$

By Theorem 7, we find that

$$s(-c,r) = s(r,c) - \frac{r^2+c^2-3rc+1}{12rc}.$$

Since $-d$ is congruent to r modulo c, Lemma 2 implies that

$$s(-c,r) = s(-d,c) - \frac{r^2+c^2-3rc+1}{12rc}.$$

Hence,

$$\omega(a,b,c,d) = \Lambda(a,b,c,d) + 12s(-d,c),$$

where

$$\begin{aligned}\Lambda(a,b,c,d) &= \frac{u+c+qr}{r} - 3 - \frac{r^2+c^2-3rc+1}{rc} \\ &= \frac{uc+cqr-r^2-1}{rc} \\ &= \frac{c(aq-b)-1+dr}{rc} \\ &= \frac{a(cq-d)+dr}{rc} \\ &= \frac{a+d}{c}.\end{aligned}$$

This proves (21). Hence, the theorem is proved. □

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