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Survey on the surfaces generated by generalized analytic functions

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Abstract

In this survey some fundamental results from the theory of generalized analytic functions are presented. Then, based on that theory, we develop the theory of the so-called Bohr-Riemann surfaces.

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Introduction

In the beginning of the last century the works of H. Bohr laid the foundations of the theory of analytic almost periodic functions. However, many natural problems in the theory still remain open. This particularly applies to meromorphic almost periodic functions for which there are just a few remarkable results obtained. Lately an intensively developing theory of differential equations with almost periodic coefficients again raised the issue of studying both analytic and meromorphic almost periodic functions. Quite useful and convenient tools for such studies are the generalized analytic functions which were introduced by R. Arens and I.M. Singer (see [1]). The method suggested by those authors not only allowed to give a new treatment of the well-known theorems from the theory of analytic almost periodic functions (such as the theorems of H. Bohr, A. Besicovitch, B. Levin, et al) but also enabled to take a look at the theory of almost periodic functions and at the theory of analytic functions in the unit disc from a unified point of view. In short, as we know the theory of analytic functions in the unit disc considers the functions that are represented by the series

$f(z) = f(r \cdot e^{i\theta}) = \sum_{n\in\mathbb{Z}_+} a_n z^n = \sum_{n\in\mathbb{Z}_+} a_n r^n \cdot e^{in\theta}$, where $r = |z| \le 1$ and $\theta \in [0, 2\pi)$. Thus, an analytic function is expressed as the power series in the variable $r \cdot e^{i\theta}$ over the positive elements of an additive group of integers $\mathbb{Z}$. Based on the fact that in that representation $e^{i\theta}$ belongs to the unit circle $\mathbb{T}$ of the complex plane and $\mathbb{T}$ is the group of characters of $\mathbb{Z}$ the theory of generalized analytic functions considers the functions which are represented in the form of the series above, where the group $\mathbb{Z}$ is replaced by an arbitrary additive subgroup Γ of the group of real numbers $\mathbb{R}$ and the unit circle $\mathbb{T}$ is replaced by the group of characters of Γ. Clearly, if Γ is isomorphic to $\mathbb{Z}$ then the obtained theory is identical to the classical function theory of one complex variable but in the case when Γ is a dense subgroup of $\mathbb{R}$ the theory of generalized analytic functions which will be developed in this work significantly differs from its classical prototype. However, the search of new features while applying the classical scenarios continues to be one of the most important aims of the theory of generalized analytic functions. In this work we develop one of the applications of the theory of generalized analytic functions to the classical theory of Riemann surfaces.

The work consists of three parts. In the first part the notions of a generalized plane and generalized analytic functions are introduced. Then the structure of a generalized plane is described in detail and the behaviour of generalized analytic functions are studied. In the second part generalized meromorphic functions are observed and, among other results, the theorem about the factorization of generalized meromorphic functions is proved. Note that the results of the first two chapters are mainly taken from the works [2], [3] and [4]. The last part is devoted to the investigation of Bohr-Riemann surfaces. Particularly, the group structures as well as the local geometric structures of the Bohr-Riemann surfaces studied.

1 Generalized analytic functions

1.1 Basic concepts and auxiliary results

In this section we present some definitions and results from the theory of uniform algebras and from the theory of analytic functions which will be used later in the work. They are mainly taken from the notable books [5], [6] and [7].

A Banach algebra is an algebra A over the field of complex numbers $\mathbb{C}$ which is a Banach space with $||fg|| \le ||f||||g||$ for $f, g \in A$. An example of a Banach algebra is the algebra $C(X)$, i.e., the algebra of continuous functions on a compact set X with the sup–norm. A closed algebra $A \subset C(X)$ is called uniform algebra if it contains all constant functions and separates the points of a compact set X. The classical non–trivial example of a uniform algebra is a disk algebra, i.e., the algebra of all continuous functions on the closed unit disc which are analytic on the interior of unit disc. This example allows to demonstrate many of the results from the theory of uniform algebras.

Each point $x \in X$ determines a multiplicative functional on A which is given as the value at x. The family of all multiplicative functionals on A is denoted by M_A. Obviously, $X \subset M_A$. The kernel of a multiplicative functional is a maximal ideal, and, hence, a closed ideal of an algebra A. Therefore, each multiplicative functional is continuous. Moreover, it can be shown that M_A is contained in the unit ball of the dual space of A and is compact in a weak* topology.

Let I be a maximal ideal of A. By Gelfand–Mazur theorem the quotient algebra A/I is isomorphic to the field of complex numbers $\mathbb{C}$. This result allows to uniquely recover a multiplicative functional on A via maximal ideal. That is why M_A is also called the space of maximal ideals of A.

A homomorphism $A \to C(M_A)$, which sends each $f \in A$ to a function $\tilde{f} \in C(M_A)$ defined as $\tilde{f}(m) = m(f), m \in M_A$, is called the Gelfand transform. The obtained algebra $\widetilde{A}$, which consists of all such functions $\tilde{f}$ with $f \in A$, is a uniform algebra which is isometrically isomorphic to the algebra A.

A subset E of M_A is called a boundary of A if each function from $\widetilde{A}$ attains its maximum modulus on E. By Shilov's theorem the intersection of all closed boundaries of an algebra A is also a boundary, the so called Shilov boundary, which is denoted by ∂A.

A point $x \in X$ is called a peak point for A if there is a function $f \in A$ such that $f(x) = 1$ and $|f(y)| < 1, y \neq x$. Respectively, a closed subset E of X is called a peak set if there is a function $f \in A$ such that $f(x) = 1$ for $x \in E$ and $|f(y)| < 1$ for $y \in X \setminus E$.

If X is a compact metrizable space then the intersection of all boundaries of an algebra A is also a boundary (the Choquet boundary). That set coincides with $P(A)$, the set of peak points of an algebra A. The isometricity of the Gelfand transform implies that $P(A) \subset \partial A \subset X$.

In case X is not metrizable we consider generalized peak points (p–points), which are obtained from the intersection of some family of peak sets. The intersection of peak sets is called a generalized peak set or p–set. If $F \subset X$ is a p–set for A then the restriction $A|_F$ of an algebra A to F is a uniform algebra on F.

An important class of p–sets is formed by the maximal sets of antisymmetry of an algebra A. A uniform algebra A on a compact set X is called antisymmetric if every real–valued function in A is constant. A set $F \subset X$ is called a set of antisymmetry of A if every function in A that is real–valued on F is constant on F. Compact set X can be represented as disjoint union of maximal sets of antisymmetry $\{F_\alpha\}$ where, by Bishop–Shilov theorem, each F_α is a p–set and if $f \in C(X)$ and $f|_{F_\alpha} \in A|_{F_\alpha}$ then $f \in A$.

Let $A^\perp$ be a space of regular Borel measures on X which are orthogonal to A and let $S(A)$ be the unit ball in $A^\perp$. By Bishop theorem maximal sets of antisymmetry are the supports of extremal measures from $S(A)$. The peak sets are described via orthogonal measures as follows: a set $F \subset X$ is a peak set for an algebra A if and only if the restriction of each measure $\mu \in A^\perp$ to the set F belongs to $A^\perp$.

Each continuous functional on $C(X)$ corresponds to a regular Borel measure which represents that functional. Using Hahn–Banach theorem each multiplicative functional from M_A can be extended to a state on $C(X)$. Therefore, there exist probability measures among representing measures of a multiplicative functional. More precisely, by Bishop–de Leeuw theorem for each multiplicative functional there exists a representing probability measure concentrated on $P(A)$.

From now on, by "representing measure" we shall always mean representing probability measure.

A family M_m of representing measures of a multiplicative functional $m \in M_A$ is a convex compact set. The examples of algebras for which M_m consists of a single point are the Dirichlet algebras, the algebras whose real parts are dense in the algebra of bounded continuous real functions on X. A measure $\mu \in M_m$ is called a Jensen measure if $\log|m(f)| \leq \int \log|f|d\mu$, $f \in A$. By Bishop theorem for each $m \in M_A$ there exists at least one representing Jensen measure and, moreover, by Arens theorem, there is a Jensen measure $\mu \in M_m$ concentrated on ∂A such that

$$\int \log|f|d\gamma \leq \int \log|f|d\mu,\ \gamma \in M_A, f \in A.$$

The last property of the space of maximal ideals we present here concerns the group A^{-1} of invertible elements of a uniform algebra A. By Arens–Royden theorem $A^{-1}/\exp A = H^1(M_A,\mathbb{Z})$, where $H^1(M_A,\mathbb{Z})$ is the first integral cohomology group of M_A. Recall that a character of a group G is a continuous group homomorphism from G to the unit circle of the complex plane. In case G is a connected compact abelian group by Bohr–van Kampen theorem we have that $C(G)^{-1}/\exp C(G) = \hat{G}$, where $\hat{G}$ is the group of characters of the group G.

1.2 Topologies on the generalized plane

In this section we define the notion of a generalized plane. Then the topologies arising on the generalized plane and its subsets are considered and their comparisons are investigated.

Let Γ be a subgroup of an additive group of real numbers $\mathbb{R}$ and let G be the group of characters of Γ: $G = \hat{\Gamma}$. By Pontryagin duality theorem we have that the group of characters of a group G is isomorphic to Γ: $\hat{G} \cong \Gamma$. Using G we define a Cartesian product $G \times [0,\infty)$ and glue to the point the bottom layer $G \times \{0\}$. The obtained space is called generalized plane and is denoted by $\mathbb{C}(\Gamma)$. This construction is due to Arens and Singer (see [1]).

As Γ is an additive subgroup of $\mathbb{R}$ then by the well–known dichotomy we have that there are two possible cases: either Γ is isomorphic to the group of integers $\mathbb{Z}$ or Γ is dense in $\mathbb{R}$ in the Euclidean topology τ. In the first case the constructed space $\mathbb{C}(\Gamma)$ and the theory of generalized analytic functions which we would like to develop on $\mathbb{C}(\Gamma)$ are identical to their classical prototype, that is, to the function theory of one complex variable on the complex plane $\mathbb{C}$. That is why in what follows we assume, unless stated otherwise, that the group

Γ is dense in $\mathbb{R}$ in the Euclidean topology τ. There are many examples of such subgroups, e.g., $\Gamma = \mathbb{Q}$, an additive group of rational numbers, $\Gamma = \Gamma^\alpha = \{m + \alpha n, m, n \in \mathbb{Z}\}$, where α is a positive irrational number, or $\Gamma = \mathbb{R}_d$, the group of real numbers with the discrete topology, but we will only use the density of Γ in $\mathbb{R}$ and hence will not be interested in the precise structure of Γ.

For brevity of notation, we shall also denote the generalized plane $\mathbb{C}(\Gamma)$ by Δ.

Let $\pi : G \times [0, \infty) \to \Delta$ be a canonical projection. Then the elements of Δ are the points $\pi(\alpha, r) = (\alpha, r)$, with $\alpha \in G, r > 0$, and $* = \pi(G \times \{0\})$. The space Δ can be also canonically identified with the space $\mathcal{C} = \{\alpha r : \alpha \in G, r \in [0, \infty)\}$ – the analogue of the complex plane $\mathbb{C}$ which consists of the homomorphisms $\alpha r : \Gamma \to \mathbb{C} : a \mapsto \alpha(a) r^a$. Usually it is more convenient to take $\Delta = \mathcal{C}$, in which case the representation $s = \alpha r$ of an element $s \in \Delta$ is called a polar decomposition and the number r is called a modulus of s. As the null element $*$ essentially differs from the other elements of the space Δ, it makes sense to define a space $\Delta^0 = \Delta \setminus \{*\}$, the so called punctured generalized plane. Obviously, $\Delta^0 = G \times (0, \infty)$ and Δ^0 can be canonically identified with the space $\{\alpha r : \alpha \in G, r \in (0, \infty)\}$.

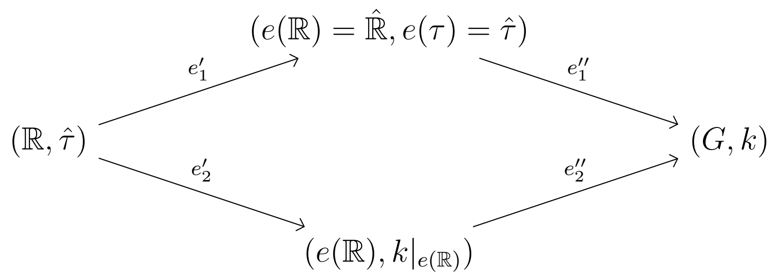
Let us now pass to the topologies which arise on Δ. Let $\{T\}$ be some basis of open sets of the unit circle $\mathbb{T}$ of the complex plane $\mathbb{C}$ and let $\mathcal{F}$ be a collection of all finite subsets of Γ. Define $P(F, T) = \{\chi \in G |\ \chi(F) \subseteq T\}$. The family $\{P(F, T), F \in \mathcal{F}, T \in \{T\}\}$, is a basis of some topology in G, which will be denoted by k. Then the topology on Δ would be the standard factor topology $\tau_\Delta = \{U \subset \Delta : \pi^{-1}(U) \in k \times \tau_{[0,\infty)}\}$, where $\tau_{[0,\infty)}$ is a restriction of the Euclidean topology τ to $[0, \infty)$. As a basis of the topology τ_Δ could be taken the family of sets $\mathcal{B} = \{\pi(G \times [0, r))\}_{r>0} \cup \pi(\textit{any basis in } G \times (0, \infty))$, where the first component in this union is a basis of open neighbourhoods of the element $* \in \Delta$. Similarly, we define the topology $\tau_{\Delta^0} \cong k \times \tau_{(0,+\infty)}$ on Δ^0. Canonical projection π is not open (as the topology k is not trivial), but it is a closed mapping which induces a homeomorphism $\pi|_{G\times(0,\infty)} : (G \times (0, \infty), k \times \tau_{(0,+\infty)}) \to (\Delta^0, \tau_{\Delta^0})$. The space Δ is then a locally compact Hausdorff space.

Let us now consider the mapping $e : \mathbb{R} \to G : e(t) = e_t$, where $e_t(a) = e^{iat}, a \in \Gamma$. The density of Γ in $\mathbb{R}$ implies that e is injective. Indeed, if $e_{t_1} = e_{t_2}$ with $t_1, t_2 \in \mathbb{R}, t_1 \neq t_2$, then $e^{iat_1} = e^{iat_2}$ for all $a \in \Gamma$. As Γ is dense in $\mathbb{R}$ and $e_{t_i}, i = 1, 2$, are both continuous on $(\mathbb{R}, \tau)$, we get that $e^{iat_1} = e^{iat_2}$ for all $a \in \mathbb{R}$, and, therefore, $t_1 = t_2$. This argumentation can be also used as a justification of an equality $e(\mathbb{R}) = \hat{\mathbb{R}}$ of two groups of characters with different domains (Γ and $\mathbb{R}$, respectively). In other words, the equality $e(\mathbb{R}) = \hat{\mathbb{R}}$ assigns to an element $e_t \in e(\mathbb{R}), t \in \mathbb{R}$, the element from $\hat{\mathbb{R}}$ which (by Pontryagin duality) corresponds to the number $t \in \mathbb{R}$. The proof of density of $e(\mathbb{R})$ in G is similar and is based on the fact that $e(\mathbb{R})$ separates the points of a group Γ (see [8], p. 55).

The space $\Delta^0 = G \times (0, \infty)$ which has been canonically identified with the space $\{\alpha r : \alpha \in G, r \in (0, \infty)\}$ is a locally compact abelian group under the coordinate-wise multiplication with the unit element $e_0 \cdot 1 = e(0)$.

There are two topologies arising on $e(\mathbb{R})$: the restriction $k|_{e(\mathbb{R})}$ of a finite-open topology k on G and the topology $\hat{\tau}$ which arises as a compact-open topology on $e(\mathbb{R}) = \hat{\mathbb{R}}$. Since each finite set is compact we get that the topology $\hat{\tau}$ is stronger than $k|_{e(\mathbb{R})}$. As a basis of neighbourhoods at the unit element $e_0 \in e(\mathbb{R})$ which defines $\hat{\tau}$ can be taken the family $\{P_\varepsilon\}_{\varepsilon\in(0,\pi)}$ of the sets $P_\varepsilon = \{e_t : e_t([-1,1]) \subset V_\varepsilon\}$, where $V_\varepsilon = \{\xi \in \mathbb{T} : \xi = e^{i\theta}, \theta \in (-\varepsilon, \varepsilon)\}$. Clearly, $P_\varepsilon = e((-\varepsilon, \varepsilon))$ and, therefore, $\hat{\tau} = e(\tau)$, the homeomorphic image of the Euclidean topology τ on $\mathbb{R}$.

The obtained topologies on $e(\mathbb{R})$ determine two different factorizations of the mapping $e : \mathbb{R} \to G$ which are presented in the following diagram:



In that diagram e'_1 is a homeomorphism, e'_2 is a continuous homomorphism and the insertion $e''_1 : e_t \mapsto e_t|\Gamma$, as well as the embedding e''_2, is continuous.

The group $e(\mathbb{R})$ is a path-connected group in both topologies as the image of a path-connected space under the continuous mappings e'_1 and e'_2. Moreover, we claim that these path-connectedness are equivalent. Clearly, a path-connectedness of $e(\mathbb{R})$ with respect to the topology $\hat{\tau}$ implies the path-connectedness with respect to a weaker topology $k|_{e(\mathbb{R})}$. Let us now prove the converse statement.

Lemma 1 *Each path $\sigma : I = [0,1] \to e(\mathbb{R})$ that is continuous with respect to the topology $k|_{e(\mathbb{R})}$ is also continuous with respect to the topology $\hat{\tau} = e(\tau)$.*

Proof. Using the fact that the family $\{P_\varepsilon\}_{\varepsilon\in(0,\pi)}$ is a basis of neighbourhoods of e_0 in the topology $\hat{\tau}$ we first prove the continuity of a path σ at the point $t_0 \in I$ with $\sigma(t_0) = e_0$. Let us fix any $\varepsilon \in (0, \pi)$ and consider the corresponding $P_\varepsilon = \{e_t : e_t([-1,1]) \subset V_\varepsilon\} = e((-\varepsilon, \varepsilon))$ which is a neighbourhood of e_0 in the topology $\hat{\tau}$. The set $Q_\varepsilon = \{e_t : e_t(1) \subset V_\varepsilon\} = \bigcup_{n\in\mathbb{Z}} e(I_n)$ is then a neighbourhood of e_0 in the topology $k|_{e(\mathbb{R})}$, where $I_n = (2\pi n - \varepsilon, 2\pi n + \varepsilon), n \in \mathbb{Z}$. Obviously, $P_\varepsilon \subset Q_\varepsilon$.

We have that σ is continuous at t_0 with respect to the topology $k|_{e(\mathbb{R})}$, therefore there exists $\delta > 0$ such that $\sigma(I_\delta) \subset Q_\varepsilon$, where $I_\delta = (t_0 - \delta, t_0 + \delta) \cap I$. We want to show that $\sigma(I_{\delta/2}) \subset P_\varepsilon$.

Indeed, the continuum $\sigma(\overline{I}_{\delta/2})$, which is contained in $\sigma(I_\delta)$, is covered by the set $\tilde{Q}_\varepsilon = \{e_t : e_t(1) \subset \overline{V}_\varepsilon\} = \bigcup_{n\in\mathbb{Z}} e(\overline{I}_n)$ which is a countable union of compact and, therefore, closed sets $e(\overline{I}_n)$. By Sierpinski's theorem (see [9]. p. 526), at most one of these sets is non-empty. As the set $\sigma(\overline{I}_{\delta/2})$ certainly intersects with $e(\overline{I}_0)$, then $e(\overline{I}_0)$ is the mentioned unique non-empty set. Hence, $\sigma(I_{\delta/2}) \subset e(\overline{I}_0) \cap Q_\varepsilon = e(I_0) = P_\varepsilon$, as desired.

The general case is reduced to the considered situation via transitions from σ to $\tilde{\sigma}(t) = \sigma(t_0)^{-1}\sigma(t)$ and back again, using the fact that the shifts by $\sigma(t_0)^{-1}$ and $\sigma(t_0)$ are topological automorphisms of $e(\mathbb{R})$ in both topologies $k|_{e(\mathbb{R})}$ and $\hat{\tau}$.

Lemma is proved. □

The mapping $e : \mathbb{R} \to G$ generates an embedding

$$\varphi : \mathbb{C} \to \Delta^0 : z = t + iy \mapsto \varphi_z = e_t \cdot e^{-y}.$$

The transition from e to φ complexifies the above diagram keeping the properties of the mappings in it.

A topology $\tau_{\Delta^0}|_{\varphi(\mathbb{C})}$ which is induced on $\varphi(\mathbb{C})$ from a topology $\tau_{\Delta^0} \cong k \times \tau_{(0,+\infty)}$ on Δ^0 is weaker than the topology $\tau_\varphi = \varphi(\tau_c)$, the homeomorphic image of the Euclidean topology τ_c on $\mathbb{C}$. The topology $\varphi(\tau_c)$ has also two other equivalent descriptions: it emerges as a product of topologies $\hat{\tau} \times \tau_{(0,\infty)}$ with $\{P_\varepsilon \times (e^{-\delta}, e^{\delta})\}_{\varepsilon\in(0,\pi),\delta>0}$ being the basis of neighbourhoods of the unit element $(e_0, 1) \cong \varphi(0)$ and as a compact–open topology on $\varphi(\mathbb{C})$ with the basis of neighbourhoods of the unit element formed by the sets $P_{\varepsilon,\delta} = \{\varphi_z : \varphi_z([-1, 1]) \subset V_{\varepsilon,\delta}\} = \varphi(K_{\varepsilon,\delta}), \varepsilon \in (0, \pi), \delta > 0$, where $V_{\varepsilon,\delta} = \{w = \rho e^{i\theta} : e^{-\delta} < \rho < e^{\delta}, e^{i\theta} \in V_\varepsilon\}$, and the sets $K_{\varepsilon,\delta} = \{z = t + iy : |t| < \varepsilon, |y| < \delta\}$ obviously form the basis of neighbourhoods of the zero element $z = 0$ of a group $\mathbb{C}$.

Note that since $e(\mathbb{R})$ is dense in G the image $\varphi(\mathbb{C})$ is dense in both Δ^0 and Δ.

Definition 1 *For a point $s \in \Delta^0$ the set of the form $\mathbb{C}_s = s\varphi(\mathbb{C})$ is called a plane in Δ^0 passing through s.*

Obviously, $\mathbb{C}_s$ is dense in Δ^0 for any $s \in \Delta^0$. We also define $\mathbb{C}_0 := \mathbb{C}_{\varphi(0)} = \varphi(\mathbb{C})$. Define a mapping $\varphi_s : \mathbb{C} \to \mathbb{C}_s : z \mapsto s\varphi_z$. Again there are two topologies on each plane $\mathbb{C}_s$: the topology $\tau_s := \tau_{\Delta^0}|_{\mathbb{C}_s}$ which is induced from Δ^0 and the stronger topology $\tau_{s\varphi} = s\tau_\varphi = \{sU : U \in \tau_\varphi\}$ which is inherited from $\mathbb{C}$ by the mapping φ_s.

The theory of Bohr-Riemann surfaces which will be developed in the third chapter considers the so called *thin* sets K in Δ and investigates the finite sheeted coverings of the space $\Delta^* = \Delta^0 \setminus K$. So we now pass to the situation which often arises in that theory.

Let $s \in \Delta^0$ and let K be a closed nowhere dense subset of Δ^0 such that the intersection $K \cap \mathbb{C}_s$ is a discrete set. Define $\Delta^* = \Delta^0 \setminus K$ and $\mathbb{C}_s^* = \mathbb{C}_s \cap \Delta^* = \mathbb{C}_s \setminus K$. Let us consider the preimage $\pi^{-1}(\mathbb{C}_s^*)$ under unfolded, finite-sheeted covering $\pi : X \to \Delta^*$, where X is a topological space. There are two topologies that arise on $\pi^{-1}(\mathbb{C}_s^*)$: the topology $\tau_{s,X}$ which is induced from a topology τ_X of the space X and is locally homeomorphic to $\tau_s^* = \tau_s|_{\mathbb{C}_s^*}$, and the topology $\tau_{s,\mathbb{C}}$ base of which is consisted of the path–connected components of the sets from $\tau_{s,X}$. Thus, using the characterization of $\tau_{s\varphi}$ as a topology with a base consisted of the path–connected components of the sets from τ_s we get that the restriction $\pi|_{\pi^{-1}(\mathbb{C}_s^*)}$ induces two coverings of the punctured plane $\mathbb{C}_s^*$: $\pi_{s,X} : \tau_{s,X} \to \tau_s^*$ and $\pi_{s,\mathbb{C}} : \tau_{s,\mathbb{C}} \to \tau_{s\varphi}^*$.

Theorem 1 *The path–connected components of a subspace $\pi^{-1}(\mathbb{C}_s^*)$ in the topology $\tau_{s,X}$ coincide with the path–connected components of a Riemann surface $\pi^{-1}(\mathbb{C}_s^*)$ in $\tau_{s,\mathbb{C}}$.*

Proof. Let $x \in \pi^{-1}(\mathbb{C}_s^*)$ and let C_x and D_x be the path-connected components of a pre–image $\pi^{-1}(\mathbb{C}_s^*)$ containing x in the topologies $\tau_{s,\mathbb{C}}$ and $\tau_{s,X}$ respectively. Since $\tau_{s,\mathbb{C}}$ is stronger than $\tau_{s,X}$ it follows that $C_x \subset D_x$. Let us proof the converse inclusion. Fix an arbitrary point $y \in D_x$ and connect it with x by a path $\gamma : I = [0,1] \to X$ which lies in D_x and which is continuous with respect to the topology $\tau_{s,X}$. Then $\lambda = \pi \circ \gamma : I \to \mathbb{C}_s^*$ is a continuous path from $\tau|_I$ to τ_s^*. Temporarily forgetting about the stars we get that λ is a continuous path from $\tau|_I$ to τ_s, and, therefore, $s^{-1}\lambda : I \to \mathbb{C}_0$ is a continuous path from $\tau|_I$ to $\tau_0 = \tau_{\Delta^0}|_{\mathbb{C}_0} \cong k|_{e(\mathbb{R})} \times \tau_{(0,+\infty)}$. Using the interpretation of a space Δ^0 as a Cartesian product $G \times (0,\infty)$ we get that the mapping $s^{-1}\lambda : I \to \mathbb{C}_0$ is comprised of the pair of mappings $s^{-1}\lambda(t) = (\beta(t), r(t)), t \in I$, with $\beta : I \to e(\mathbb{R})$ and $r : I \to (0,+\infty)$. But then the mapping $s^{-1}\lambda$ is continuous if and only if β is a continuous mapping from $\tau|_I$ to $k|_{e(\mathbb{R})}$ and r is a continuous mapping from $\tau|_I$ to $\tau_{(0,+\infty)}$ (see e.g. [9]. pp. 129,131). By Lemma 1 we get that the path $\beta : I \to e(\mathbb{R})$ is then continuous with respect to the topology $\hat{\tau} = e(\tau)$ as well. This, together with the arguments above, shows that the mapping $s^{-1}\lambda(t) = (\beta(t), r(t)), t \in I$, is continuous with respect to the topology $\hat{\tau} \times \tau_{(0,+\infty)} \cong \tau_\varphi$, i.e. the path λ is a continuous path from $\tau|_I$ to $\tau_{s\varphi}$, and, therefore, to $\tau_{s\varphi}^*$ as well. The continuity of the path γ with respect to the topology $\tau_{s,\mathbb{C}}$ is obtained from the local homeomorphity of π as a covering $\pi_{s,\mathbb{C}} : \pi^{-1}(\mathbb{C}_s^*) \to \mathbb{C}_s^*$ from $\tau_{s,\mathbb{C}}$ to $\tau_{s\varphi}^*$. Thus, $y \in C_x$. This completes the proof of the theorem. □

1.3 Generalized analytic functions

Let us remind that given a dense subgroup Γ of an additive group of real numbers $\mathbb{R}$ the generalized plane $\Delta = \mathbb{C}(\Gamma)$ is obtained from the Cartesian product $G \times [0,\infty)$ by gluing to the point the layer $G \times \{0\}$ where $G = \hat{\Gamma}$ is a group of characters of Γ. By Pontryagin duality theorem we have that $\hat{G} \cong \Gamma$. Let $\chi^a \in \hat{G}$ be a character corresponding to an element $a \in \Gamma$. Obviously, $\chi^a \cdot \chi^b = \chi^{a+b}$. Also for a character $\chi^a, a \in \Gamma$, the conjugate character $\overline{\chi}^a$ is defined as $\overline{\chi}^a(\alpha) = \overline{\chi^a(\alpha)}, \alpha \in G$. Each function $f \in L^1(d\sigma)$, where σ is a normalized Haar measure of a group G, has the following Fourier series representation

$$f \sim \sum_{a\in\Gamma} c_a(f)\chi^a, \quad \text{where } c_a(f) = \int_G f \cdot \overline{\chi}^a \, d\sigma.$$

The set of all $a \in \Gamma$ such that $c_a(f) \neq 0$ is called the spectrum of f and is denoted by $S(f)$. Similarly, the spectrum of a regular Borel measure μ on G is the set of all $a \in \Gamma$

such that $\int_G \overline{\chi}^a \, d\mu \neq 0$. Define $\Gamma_+ = \{a \in \Gamma : a \geq 0\}$. A function $f \in L^1(d\sigma)$ is called generalized analytic or just analytic function if $S(f) \subset \Gamma_+$ and the measure μ is called analytic if $S(\mu) \subset \Gamma_+$. With the sup-norm the space A of all continuous analytic functions on G is a uniform Dirichlet algebra. The space of maximal ideals of an algebra A coincides with the generalized unit disc $\Omega = \{s \in \Delta, |s| \leq 1\}$ which is the group of semi-characters of a semigroup Γ_+. Recall that a semi-character on a semigroup is a non-zero continuous homomorphism from a semigroup to the unit disc of the complex plane.

Each character $\chi^a, a \in \Gamma_+$, can be extended to a continuous function φ^a on Δ which acts as follows:

$$\varphi^a(s) = \chi^a(\alpha) r^a, \; s = \alpha r,$$

where $\chi^a(\alpha) = \alpha(a), \alpha \in G$ (we assume $0^a = 0, a \in \Gamma_+$). The family $\{\varphi^a\}_{a \in \Gamma_+}$ thus obtained separates the points of a space Δ.

Definition 2 *Let D be an open set in Δ. Continuous function f on D is called a generalized analytic function if for any $s \in D$ there is a neighbourhood $U \subset D$, $s \in U$, such that the restriction of f to U can be uniformly approximated by linear combinations of the functions $\varphi^a, a \in \Gamma_+$.*

The set of all generalized analytic functions on D will be denoted by $\mathcal{O}(D)$. A function from $\mathcal{O}(\Delta)$ will be called entire function. Finite linear combinations of the functions from $\{\varphi^a\}_{a \in \Gamma_+}$ will be called polynomials and the ratio of two polynomials will be called a rational function.

Let K be a compact set in Δ and let $\mathcal{P}$ be the family of all polynomials. Then the polynomially convex hull of K is defined as follows:

$$K^0 = \{s \in \Delta, |p(s)| \leq \sup_K |p|, p \in \mathcal{P}\}.$$

The set

$$K_0 = \{s \in \Delta, |p_1(s)/p_2(s)| \leq \sup_K |p_1/p_2|, p_1, p_2 \in \mathcal{P}\},$$

where p_2 does not vanish on K, is called a rationally convex hull of K. Obviously, $K \subset K_0 \subset K^0$.

Proposition 1 *Let K be a compact set in Δ. Then*

1. *K^0 is the space of maximal ideals of the uniform algebra $P(K)$ generated by polynomials on K;*

2. *K_0 is the space of maximal ideals of the uniform algebra $R(K)$ generated by rational functions of the form p_1/p_2, where p_2 does not vanish on K.*

Proof. If $s_0 \in K^0$ then the point–evaluation at s_0 is a multiplicative functional on $P(K)$, and, therefore, $s_0 \in M_{P(K)}$, where $M_{P(K)}$ is the space of maximal ideals of $P(K)$. Conversely, as the norm of each multiplicative functional $m \in M_{P(K)}$ (which is a point–evaluation) is 1 then $M_{P(K)}$ is contained in K^0.

Second statement is proved similarly. □

The question naturally arises as whether a compact set K is always rationally convex. It is known that if $\Gamma = \mathbb{Z}$ then for any compact set $K \subset \mathbb{C}$ we have $K = K_0$. However, if $\Gamma = \Gamma^\alpha = \{m + \alpha n, m, n \in \mathbb{Z}\}$, where α is a positive irrational number, then it is possible to construct a compact set in Δ which is not rationally convex. Nevertheless, the following proposition holds.

Proposition 2 *Suppose that Γ is isomorphic to some subgroup of the group of rational numbers. Then each compact set in Δ is rationally convex.*

Proof. Let K be a compact set in Δ and suppose $s_0 \in \Delta \setminus K$. Choose $a_1, ..., a_n \in \Gamma_+$ and $\varepsilon > 0$ such that an intersection of the set $U = \{s \in \Delta, |\varphi^{a_i}(s) - \varphi^{a_i}(s_0)| < \varepsilon, i = 1, ..., n\}$ with K is empty. Since Γ is isomorphic to a subgroup of the group of rational numbers then there exists an element $a_0 \in \Gamma_+$ such that $a_i = m_i a_0, m_i \in \mathbb{Z}_+, i = 1, ..., n$. Therefore, for sufficiently small $\varepsilon' > 0$, the set $V = \{s \in \Delta, |\varphi^{a_0}(s) - \varphi^{a_0}(s_0)| < \varepsilon'\}$ is contained in U. Hence, the function $1/(\varphi^{a_0}(s) - \varphi^{a_0}(s_0))$ belongs to $R(K)$ which means that $s_0 \notin K_0$. Thus, $K = K_0$. □

We now describe the local structure of the algebras of generalized analytic functions. Let us remind that we assume, unless stated otherwise, that the group Γ is dense in $\mathbb{R}$. Also recall that for each $t \in \mathbb{R}$ a character $e_t \in G$ of a group Γ acts as $e_t(a) = e^{iat}, a \in \Gamma$, and the family $\{e_t\}_{t \in \mathbb{R}}$ is dense in G. Let $W \subset \mathbb{C}$ be a compact set and let $P(W)$ be the uniform algebra on W generated by polynomials. For a fixed $a \in \Gamma_+, a \neq 0$, the set $G_a = \{\alpha \in G, \alpha(a) = 1\}$ is a subgroup of G.

Theorem 2 *Each point from $\Delta^0 = \Delta \setminus \{*\}$ has a closed neighbourhood F which is homeomorphic to the Cartesian product $G_a \times W$ such that the uniform algebra $P(F)$ is isometrically isomorphic to the uniform algebra on $G_a \times W$ generated by the functions of the form $f \cdot g$, where $f \in P(W)$, $g \in C(G_a)$.*

Proof. Let $\omega = \{e^{i\theta} : |\theta| \leq \pi/2\}$ be an arc on the unit circle. Fix $a \in \Gamma_+, a \neq 0$ and consider the sets $K = \{\alpha \in G, \alpha(a) \in \omega\}$ and $l = \{t \in \mathbb{R}, |t| \leq \pi/2a\}$. Define a mapping $h : G_a \times l \to K$ by $h(\alpha, t) = \alpha \cdot e_t$. Let us show that h is a homeomorphism. Suppose $\beta \in K$ and $\beta(a) = e^{i\theta}, |\theta| \leq \pi/2$. Then $t = -\theta/a \in l$ and $\alpha = e_t \cdot \beta \in G_a$ which means that $\beta = h(\alpha, t) \in h(G_a \times l)$. Therefore, we can assume that $K = G_a \times l$. Now consider the neighbourhood $F = K \times [r_1, r_2], r_1 < r < r_2$ of a point $e_0 \cdot r \in \Delta^0$. Obviously, $F = G_a \times W$

where $W = l \times [r_1, r_2] \cong \{z \in \mathbb{C}, \mathrm{Re} z \in l, \mathrm{Im} z \in [-\log r_2; -\log r_1]\}$. Since $\alpha(a) = 1$ for all $\alpha \in G_a$, the function $\varphi^a(\alpha, z) = \alpha(a) \cdot e_t(a) \cdot e^{-ay} = e^{iaz}$, where $z = t + iy$, does not depend on α, is an analytic function of z and separates the points of the polynomially convex compact set W. Therefore, by Mergelyan's theorem each continuous function on W that is analytic on $\mathrm{int}\, W$ can be approximated by the polynomials on e^{iaz}. Thus, $P(W)$ can be embedded in $P(F)$. Let us show that $C(G_a)$ can be also embedded in $P(F)$. Indeed, since $\varphi^b(\alpha, z) = \alpha(b)e^{ibz}, (\alpha, z) \in G_a \times W$, and the functions $f(z) = e^{ibz}$ and $f(\alpha, z) = e^{ibz}$ are invertible in $P(W)$ and in $P(F)$, respectively, then the function $g_b(\alpha, z) = \alpha(b)$ belongs to $P(F)$. Since F is the set of maximal ideals of $P(F)$ (see Proposition 1) and the family $\{g_b\}_{b \in \Gamma_+}$ separates the points of a group G_a, by Stone-Weierstrass theorem we get that $C(G_a) \subset P(F)$. Now suppose that $g \in P(F)$. Since g can be locally approximated by the linear combinations of $\varphi^b, b \in \Gamma_+$, then for every fixed $\alpha \in G_a$ the function $g(\alpha, z)$ belongs to $P(W)$. Therefore, by Bishop-Shilov theorem (see Section 1.1) $g(\alpha, z)$ belongs to the uniform algebra $\overline{P(W) \cdot C(G_a)}$. $\square$

Thus, the space Δ^0 locally has a structure of the form $G_a \times W$, i.e., each point $s \in \Delta^0$ has a neighbourhood of the form $V \times W$, $V \subset G_a, W \subset \mathbb{C}$.

Definition 3 *An open bounded set D in Δ is called a set of uniqueness if each function $f \in \mathcal{O}(D)$ vanishes on some open subset of D.*

We finish this section with the following theorem.

Theorem 3 *(see [10]) An open bounded set D in Δ is a set of uniqueness if and only if $* \in D$.*

1.4 Endomorphisms of the algebra A

The normalized Haar measure σ of a group G can be naturally extended to a measure σ on $T_r = G \times \{r\}, r > 0$. Denote $A = P(\Omega)$, the uniform algebra on Ω generated by polynomials, where $\Omega = \{s \in \Delta, |s| \leq 1\}$ is a generalized unit disc. Each function $f \in A$ has a formal series

$$f \sim \sum_{a \in \Gamma_+} c_a(f)\varphi^a, \tag{1}$$

where $c_a(f) = \int_{T_r} f/\varphi^a d\sigma$ does not depend on the choice of $r > 0$. The converse is also true: a function $f \in C(\Omega)$ which has a formal series of the form (1) belongs to A (see [11]).

Denote $\Omega(r_1, r_2) = G \times [r_1, r_2]$ and $B = R(\Omega(r_1, r_2))$, the uniform algebra on Ω generated by rational functions, where $0 < r_1 < r_2 \leq 1$. Each function $f \in B$ also has a formal series

$$f \sim \sum_{a \in \Gamma} c_a(f)\varphi^a$$

on $\Omega(r_1, r_2)$, where $\varphi^{-a} = 1/\varphi^a, a \in \Gamma_+$.

Lemma 2 *Suppose that a function $f \in B$ does not vanish on $\Omega(r_1, r_2)$. Then there exist $a \in \Gamma$ and $g \in B$ such that $f = \varphi^a \exp g$.*

Proof. From Proposition 1 we have that $\Omega(r_1, r_2)$ is the space of maximal ideals of the algebra B. Therefore, by Arens-Royden's theorem $H^1(\Omega(r_1, r_2); \mathbb{Z}) = B^{-1}/\exp B$ (see Section 1.1). But $H^1(\Omega(r_1, r_2); \mathbb{Z}) = H^1(G; \mathbb{Z})$. Hence, by Bohr-van Kampen's theorem $H^1(\Omega(r_1, r_2); \mathbb{Z}) = \Gamma$. $\square$

The next lemma is a variant of the Phragmén–Lindelöf principle for generalized analytic functions.

Lemma 3 *Let f be a generalized analytic function on the set $\Omega^0 \setminus \{*\}$, where $\Omega^0 = \{s \in \Omega, |s| < 1\}$. Suppose that $\operatorname{Re} f(s) < c \log |s|$ for all $s \in \Omega^0 \setminus \{*\}$. Then f can be extended to an analytic function from $\mathcal{O}(\Omega^0)$.*

Proof. Note that the statement of the Lemma is obvious if f is an analytic function of one complex variable. Indeed, let us show that if a function $f(z)$ is analytic in a punctured unit disc and satisfies the conditions of the Lemma then zero is a removable singularity. We have that

$$\exp f(z) < |z|^c < 1/|z|^n \text{ for } n > |c|.$$

Therefore, the function $g(z) = z^n \exp f(z)$ can be analytically extended to the whole disc. Hence $g(z) = z^m \exp k(z)$, where $k(z)$ is an analytic function in the unit disc. Thus, $n = m$ and $k(z) - f(z) = \text{const}$.

In general case the proof goes as follows. Since f is analytic on $\Omega^0 \setminus \{*\}$ then it can be represented by the series $\sum_\Gamma c_a(f) \cdot \varphi^a$. Let us show that $c_a(f) = 0$ if $a \notin \Gamma_+$, which would mean that $f \in \mathcal{O}(\Omega^0)$. Assume to the contrary that $c_a(f) \neq 0$ for some $a \notin \Gamma_+$. As before, denote $G_a = \{\alpha \in G, \alpha(a) = 1\}$ and let γ be the Haar measure of the group G_a. Then the function $g(\alpha \cdot r) = \int_{G_a} f(\alpha \cdot \beta \cdot r) d\gamma(\beta)$ is also analytic on $\Omega^0 \setminus \{*\}$ and has a series of the form $\sum_{n \in \mathbb{Z}} c_{na}(f) \cdot \varphi^{na}$, $c_a(f) \neq 0$. Hence the function g could be considered as an analytic function of φ^{-a}. Clearly $\operatorname{Re} g \leq c \log |\varphi^{-a}|$. Therefore, g can be analytically continued to the whole Ω^0, i.e., $c_a(f) = 0$. $\square$

Theorem 4 *Suppose that a function $f \in A$ does not vanish on $\Omega \setminus \{*\}$. Then there exist $a \in \Gamma_+$ and $g \in A$ such that $f = \varphi^a \exp g$.*

Proof. Since the set Ω, which is the space of maximal ideals of the algebra A, can be shrunk to the point $* \in \Omega$ then $H^1(\Omega; \mathbb{Z})$ is trivial. Therefore, if f does not vanish on Ω then by Arens–Royden theorem $f = \exp g$ for some $g \in A$. Now suppose that $f(*) = 0$. By Lemma 2 there exist $a \in \Gamma_+$ and generalized analytic function g on $\Omega \setminus \{*\}$ such that $f = \varphi^a \cdot \exp g$ on $\Omega \setminus \{*\}$. Boundedness of f implies that there exists a constant $c > 0$ such that $\operatorname{Re} g(s) < c \log s$ for all $s \in \Omega \setminus \{*\}$. Applying Lemma 3 we get that $g \in A$. $\square$

Corollary 1 *Suppose that for a function $f \in A$ there exists a number $r \leq 1$ such that f does not vanish on $\Omega_r \setminus \{*\}$, where $\Omega_r = \{s \in \Omega, |s| \leq r\}$ is a generalized disk of radius r. Then there exist $a \in \Gamma_+$ and $g \in A, g(*) \neq 0$, such that $f = \varphi^a \cdot \exp g$.*

Corollary 2 *Suppose that a function $f \in A$ does not vanish on Ω^0. Then there exists a function $k \in \mathcal{O}(\Omega^0)$ such that $f = \exp k$ on Ω^0.*

Definition 4 *A mapping $\mathcal{H} : \Omega \to \Omega$ is called holomorphic if for each $f \in A$ the function $f \circ \mathcal{H}$ belongs to A.*

Let us remind that unless stated otherwise, we suppose that the group Γ is not isomorphic to $\mathbb{Z}$.

Theorem 5 *Suppose that $\mathcal{H}$ is a holomorphic mapping and $\mathcal{H}(*) = *$. Then there exists a number $b \in \Gamma_+, b \neq 0$, such that $|\mathcal{H}(s)| < |s|^b$.*

Proof. If $\mathcal{H}$ is a trivial mapping then the statement of the theorem is obvious. Suppose that $\mathcal{H}$ is a non–trivial mapping. Let us show that if $s_0 \neq *, |s_0| < 1$, then $\mathcal{H}(s_0) \neq *$. Assume to the contrary that there exists $s_0 \neq *, |s_0| < 1$, such that $\mathcal{H}(s_0) = *$. Let $F = G_a \times W$ be a neighbourhood of s_0. Then for each $a \in \Gamma_+, a \neq 0$, the function $g^a(\alpha, z) = \varphi^a \circ \mathcal{H}(\alpha, z)$ is an analytic function of z which vanishes at the point $s_0 = (\alpha_0, z_0) \in F$. Since for every $n \in \mathbb{Z}_+$ there exist $a_1, ..., a_n \in \Gamma_+ \setminus \{0\}$ such that $g^a = \prod_{i=1}^n g^{a_i}$ we get that the order of zero of a function g^a at s_0 is infinite which is impossible. Therefore, $\mathcal{H}(s_0) \neq *$. Now applying the Corollary 1 to the function $\varphi^a \circ \mathcal{H}$ we get that

$$|\mathcal{H}(s)|^a = |\varphi^a \circ \mathcal{H}(s)| \leq |\varphi^d(s)| = |s|^d,$$

where $d \in \Gamma_+, d \neq 0$. It remains to take $b = d/a \in \Gamma_+$ to complete the proof. □

Remark. Each holomorphic mapping from the unit disc of the complex plane to itself is determined by some function from a disc algebra $A(D)$ with modulus not exceeding 1. Therefore, if a function $f \in A(D)$ satisfies the conditions of the Schwarz lemma then it can be considered both as a holomorphic mapping and as a function from $A(D)$ that vanishes at the origin. Theorem 5 is a direct generalization of Schwarz's lemma with assumption that f is a holomorphic mapping. And if f is regarded as a function we can again get a generalization of Schwarz's lemma (H.Bohr's approach). Indeed, let $f \in A, f(*) = 0, |f| \leq 1$ and $b \leq \inf S(f)$, where $S(f)$ is the spectrum o f. Then $|f(s)| \leq |\varphi^b(s)|$. Indeed, we have that $S(f/\varphi^b) \in \Gamma_+$. Therefore, there exists $g \in A, g \leq 1$ such that $f = \varphi^b \cdot g$. Hence $|f(s)| \leq |\varphi^b(s)|$.

Now we pass to the description of the endomorphisms of an algebra A. Note that there exists one-to-one correspondence between endomorphisms and holomorphic mappings: each endomorphism $\mathcal{H}^0$ that acts on A generates a holomorphic mapping and vice versa.

Recall that if $\sigma(\Gamma_+) \subset \Gamma_+$ then the endomorphism σ on a group Γ is an order–preserving endomorphism.

Theorem 6 *Each endomorphism of the algebra A is given by a triple (σ, V_α, k) where σ is an order–preserving endomorphism of Γ, V_α is the operator of a shift by $\alpha \in G$ and $k \in \mathcal{O}(\Omega^0)$ with $\operatorname{Re} k \leq 0$ and $\exp k \in A$. Conversely, every such triple (σ, V_α, k) determines an endomorphism of A.*

Proof. Let $\mathcal{H}^0$ be an endomorphism of A and let $\mathcal{H}$ be the corresponding holomorphic mapping. Assume first that $\mathcal{H}(*) = *$. Since $\mathcal{H}(s) \neq *$ for $s \in \Omega, 0 < |s| < 1$ (see the proof of previous theorem) then, from the Corollary 1 we have that

$$\mathcal{H}^0(\varphi^a) = \varphi^{\sigma(a)} \cdot g_a,\ a \in \Gamma_+. \tag{2}$$

Since $\mathcal{H}^0(\varphi^a \cdot \varphi^b) = \mathcal{H}^0(\varphi^a) \cdot \mathcal{H}^0(\varphi^b)$ then σ is an endomorphism of Γ_+ which can be continued to an order–preserving endomorphism of Γ and $g_a \in A$ does not vanish on Ω^0, with $g_{a+b} = g_a \cdot g_b$. Hence, for every fixed $s \in \Omega^0$ the function $\psi(a) = g_a(s)$ determines a semi-character of a semigroup Γ_+. Since the family of semi-characters of a semigroup Γ_+ coincides with Ω there exist $\alpha \in G$ and $r > 0$ such that $g_a(*) = \alpha(a) r^a$ for all $a \in \Gamma_+$. The function $f_a = \overline{\alpha(a)} \cdot g_a, a \in \Gamma_+$, does not vanish on Ω^0, hence, by Corollary 2, there exists a function $k \in \mathcal{O}(\Omega^0)$ such that $f_a = \exp ak, a \in \Gamma_+$. Thus, $\mathcal{H}^0 \varphi^a = \varphi^{\sigma(a)} \cdot \alpha(a) \cdot \exp ak$, i.e., $\mathcal{H}^0$ is determined by the triple (σ, V_α, k). In case $\mathcal{H}(*) \neq *$ an endomorphism σ in (2) is identically zero (i.e. $\sigma(a) \equiv 0$ for all $a \in \Gamma_+$). Therefore, $\mathcal{H}^0$ is determined by $(0, V_\alpha, k)$.

Now suppose that we are given a triple (σ, V_α, k). For any fixed $s \in \Omega$ define a character τ_s of a semigroup Γ_+ as follows: $\tau_s(a) = \varphi^{\sigma(a)}(s) \cdot \alpha(a) \cdot \exp ak(s)$. Since $\operatorname{Re} k(s) < 0$ and the space of semi–characters of Γ_+ coincides with Ω then there exist $\beta_s \in G$ and $r_s \in [0, 1]$ such that $\tau_s(a) = \beta_s(a) \cdot r_s^a, a \in \Gamma_+$. Define a mapping $\mathcal{H} : \Omega \to \Omega$ with $\mathcal{H}(s) = \beta_s \cdot r_s, s \in \Omega$. Obviously, the function $\varphi^a \circ \mathcal{H}(s) = \beta_s(a) \cdot r_s^a = \varphi^{\sigma(a)}(s) \cdot \alpha(a) \cdot \exp ak(s)$ belongs to A. As the family $\{\varphi^a\}, a \in \Gamma_+$, generates A we get that $\mathcal{H}$ is a holomorphic mapping. □

Corollary 3 *Each isometric endomorphism of A is determined by a triple $(\sigma, V_\alpha, 0)$, where σ is an order–preserving endomorphism of Γ and V_α is the operator of a shift by $\alpha \in G$.*

Proof. As $\mathcal{H}^0$ is isometric then $\mathcal{H}(T_1) = T_1$, where $T_1 = G \times \{1\}$. Hence the function $\exp ak(s) \in A$ does not vanish on Ω and $|\exp ak(s)| \equiv 1$ on T_1. Since $H^1(\Omega; \mathbb{Z})$ is trivial we get that $k(s) \in A$ with $\operatorname{Re} k(s) \equiv 0$ on T_1. Therefore, $k \equiv 0$. □

Corollary 4 *(see [12]) Each automorphism of A is determined by a pair (σ, V_α), where σ is an order–preserving endomorphism of Γ and V_α is the operator of a shift by $\alpha \in G$.*

1.5 Differentiation of generalized analytic functions

In this section we introduce the notion of differentiation of generalized analytic functions and prove that it is well-defined.

Let $D \subset \Omega$ be an open set and let $\mathcal{O}(D)$ be the algebra of generalized analytic functions on D. Define a Frechet topology on $\mathcal{O}(D)$ as follows: for a compact set $F \subset D$ let $||f||_F = \sup_F |f|, f \in \mathcal{O}(D)$. In this topology $\mathcal{O}(D)$ is a Frechet algebra.

Definition 5 *Continuous linear functional $\mathbb{D}_s$ on $\mathcal{O}(D)$ is called a point differentiation at $s \in D$ if $\mathbb{D}_s(f \cdot g) = \mathbb{D}_s(f) \cdot g(s) + f(s) \cdot \mathbb{D}_s(g)$ for any $f, g \in \mathcal{O}(D)$.*

Let E_s be the linear space of all point differentiations at $s \in D$ on a Frechet algebra $\mathcal{O}(D)$.

Theorem 7

$$\dim E_s = \begin{cases} 1, s \neq *, \\ 0, s = *. \end{cases}$$

Proof. Suppose that $\mathbb{D}_s$ is a point differentiation at $s \in D$ on $\mathcal{O}(D)$. If $s \neq *$ define an additive function k_s on Γ_+ by $k_s(a) = \mathbb{D}_s(\varphi^a)/\varphi^a(s)$. The continuity of $\mathbb{D}_s$ implies that $|k_s(a)| < c \cdot n^a \cdot |s|^{-a}$ for some $c > 0$ and $n \in \mathbb{Z}_+$. Therefore, if we define a natural topology on Γ that is induced from $\mathbb{R}$, then k_s can be extended to an additive, continuous function on the whole Γ. But all such functions are linear. Hence, there exists $z_s \in \mathbb{C}$ such that $k_s(a) = z_s \cdot a, a \in \Gamma$.

Suppose now that $s = *$. Obviously, $\mathbb{D}_*(\varphi^0) = 0$. Furthermore, since for any $a \in \Gamma_+ \setminus \{0\}$ there exist $b, c \in \Gamma_+ \setminus \{0\}$ such that $a = b + c$ we get that $\mathbb{D}_*(\varphi^a) = \mathbb{D}_*(\varphi^b \cdot \varphi^c) = 0, a \in \Gamma_+ \setminus \{0\}$. Therefore, $\mathbb{D}_* \equiv 0$. $\square$

Definition 6 *Continuous linear operator $\mathbb{D} : \mathcal{O}(D) \to \mathcal{O}(D)$ is called differentiation if $\mathbb{D}(f \cdot g) = \mathbb{D}(f) \cdot g + f \cdot \mathbb{D}(g)$ for any $f, g \in \mathcal{O}(D)$.*

Clearly, if $f \in \mathcal{O}(D)$ and $\mathbb{D}$ is an operator of differentiation then $\mathbb{D}_f = f \cdot \mathbb{D}$ is also a differentiation. Therefore, the space of all differentiations on $\mathcal{O}(D)$ is an $\mathcal{O}(D)$-module. Let us find the $\mathcal{O}(D)$-dimension of that module. If $D = G_a \times W$, where $W \subset \mathbb{C}$ is an open set and $G_a = \{\alpha \in G, \alpha(a) = 1\}, a \in \Gamma_0$, then $\mathcal{O}(G_a \times W)$ is the algebra of all functions continuous on $G_a \times W$ that are analytic as functions of $z \in W$. Therefore, for each function $f \in \mathcal{O}(G_a \times W)$ we can define a derivative

$$f'(\alpha, z) = \frac{1}{2\pi i} \int_\Lambda \frac{f(\alpha, \xi)}{(\xi - z)^2} d\xi, \tag{3}$$

where Λ is a smooth contour around the point $z \in W$. Since $\varphi^a(\alpha, z) = \alpha(a) e^{iaz}, a \in \Gamma_+$, then $(\varphi^a)' = ia\varphi^a$. We then define a derivative $\mathbb{D}(f)$ of a function $f \in \mathcal{O}(D)$ as

$$\mathbb{D}(f) = -if'$$

on $D \setminus \{*\}$ and $\mathbb{D}(f)(*) = 0$. From (3) it follows that $\mathbb{D}$ is a linear operator on $\mathcal{O}(D)$.

Theorem 8 *The operator $\mathbb{D} : \mathcal{O}(D) \to \mathcal{O}(D)$ is continuous in Frechet topology.*

Proof. Suppose that a sequence $\{f_n\}_1^\infty \subset \mathcal{O}(D)$ converges in a Frechet topology to a function $f \in \mathcal{O}(D)$ and $F \subset D$ is a compact set. Without loss of generality, assume that there exists an open set $W \subset \mathbb{C}$ with a smooth boundary $\Lambda = \partial W$ such that $F \subset G_a \times W \subset G_a \times \overline{W} \subset D$. Now if $E \subset W$ is a compact set such that $F \subset G_a \times E$ then

$$||\mathbb{D}(f_n) - \mathbb{D}(f_m)||_F = \sup_F |f'_n - f'_m| \leq \frac{1}{2\pi} \sup_{G_a} \sup_E \left| \int_\Lambda \frac{f_n(\alpha, \xi) - f_m(\alpha, \xi)}{(\xi - z)^2} d\xi \right| \leq d||f_n - f_m||_{G_a \times E},$$

where $d = \sup_{z \in \Lambda} \sup_{\xi \in E} \left| \frac{1}{(\xi - z)^2} \right|$. $\square$

Theorem 9 *Suppose that $\widetilde{\mathbb{D}} : \mathcal{O}(D) \to \mathcal{O}(D)$ is a differentiation. Then $\widetilde{\mathbb{D}} = g \cdot \mathbb{D}$ for some $g \in \mathcal{O}(D)$.*

Proof. Let $s \in D, s \neq *$. Then $\widetilde{\mathbb{D}}_s(f) := \widetilde{\mathbb{D}}(f)(s)$ as well as $\mathbb{D}_s(f) := \mathbb{D}(f)(s)$ is a point differentiation at s on $\mathcal{O}(D)$. By Theorem 7 there exists a number $z_s \in \mathbb{C}$ such that $\widetilde{\mathbb{D}}_s = z_s \cdot \mathbb{D}_s$. Define a function $g(s) = z_s$. Then $\widetilde{\mathbb{D}}(\varphi^a)(s) = g(s) \cdot \mathbb{D}(\varphi^a)(s) = a \cdot g(s) \cdot \varphi^a(s)$, $a \in \Gamma$. Hence, $g(s) \in \mathcal{O}(D \setminus \{*\})$ and the theorem is proved for the case $* \notin D$. In case $* \in D$ we have that $\Omega_r \subset D$ for some $r > 0$. Let us show that $* \in \Omega_r$ is a removable singularity for a function $\psi = g|_{\Omega_r \setminus \{*\}}$. It is sufficient to show that the spectrum $S(\psi)$ of a function ψ is contained in Γ_+ (see Theorem 20). Indeed, since for any $a \in \Gamma_+$ we have that $S(a \cdot \psi \cdot \varphi^a) \subset \Gamma_+$ and $S(a \cdot \psi \cdot \varphi^a) = \{b \in \Gamma, b = a + c, c \in S(\psi)\}$ then $S(\psi) \subset \Gamma_+$. $\square$

Theorem 10 *Suppose that a function $f \in \mathcal{O}(\Omega^0)$ has formal series of the form $f \sim \sum_{a \in \Gamma_+} c_a(f) \varphi^a$. Then the formal series of $\mathbb{D}(f)$ is of the form $\sum_{a \in \Gamma_+} a \cdot c_a(f) \cdot \varphi^a$.*

Note that if $\Gamma = \mathbb{Z}$ then for every $f \in \mathcal{O}(\Omega)$ the operator equation $\mathbb{D}(g) = f$ has a solution

$$g(z) = \int_{z_0}^{z} f(\xi) d\xi, \; z_0 \in \Omega.$$

In case $\Gamma \neq \mathbb{Z}$ this is not true. Indeed, let $\{a_n\}$ be a sequence of elements from Γ such that $0 < a_n \cdot 2^n < 1/2^n$. Then $f = \sum (1/2^n) \varphi^{a_n}$ is an entire function but the operator equation $\mathbb{D}(g) = f$ has no solution.

In conclusion of this section we note that for any positive integer n the n-th derivative of a function $f \in \mathcal{O}(D)$ can be defined inductively by $\mathbb{D}^n(f) = \mathbb{D}(\mathbb{D}^{n-1}(f))$.

1.6 Jensen's formula

In this section we suggest a method to count the number of zeros of a generalized analytic function. Let $A = P(\Omega)$. The representing measure of a point $* \in \Omega$ concentrated on $T_1 = G \times \{1\}$ coincides with the normalized Haar measure σ of a group G.

Theorem 11 *(see [13]) Suppose $f \in A$ is not identically zero and let m_s be a representing measure of a point $s \in \Omega, |s| < 1$, concentrated on T_1. Then $\int_{T_r} \log|f| dm_s > -\infty$.*

Proof. Since the restriction $A|_{T_1}$ is a Dirichlet algebra then m_s is the unique representing measure of $s \in \Omega$ concentrated on T_1. Assume first that $s \neq *$. Let $F = G_a \times W$ be a closed neighbourhood of a point $s = (\alpha_0, z_0)$. Since for a fixed $\alpha \in G_a$ $f(\alpha, z)$ is analytic as a function of z then we can choose W such that $f(\alpha_0, z)$ does not vanish on $\{\alpha_0\} \times \partial W$. Hence $\int_{\{\alpha_0\}\times\partial W} \log f d\widetilde{m}_s > -\infty$, where $\widetilde{m}_s$ is a representing measure (for the algebra A) of a point s concentrated on $\{\alpha_0\} \times \partial W$. Therefore, by Arens Theorem (see Section 1.1),

$$\int \log|f| dm_s > \int \log|f| d\widetilde{m}_s > -\infty. \tag{4}$$

The group G naturally acts on Ω by $\beta \cdot s = (\alpha \cdot \beta) \cdot r$, where $\beta \in G$ and $s = \alpha \cdot r$. Since G is compact and $\int \log|f| dm_s$ is continuous as a function of s then (4) implies that

$$d = \inf_{\alpha \in G} \int \log|f| dm_{\alpha \cdot s} > -\infty.$$

Hence, for $\varepsilon > 0$

$$\begin{aligned}\int \log(|f| + \varepsilon) d\sigma &= \int \log(|f(\alpha \cdot \beta)| + \varepsilon) d\sigma(\alpha) \\ &= \iint (\log|f(\alpha \cdot \beta)| + \varepsilon) d\sigma(\alpha) dm_s(\beta) \\ &= \iint \log(|f(\beta)| + \varepsilon) dm_{\alpha \cdot s}(\beta) d\sigma(\alpha) \geq d.\end{aligned}$$

□

Definition 7 *We say that a function $f(r)$ on $\mathbb{R}$ is a convex function of $\log r$ if for any $r_0 \in \mathbb{R}$ and any $r_1, r_2 \in \mathbb{R}$ such that $r_0^2 = r_1 r_2$ we have that $f(r_0) \leq \frac{1}{2}(f(r_1) + f(r_2))$.*

Theorem 12 *Suppose $f \in A$. Then $\Phi(r) = \int_G \log|f(\alpha \cdot r)| d\sigma(\alpha)$ is a convex function of $\log r$.*

Proof. Fix $r_0 \in (0, 1)$. Let $r_1, r_2 \in (0, 1), r_1 < r_2$, be such that $r_0^2 = r_1 r_2$. Define a homeomorphism τ on $\Omega(r_1, r_2)$ by $\tau(\alpha \cdot r) = \alpha^{-1} \cdot r_0^2/r$. Obviously, $s_0 = e_0 \cdot r_0 \in \Omega(r_1, r_2)$ is the unique fixed point of that homeomorphism. Since $\varphi^a \circ \tau = r_0^{2a} \cdot \varphi^{-a}$ then τ generates an automorphism on $R(\Omega(r_1, r_2))$. If m_{s_0} is a representing Jensen measure of a point s_0 for $R(\Omega(r_1, r_2))$ concentrated on $T_{r_1} \cup T_{r_2}$ then $m_{s_0} \circ \tau$ is a representing Jensen measure of s_0 as well. Therefore, the measure $\widetilde{m} = (m_{s_0} + m_{s_0} \circ \tau)/2$ is a τ-invariant representing Jensen measure. Hence, $||\widetilde{m}_{T_{r_1}}|| = ||\widetilde{m}_{T_{r_2}}|| = 1/2$. Thus,

$$\Phi(r_0) = \int_G \log|f(\alpha \cdot r_0)| d\sigma(\alpha) \leq \int_G \int_{T_{r_1} \cup T_{r_2}} \log|f(\alpha \cdot s)| d\widetilde{m}(s) d\sigma(\alpha).$$

As for $s \in T_{r_i}$ $\int_G \log|f(\alpha \cdot s)| d\sigma(\alpha) = \Phi(r_i), i = 1, 2$, then, using Fubini's theorem, we get that $\Phi(r_0) \leq \Phi(r_1)/2 + \Phi(r_2)/2$ and, therefore, $\Phi(r)$ is a convex function of $\log r$. □

Convexity of $\Phi(r)$ implies the existence of a positive non-decreasing function $n(r,f)$ such that

$$\Phi(r_2)-\Phi(r_1)=\int_{r_1}^{r_2}\frac{n(r,f)}{r}dr.$$

Definition 8 *The number $n(r,f)=r\cdot\Phi'(r+0)$ will be called a measure of the set of zeros of a function f on Ω_r.*

Let us discuss the given definition. Fix $a\in\Gamma_+, a\neq 0$, and denote $G_a=\{\alpha\in G,\alpha(a)=1\}$ and $W=\{z\in\mathbb{C}, 0<\mathrm{Re}z\leq 2\pi/a, b_1\leq \mathrm{Im}z\leq b_2\}$. Consider a mapping $h: G_a\times W\to\Omega(r_1,r_2): h(\alpha,z)=\alpha\cdot e_t\cdot e^{-y}$, where $r_1=-\log b_2, r_2=-\log b_1$ and $z=t+iy$. As in the proof of Theorem 2 it can be shown that h is a one-to-one mapping and $\varphi^a\circ h(\alpha,z)$ is analytic as a function of $z\in\mathrm{int}\,W$. Suppose that $f\in R(\Omega(r_1,r_2))$ does not vanish on $h(G_a\times\partial W)$. Then, by the argument principle, the number of zeros of a function $f(\alpha,z)=f\circ h(\alpha,z)$ on $\{\alpha\}\times W$ coincides with

$$n_a(f,\alpha)=\frac{1}{2\pi}\left\{\int_0^{2\pi/a}\frac{d\arg f(\alpha,t+ib_1)}{dt}dt-\int_0^{2\pi/a}\frac{d\arg f(\alpha,t+ib_2)}{dt}dt\right.$$

$$\left.-\mathrm{Im}\int_{b_1}^{b_2}\frac{f'(\alpha,0+iy)}{f(\alpha,0+iy)}dy+\mathrm{Im}\int_{b_1}^{b_2}\frac{f'(\alpha,2\pi/a+iy)}{f(\alpha,2\pi/a+iy)}dy\right\}.$$

Since $e_{2\pi/a}\in G$ and $h(\alpha\cdot e_{2\pi/a},0)=h(\alpha,2\pi/a)$ then

$$f(\alpha\cdot e_{2\pi/a},0+iy)=f(\alpha,2\pi/a+iy)$$

which means that the last two terms in the above equation cancel each other out.

Furthermore, the Cauchy–Riemann conditions give

$$\frac{d\arg f(\alpha,t+iy)}{dt}=-\frac{d\log|f(\alpha,t+iy)|}{dy}.$$

Therefore,

$$\int_{G_a}n_a(f,\alpha)d\gamma=\frac{1}{2\pi}\left\{\int_{G_a}\int_0^{2\pi/a}\left.\frac{d\log|f(\alpha,t+iy)|}{dy}\right|_{y=b_1}d\gamma dt\right.$$

$$\left.-\int_{G_a}\int_0^{2\pi/a}\left.\frac{d\log|f(\alpha,t+iy)|}{dy}\right|_{y=b_2}d\gamma dt\right\},$$

where γ is the normalized Haar measure of a group G_a.

Since $h(G_a\times l)=G\times\{e^{-b}\}\subset\Omega$, where $l=\{z\in\mathbb{C},0<\mathrm{Re}z\leq 2\pi/a, b_1\leq\mathrm{Im}z=b\}$, then the pre-image of the Haar measure σ of a group G under the mapping h is a measure $a/2\pi d\gamma\times dt$ on $G_a\times l$. Therefore,

$$\frac{a}{2\pi}\int_{G_a}\int_0^{2\pi/a}\left.\frac{d\log|f(\alpha,t+iy)|}{dy}\right|_{y=b_1}d\gamma dt=-\left.\frac{d\Phi(r)}{d\log r}\right|_{r=r_1}=-\Phi'(r_1)\cdot r_1.$$

Hence,

$$\Phi'(r_2)r_2 - \Phi'(r_1)r_1 = a \cdot \int_{G_a} n_a(f, \alpha) d\gamma. \tag{5}$$

This formula implies the following theorem.

Theorem 13 *Suppose that for some* $f \in R(\Omega(r_1, r_2))$ $\Phi(r)$ *is a linear function of* $\log r$. *Then there exist* $a \in \Gamma$ *and* $g \in R(\Omega(r_1, r_2))$ *such that* $f = \varphi^a \cdot \exp g$.

Proof. Since $\Phi(r) = c \log r + d$ then the left side of (5) is zero which means that the right side of (5) is also zero. Therefore, f does not vanish on $R(\Omega(r_1, r_2))$. Now applying Arens–Royden's theorem to the algebra $R(\Omega(r_1, r_2))$ and noting that $\Omega(r_1, r_2)$ is the space of maximal ideals of $R(\Omega(r_1, r_2))$ and $H^1(\Omega(r_1, r_2)); \mathbb{Z}) = H^1(G; \mathbb{Z}) = \Gamma$ we get the desired expression for f. □

Theorem 14 *Let* $f \in P(\Omega)$. *Then*

1. $n(r, f)$ *is a positive non–decreasing function;*
2. *if* $n(r, f) = const, r \in (0, 1]$, *then* $a = n(r, f) \in \Gamma_+$ *and there exists a function* $g \in P(\Omega)$ *such that* $f = \varphi^a \cdot \exp g$;
3. *if* f *does not vanish on* $T_r, 0 < r < 1$, *then* $n(r, f) = \int_{T_r} \frac{\mathbb{D}(f)}{f} d\sigma$;
4. *if* $g \in P(\Omega)$ *is such that* $|f| > |g|$ *on* T_r *then* $n(r, f + g) = n(r, f)$.

Proof. (1) is obvious. (2) follows from Theorem 13. (3) If the function f does not vanish on T_r then $f = \varphi^a \cdot \exp g$ on some neighbourhood of T_r, where $a = n(r, f)$. Therefore,

$$\int_{T_r} \frac{\mathbb{D}(f)}{f} d\sigma = \int_{T_r} \frac{\varphi^a \cdot \mathbb{D}(g) \cdot \exp g + a\varphi^a \cdot \exp g}{\varphi^a \cdot \exp g} d\sigma = a$$

as $\int_{T_r} \mathbb{D}(g) d\sigma = 0$. (4) Since $|f| > |g|$ on T_r then $|(f+g)/f - 1| < 1$ on some neighbourhood of T_r. Therefore, there exists $h \in R(\Omega(r - \varepsilon, r + \varepsilon))$ such that $(f+g)/f = \exp h$. This means that $\Phi(r_1, f+g) = \Phi(r_1, f) + \int_{T_{r_1}} \operatorname{Re} h d\sigma$. But $\int_{T_{r_1}} \operatorname{Re} h d\sigma = \operatorname{Re} c_0(h)$, where $c_0(h)$ is the zero Fourier coefficient of a function h which does not depend on r_1. Hence, $n(r, f+g) = n(r, f)$. □

Let f be a generalized analytic function. Jensen's formula allows to calculate the modulus of the first Fourier coefficient of f. So it is natural to assume that $a = \inf S(f) \in S(f)$. Let

$$N(r, f) = \int_0^r \frac{n(t, f) - a}{t} dt.$$

Theorem 15

$$\log |c_a(f)| = \int_{T_r} \log |f| d\sigma - N(r, f) - a \log r.$$

Proof. Since $a \in S(f)$ there exists $r_0 > 0$ such that f does not vanish on $\Omega_{r_0} \backslash \{*\}$. Therefore on Ω_r we have that $f = \varphi^a \cdot \exp g = c_a(f)\varphi^a \cdot \exp(g - g(*))$ with $g \in P(\Omega_r)$. Hence,

$$\Phi(r_0) = \log |c_a(f)| + a \log r_0.$$

Now to complete the proof it remains to note that $N(r, f) = \Phi(r) - \Phi(r_0)$. □

2 Generalized meromorphic functions

2.1 On the singularities of generalized analytic functions

In this section we state the theorems on analytic continuation of generalized analytic functions.

Theorem 16 *Let D be an open set in Ω and let $s \in D, s \neq *$. Then every analytic function on $D \setminus \{s\}$ can be analytically continued to some function from $\mathcal{O}(D)$.*

Proof. Let $F = V \times W \subset D$ be a closed neighbourhood of a point $s = (\alpha_0, z_0)$ with $V \subset G_a$ and $W \subset \mathbb{C}$. Then the function $f \in \mathcal{O}(D \setminus \{s\})$ is continuous on $\partial F = V \times \partial W$ and by the maximum principle for the functions of one complex variable we have that

$$\sup_{\{\alpha\}\times W} |f| = \sup_{\{\alpha\}\times\partial W} |f| \leq \sup_{V\times\partial W} |f| = M < \infty,\ \alpha \in V, \alpha \neq \alpha_0.$$

Since G_a is a perfect set then for any $z \neq z_0$ and any $\varepsilon > 0$ there exists $\alpha \in V$ such that $|f(\alpha_0, z) - f(\alpha, z)| < \varepsilon$. Therefore, $|f(\alpha_0, z)| \leq M, z \neq z_0$. Thus, $f(\alpha_0, \cdot)$ is a bounded analytic function of one complex variable on $W \backslash z_0$ and, hence, f can be analytically extended to $\{\alpha_0\} \times W$. Thus, f can be extended to some function from $\mathcal{O}(D)$. □

In case $s = * = G \times \{0\}$ the previous theorem does not hold. However we still have the following result.

Theorem 17 *Let $D \ni *$ be an open set in Ω. Then every bounded analytic function on $D \setminus \{*\}$ can be analytically continued to some function from $\mathcal{O}(D)$.*

Proof. Let $r_0 > 0$ be such that the set $\Omega_{r_0} = \{s \in \Omega, |s| < r_0\}$ is contained in D. For any $a \in \Gamma$ and $0 < r < r_0$ the Fourier coefficient $c_a(f) = \int_{T_r} f \cdot \varphi^{-a} d\sigma$ of a function f satisfies the inequality $|c_a(f)| r^a \leq \sup_{T_r} |f|$ where $T_r = G \times \{r\}$. Boundedness of f implies that if $a < 0$ then $c_a(f) = 0$. Thus, the spectrum $S(f)$ does not contain any negative element from Γ. Therefore (see Theorem 20), f can be continued to a function from $\mathcal{O}(\Omega_r)$. □

Let us now introduce the notion of a thin set, which is an analogue of sets of zeros and poles of analytic functions of one complex variable. Here we again use the fact that each point from $\Delta \setminus \{*\}$ has a neighbourhood U of the form $U = V \times W$ where $V \subset G_a$ and $W \subset \mathbb{C}$.

Definition 9 *Let D be an open set in Δ. A subset $F \subset D$ is called a thin set if it is closed in D and the following conditions hold:*

1. *for each $s \in D, s \neq *$, there exist a neighbourhood $U \subset D, U = V \times W$, and a function $f \in \mathcal{O}(U), f \not\equiv 0$, such that $f = 0$ on $F \cap U$ and f is not identically zero on each $W_\alpha = \{\alpha\} \times W, \alpha \in V$;*

2. *if $* \in D$ then there exists a function $f \in \mathcal{O}(\Omega_r), \Omega_r \subset D, f \not\equiv 0$, such that $f = 0$ on $\Omega_r \cap F$.*

Remark. Clearly, if F is a thin set in D and D_1 is an open subset of D then $F \cap D_1$ is a thin set in D_1.

Slightly modifying the proof of Theorem 16 one can prove the following theorem.

Theorem 18 *Let $D \subset \Omega$ be an open set and suppose F is a thin subset of D. Then every bounded function from $\mathcal{O}(D \setminus F)$ can be uniquely extended to some function from $\mathcal{O}(D)$.*

Theorem 19 *Let F be a thin subset of an open set $D \subset \Omega$ and let f be an analytic function on $D \setminus F$. Then the behaviour of the function f near each point $s_0 \in F$ can be one of the following:*

1. *$f(s_\lambda)$ tends to a finite limit as $s_\lambda \to s_0$;*

2. *$|f(s_\lambda)|$ tends to ∞ for any $s_\lambda \to s_0$;*

3. *in every neighbourhood of s_0 the function f takes on values arbitrarily close to any number.*

A point satisfying 1), 2) or 3) will be called, respectively, removable singularity, pole or essential singularity.

Behaviour of a function f near $* \in \Omega$ is determined by the following property of a spectrum.

Theorem 20 *Suppose $f \in \mathcal{O}(\Omega^0 \setminus \{*\})$ and $a_0 = \inf S(f)$. Then*

1. *if $a_0 > 0$ then $*$ is a removable singularity;*

2. *if $a_0 < 0$ and $a_0 \in S(f)$ then $*$ is a pole;*

3. *if $a_0 < 0$ and $a_0 \notin S(f)$ then $*$ is an essential singularity.*

Proof. 1) If $S(f) \subset \Gamma_+$ then on $T_r, 0 < r < 1$, f can be uniformly approximated by polynomials. But T_r is the Shilov boundary of an algebra $P(\Omega_r)$. Therefore, $*$ is a removable singularity of f. 2) The spectrum of a function $g = \varphi^{a_0} \cdot f$ is contained in Γ_+ and $0 \in S(g)$. Hence, $g(*) \neq 0$. Therefore, $*$ is a pole of f. 3) Assume that $*$ is not an essential singularity of f. Let $z \notin \overline{f(\Omega_r^0 \setminus \{*\})}, 0 < r < 1$. Then the function $1/(f-z)$ is bounded on $\Omega_r^0 \setminus \{*\}$ and does not vanish there. By Theorem 4 there exist $g \in \mathcal{O}(\Omega_r^0)$ and $a \in \Gamma_+$ such that $1/(f-z) = \varphi^a \cdot \exp g$. Hence, $f = \varphi^{-a} \cdot \exp(-g) + z$ and as $0 \in S(\exp(-g)) \subset \Gamma_+$ then $-a \in S(f)$ and $-a = \inf S(f)$. We have thus arrived at a contradiction with (3), which means that $*$ is an essential singularity of f. □

2.2 Auxiliary results

In this section, based on the function theory of one complex variable, we elaborate the methods for further investigations of generalized analytic functions.

Without loss of generality, assume that $2\pi \in \Gamma$. On a locally compact group $K \times \mathbb{R}$, where $K = \{\alpha \in G, \alpha(2\pi) = 1\}$, consider an algebra B of bounded continuous functions which for any fixed $\alpha \in K$ can be continuously extended to a bounded analytic function in the upper half–plane. Let us describe some properties of that algebra.

1. We begin with the measures which are orthogonal to B. Let A be a trace of the disk algebra on the unit circle and let H^1 be a Hardy space which is a closure of A in $\mathcal{L}^1$ norm with respect to the Lebesgue measure. Also let H_0^1 be the family of functions from H^1 whose analytic continuations vanish at the origin. By F. and M. Riesz theorem the space of all measures on the unit circle which are orthogonal to disk algebra coincides with H_0^1. Here we present an analogue of that theorem for the algebra B.

Note that the conformal mapping $w(z) = \frac{i-z}{i+z}$ of an upper half–plane to the unit disc generates a mapping from the Hardy space H^1 to $\mathcal{H}^1$ – the space of integrable functions on $\mathbb{R}$ with respect to the measure $\frac{dx}{1+x^2}$. Each function from $\mathcal{H}^1$ can be analytically continued to an analytic function in the upper half–plane. The image of H_0^1 in $\mathcal{H}^1$ under the mapping w will be denoted by $\mathcal{H}_0^1$.

Let $\mathcal{H}_0^1(K \times \mathbb{R})$ be a space of measures of the form $f(\alpha, x) \cdot \nu \times \frac{dx}{1+x^2}$, where ν is some probability measure on K and the function $f(\alpha, x) \in \mathcal{L}^1(\nu \times \frac{dx}{1+x^2})$ for almost all fixed $\alpha \in K$ (with respect to the measure ν) belongs to $\mathcal{H}_0^1$.

Lemma 4 *Suppose $\mu \in \mathcal{H}_0^1(K \times \mathbb{R})$. Then for any $g \in B$ $\int_{K\times\mathbb{R}} g\, d\mu = 0$.*

Proof. We have

$$\int_{K\times\mathbb{R}} g d\mu = \int_K \left(\int_{\mathbb{R}} g(\alpha, x) \cdot f(\alpha, x) \frac{dx}{1+x^2} \right) d\nu.$$

Since $\frac{1}{\pi(1+x^2)}$ is the Poisson kernel of the point $i \in \mathbb{C}_+$ and for almost all fixed $\alpha \in K$ an analytic continuation of f to the upper half–plane vanishes at i then the inner integral is zero for almost all $\alpha \in K$ with respect to the measure ν. □

By the Phragmén–Lindelöf principle B is a Banach algebra (even uniform) with respect to the uniform norm on $K \times \mathbb{R}$. Each point from $K \times \mathbb{R}$ is a generalized peak point for B, therefore, $K \times \mathbb{R}$ is contained in ∂B, the Shilov boundary of an algebra B. Let $B^\perp$ be a space of all regular Borel measures on ∂B which are orthogonal to the algebra B.

Lemma 5 *Suppose $\mu \in B^{\perp}$. Then there exist mutually singular measures $\mu_1, \mu_2 \in B^{\perp}$ such that*

1. $\mu = \mu_1 + \mu_2$
2. $\mu_1 \in \mathcal{H}_0^1(K \times \mathbb{R})$ *and* $\operatorname{supp} \mu_2 \subset \partial B \setminus K \times \mathbb{R}$.

Proof. The algebra $C(K)$ of all continuous functions on K can be embedded into B by assigning to a function $g \in C(K)$ the function $g(\alpha, x) = g(\alpha)$. Therefore the set $F_\alpha, \alpha \in K$, which is obtained as a closure of the set $\{\alpha\} \times \mathbb{R} = \mathbb{R}_\alpha$ in ∂B, is a maximal set of antisymmetry of an algebra B. Hence, by Bishop theorem, the support of each extremal measure from the unit ball $S_0 \subset B^{\perp}$ is contained in some $F_\alpha, \alpha \in K$ (see Section 1.1). Let $B_\alpha^{\perp}$ be a set of the measures from S_0 whose support lie in $F_\alpha, \alpha \in K$. Obviously, $B_\alpha^{\perp}$ is a compact set in a weak* topology. Since $F_\alpha \cap F_\beta = \varnothing, \alpha \neq \beta$, then for any measure $\mu \in B^{\perp}$ there is a probability measure ν on K such that

$$\mu = \int_K \mu_\alpha \, d\nu, \quad \mu_\alpha \in B_\alpha^{\perp}. \tag{6}$$

Indeed, convex combination of extremal measures from S_0 is contained in the convex combination of the measures from $B_\alpha^{\perp}, \alpha \in K$. Therefore, using the Krein–Milman theorem and the fact that F_α is a set of antisymmetry we get that (6) holds. By F. and M. Riesz theorem (see [5], pp. 66-67) the measure μ_α can be represented as the sum of two measures ζ_α and ξ_α from $B_\alpha^{\perp}$ where $\xi_\alpha = f \cdot \frac{dx}{1+x^2}$, with $f \in \mathcal{H}_0^1$, and $\operatorname{supp} \zeta_\alpha \subset F_\alpha \setminus \mathbb{R}_\alpha$. It remains to take

$$\mu_1 = \int_K \xi_\alpha d\nu \quad \text{and} \quad \mu_2 = \int_K \zeta_\alpha d\nu.$$

□

Corollary 5 *Let F be a compact subset of $K \times \mathbb{R}$. Then the restriction of the algebra B to F coincides with $C(F)$ if and only if for each $\alpha \in K$ the set $F_\alpha \cap \mathbb{R}_\alpha$ has a Lebesgue measure zero.*

2. Here we prove some facts about the zero sets of the algebra B that will be used below in this chapter.

Let M_B be the set of maximal ideals of the algebra B. Obviously, $K \times \mathbb{C}_+ \subset M_B$, where $\mathbb{C}_+ = \{z \in \mathbb{C}, \mathrm{Im} z \geq 0\}$. For $n \in \mathbb{Z}$ define $\mathbb{C}_n = \{z \in \mathbb{C}_+, n < \mathrm{Re} z \leq n+1\}$.

Hereafter the algebra B will be identified with its extension on $K \times \mathbb{C}_+$.

Lemma 6 *Let V and W be open sets in K and $\mathbb{C}_0$, respectively, and let f be a continuous function on $V \times W$ which is analytic as a function of $z \in W$. Suppose that there exists an open convex set $W_0 \Subset W$ such that the set $N(f) = \{s \in V \times W, f(s) = 0\}$ is contained in $K \times W_0$. Then for each $s \in N(f)$ there exist a neighbourhood $U = V \times W$ and a function $g \in B$ such that $N(g) \subset K \times W_0$ and $N(f) \cap U = N(g) \cap U$.*

Proof. Let us denote by $N(f_\alpha)$ the set of zeros of a function $f_\alpha(z) = f(\alpha, z)$ (counting multiplicities). Then $N(f_\alpha)$ is finite and $N(f_\alpha) \subset W_0$. Also, if two elements $\alpha, \beta \in V$ are "close" to each other then, since f continuous, Rouche's theorem implies that the sets $N(f_\alpha)$ and $N(f_\beta)$ are also "close" to each other. Therefore, if $N(f_\alpha) = \{z_1(\alpha), ..., z_{n(\alpha)}(\alpha)\}$ then the function $p = p(\alpha, z) = \prod_1^{n(\alpha)} (z - z_i(\alpha))$ is continuous on $V \times W$ with $N(p) = N(f)$.

Fix a point $s = (\alpha, z_i(\alpha)) \in N(p)$ and let h be a positive continuous function on K with $0 \leq h \leq 1$ such that $h = 1$ on a neighbourhood $V_0 \Subset V$ of $\alpha \in K$ and $h = 0$ on $K \setminus V$. Then, since W_0 is convex, the function

$$g = g(\alpha, z) = \frac{\prod_1^{n(\alpha)}(z - (1 - h(\alpha))z_0 - h(\alpha)z_i(\alpha))}{\prod_1^{n(\alpha)}(z - (1 - h(\alpha))\overline{z}_0 - h(\alpha)\overline{z}_i(\alpha))}, \quad z_0 \in W_0,$$

belongs to the algebra B. Now taking $U = V_0 \times W$ we get that $N(g) \subset K \times W_0$ and $N(g) \cap U = N(p) \cap U$. $\square$

Corollary 6 *Under the conditions of the previous lemma, for any $s \in N(f)$ there exist a neighbourhood $U = V \times W$ and a family of functions $\{g_\lambda\} \subset B$ such that $N(g_\lambda) \subset K \times W_0$, $N(f) \cap U = N(g_\lambda) \cap U$ and $\bigcap_\lambda N(g_\lambda) = \overline{N(f) \cap U}$.*

As $K = \{\alpha \in G, \alpha(2\pi) = 1\}$ then for every $n \in \mathbb{Z}$ the element $e_n \in G$ with $e_n(a) = e^{ian}, a \in \Gamma$, belongs to K. The subgroup $\mathcal{F} = \{(e_n, -n), n \in \mathbb{Z}\}$ of a group $K \times \mathbb{R}$ acts on $K \times \mathbb{C}_+$ in the following way: $(e_n, -n) \cdot (\alpha \cdot z) = (\alpha \cdot e_n, z - n)$. Given a set $F \subset K \times \mathbb{C}_0$ the set $F \cdot (e_n, -n)$ we will be denoted by F_n. Obviously, $F_n \subset K \times \mathbb{C}_n$.

Lemma 7 *Suppose that the zero set $N(g)$ of a function $g \in B$ is contained in $K \times W_0$, where $W_0 \Subset \mathbb{C}_0$. Then there exists a function $b \in B$ such that $|b||_{K \times \mathbb{R}} = 1$ and $N(b) = \bigcup_{-\infty}^{+\infty} N(g)_n$.*

Proof. Again if we consider the function $g_\alpha(z) = g(\alpha, z)$ then the zero set $N(g_\alpha)$ is finite. Suppose that $N(g_\alpha) = \{z_1(\alpha), ..., z_{n(\alpha)}(\alpha)\}$. Consider two functions $p(\alpha, z) = \prod_1^{n(\alpha)}(z - z_i(\alpha))$ and $\overline{p}(\alpha, z) = \prod_1^{n(\alpha)}(z - \overline{z}_i(\alpha))$. For $n \in \mathbb{Z}$ define

$$p_n(\alpha, z) = p(\alpha \cdot e_n, z - n), \quad \overline{p}_n(\alpha, z) = \overline{p}(\alpha \cdot e_n, z - n),$$

$$c_n(\alpha) = \overline{p}_n(\alpha, 0)/p_n(\alpha, 0), \quad b_n(\alpha, z) = c_n(\alpha)\overline{p}_n(\alpha, z)/p_n(\alpha, z).$$

Then, obviously, $b_n(\alpha, z) \in B, N(b_n) = N(g)_n, b_n(\alpha, 0) = 1$ and $|b_n(\alpha, x)| = 1, x \in \mathbb{R}$. Let us show that $b(\alpha, z) = \prod_{-\infty}^{\infty} b_n(\alpha, z)$ is the desired function. Indeed, let $W_n = \{z \in \mathbb{C}_n, z - n \in W_0\}, n \in \mathbb{Z}$, and define $V = \bigcup_{-\infty}^{\infty} W_n$. Since $\operatorname{card} N(g_\alpha) = n(\alpha) < \infty$ then, as K is compact and $n(\alpha)$ is continuous, there exists a natural number m such that

$$\sup_K n(\alpha) < m.$$

For any set $E \subset V$ with $\operatorname{card}(E \cap W_n) \leq m$, $n \in \mathbb{N}$, there exists a Blaschke product on $\mathbb{C}_+$ whose zero set, counting multiplicities, is E (see [7], p. 62). Let $\{b(z)\}_{(n)}, b(0) = 1$, be the family of all Blaschke products in upper half-plane which have no more than m zeros, counting multiplicities, all being contained in $\overline{W}_n$. Fix $\delta > 0$ and denote $\varepsilon_n(\delta) = \sup_{\{b\}_n} \sup_{x\in[-\delta,\delta]} |1 - b(x)|$. Since $\overline{W}_n$ is compact then for any $n \in \mathbb{Z}$ there exists a function $b_n \in \{b\}_n$ such that $\varepsilon_n(\delta) = \sup_{x\in[-\delta,\delta]} |1-b_n(x)|$. As the Blaschke product $\prod_{-\infty}^{\infty} b_n$ exists we have that $\sum_{-\infty}^{\infty} \varepsilon_n(\delta) < \infty$ and since for every fixed $\alpha \in K$ the function $b_n(\alpha, z)$ belongs to the family $\{b\}_n$ then $\sup_{K\times[-\delta,\delta]} |1 - b_n(\alpha, z)| < \varepsilon_n(\delta)$. Therefore, $b(\alpha, z) = \prod_{-\infty}^{\infty} b_n(\alpha, z) \in B$. □

2.3 Analytic A–measures

Denote $A = P(\Omega)$. Since the uniform algebra on G generated by the characters $\chi^a, a \in \Gamma$, is isometrically isomorphic to the algebra A we will then identify those two algebras.

Let $M(G)$ be a space of regular Borel measures on G. A measure $\mu \in M(G)$ is orthogonal to A ($\mu \in A^{\perp}$) if and only if the spectrum $S(\mu)$ of a measure μ is contained in Γ_+. Such measures are called analytic A–measures. They have been studied in the works [14], [15], [16]. In this section we explore the measures on $\Omega \setminus \{*\}$ which are orthogonal to A. Consider a group homomorphism $\Phi : K \times \mathbb{R} \to G$, $\Phi(\alpha, t) = \alpha \cdot e_t$. If $\beta \in G$ and $\beta(2\pi) = e^{i\theta_0}$ then $\beta \cdot e_t \in K$ where $t = (2\pi - \theta_0)/2\pi$. Therefore, Φ is a surjection and for every $n \in \mathbb{Z}$ the set $F_n = K \times [n, n + 1)$ is a fundamental region of Φ. The mapping Φ can be extended to a mapping $\Phi : K \times \mathbb{C}_+ \to \Omega \setminus \{*\}$ with $\Phi(\alpha, z) = \alpha \cdot e_t \cdot e^{-y}$. Then, obviously, for every $n \in \mathbb{Z}$ the set $K \times \mathbb{C}_n$ is a fundamental region of the extended mapping Φ, where $\mathbb{C}_n = \{z \in \mathbb{C}_+, n \leq \operatorname{Re} z < n+1\}$, and a group $\mathcal{F} = \{(e_n, -n)\}_{n\in\mathbb{Z}} \subset K \times \mathbb{R}$ is a kernel of Φ.

Let M be a space of locally finite measures on $K \times \mathbb{C}_+$ which are invariant under shifts by the elements of $\mathcal{F}$. The space M is uniquely determined by the measures with supports contained in $K \times \mathbb{C}_0$ – the fundamental region of Φ. Hence, the following lemma holds.

Lemma 8 *Let M_* be the space of measures on $\Omega \setminus \{*\}$. Then the mapping Φ generates a linear bijective operator $\Phi^* : M_* \to M$ with $\Phi^*(\mu_*)(\cdot) = \mu_*(\Phi(\cdot))$, $\mu_* \in M_*$.*

Let $C(K \times \mathbb{C}_+)$ be a space of bounded continuous functions on $K \times \mathbb{C}_+$ with dual space $M(K \times \mathbb{C}_+)$ which is the space of measures on $K \times \mathbb{C}_+$. Define a mapping $\psi : M_* \to M(K \times \mathbb{C}_+)$ by $\psi(\xi) = 1/(i + z)^2 \cdot \Phi^*(\xi)$, $\xi \in M_*, z \in \mathbb{C}_+$.

Theorem 21 *Suppose $\mu \in A^{\perp}$ and $\operatorname{supp} \mu \subset \Omega(r, 1) = G \times [r, 1]$. Then $\psi(\mu) \in B^{\perp}$.*

To prove this theorem we need the following lemmas.

Lemma 9 *Assume that a net of measures $\{\xi\} \subset M_*$ with $\operatorname{supp} \xi \subset \Omega(r, 1)$ converges weak* to a measure $\zeta \in M_*$. Then the net of measures $\{\psi(\xi)\}$ converges weak* to the measure $\psi(\zeta)$. Furthermore, $||\xi||/(1 - \log r)^2 \leq ||\psi(\xi)|| \leq 2||\xi||$ where the norms $|| \cdot ||$ are taken in the corresponding spaces.*

Proof. Since $F_n = K \times \mathbb{C}_n, n \in \mathbb{Z}$, is a fundamental region of the mapping Φ then for any $\xi \in M_*$ we have that $||\xi|| = ||\Phi^*(\xi)|_{F_n}||$. As $\Omega(r,1) = \Phi(K \times W)$, where $W = \{z \in \mathbb{C}_+, \mathrm{Im} z \leq -\log r\}$, then

$$||\Phi^*(\xi)|_{F_n}||/((n+1)^2 + (1-\log r)^2) \leq ||\psi(\xi)|_{F_n}|| \leq ||\psi(\xi)|| \leq 2||\xi||.$$

But $K \times \mathbb{C}_+ = \bigcup_{-\infty}^{\infty} F_n$. Hence, $||\xi||/(1-\log r)^2 \leq ||\psi(\xi)|| \leq 2||\xi||$. The first part of the lemma follows from Lemma 4. □

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a basis of neighbourhoods of the unit element of a group G and let $\{\psi_{\delta,\lambda}\}_{\substack{\delta>0 \\ \lambda \in \Lambda}}$ be a system of positive continuous functions on G such that the following conditions hold:

1. the spectrum $S(\psi_{\delta,\lambda})$ of a function $\psi_{\delta,\lambda}$ is finite;

2. $\psi_{\delta,\lambda}(\alpha) < \delta$ if $\alpha \notin U_\lambda$;

3. $\int_G \psi_{\delta,\lambda} d\sigma = 1$, where σ is the normalized Haar measure of a group G.

   Using the system $\{\psi_{\delta,\lambda}\}$ we define a family of Bochner-Fejer operators $\{K_{\delta,\lambda}\}$ as follows:

$$K_{\delta,\lambda}(f)(\alpha) = \int_G f(\alpha \cdot \beta^{-1}) \psi_{\delta,\lambda}(\beta) d\sigma(\beta).$$

4. $K_{\delta,\lambda}$ is a continuous operator on $C(G)$ with $||K_{\delta,\lambda}|| = \sup_{||f|| \leq 1} ||K_{\delta,\lambda}(f)|| = 1$, $\inf_{\delta,\lambda} ||K_{\delta,\lambda}(f) - f|| = 0$ and $S(K_{\delta,\lambda})$ is finite and is contained in $S(f)$.

   Since $M(G)$ is a dual space of $C(G)$, the operator $K_{\delta,\lambda}$ generates an adjoint operator $K^*_{\delta,\lambda} : M(G) \to M(G)$ with

$$\int_G K_{\delta,\lambda}(f) d\mu = \int_G f dK^*_{\delta,\lambda}(\mu).$$

   On the family of pairs $\{(\delta,\lambda)\}_{\substack{\delta>0 \\ \lambda \in \Lambda}}$ one can define an order in the following way: $(\delta_1, \lambda_1) < (\delta_2, \lambda_2)$ if $\delta_2 < \delta_1$ and $U_{\lambda_2} \subset U_{\lambda_1}$. Having this order a family of operators $\{K^*_{\delta,\lambda}\}$ becomes a net.

From the conditions 1)–4) it follows that for any $\mu \in M(G)$ that:

5. the spectrum $S(K^*_{\delta,\lambda}(\mu))$ of a measure $K^*_{\delta,\lambda}(\mu) \in M(G)$ consists of finite number of elements and is contained in $S(\mu)$;

6. $||K^*_{\delta,\lambda}(\mu)|| \leq ||\mu||$ and the net of measures $\{K^*_{\delta,\lambda}(\mu)\}$ converges weak* to the measure μ;

From (5) and (6) it follows that the measure $K^*_{\delta,\lambda}(\mu)$ is absolutely continuous with respect to σ and, therefore, each measure from $M(G)$ can be approximated in the weak* topology by equibounded net of measures which are absolutely continuous with respect to σ.

Let γ be the normalized Haar measure of a compact group K. Let us show that

$$\Phi^*(\sigma) = \gamma \times dx = \mu. \tag{7}$$

Since $\Phi^*(\sigma)$ and μ are invariant under shifts by the elements of a group $\mathcal{F}$ then it is sufficient to show that their restrictions to $K \times [0,1)$ coincide. If $\chi^b, b \in \Gamma$, is a character of a group G then

$$\int_{K\times[0,1)} \chi^0 \circ \Phi d\mu = \int_K d\gamma \int_{[0,1)} dx = 1 \quad \text{and}$$

$$\int_{K\times[0,1)} \chi^b \circ \Phi d\mu = \int_K \chi^b(\alpha) d\gamma(\alpha) \int_{[0,1)} e^{ibx} dx = 0 \quad \text{if} \quad b \neq 0.$$

Indeed, first equality is obvious, second equality is also obvious if $b = 2\pi n, n \in \mathbb{Z} \setminus \{0\}$. In case $b \neq 2\pi n$ since $K = \{\alpha \in G, \alpha(2\pi) = 1\}$, then χ^b is a non–trivial character of a group K. Hence, $\int_K \chi^b(\alpha) d\gamma(\alpha) = 0$. The measure $\Phi^*(\sigma)$ also satisfies the above equations, therefore, the uniqueness of Haar measure implies (7).

Lemma 10 *Let $\mu \in A^\perp$ with* $\operatorname{supp} \mu \subset G$. *Then $\psi(\mu) \in \mathcal{H}^1_0(K \times \mathbb{R})$.*

Proof. Since $\mu \in A^\perp$, $\operatorname{supp} \mu \subset G$, then $S(\mu) \subset \Gamma_+$. Therefore $\mu_{\delta,\lambda} \in A^\perp$, where $\mu_{\delta,\lambda} = K^*_{\delta,\lambda}(\mu)$ (see part (5.) above). As the set $S(\mu_{\delta,\lambda})$ is finite, then there exists a polynomial $p = \sum\limits_{i=1}^{m} c_i \varphi^{b_i} \in A$ such that $\mu_{\delta,\lambda} = p \cdot \sigma$. The function

$$p(\alpha, x) = p \circ \Phi(\alpha, x) = \sum_{i=1}^{m} c_n \cdot \alpha(b_n) \cdot e^{ib_n x}$$

belongs to B. Therefore, for every fixed α $\frac{1+x^2}{(i+x)^2} p(\alpha, x) \in \mathcal{H}^1_0$ (see Section 2.2). Hence, from (7) we have $\psi(\mu_{\delta,\lambda}) \in \mathcal{H}^1_0(K \times \mathbb{R})$. A net of equibounded measures $\{\mu_{\delta,\lambda}\}$ converges weak* to the measure $\mu \in A^\perp$. Therefore, the net of measures $\{\psi(\mu_{\delta,\lambda})\}$ is also equibounded (Lemma 9) and converges weak* to $\psi(\mu) \in \mathcal{H}^1_0(K \times \mathbb{R})$. □

Let ξ be a representing measure of a point $(\alpha_0, z_0) \in K \times \mathbb{C}_0$ concentrated on $\mathbb{R}_{\alpha_0} = \{\alpha_0\} \times \mathbb{R}$. Then ξ is of the form of the Poisson kernel for the upper half–plane: $\xi = \frac{y_0 dx}{\pi((x-x_0)^2 + y_0^2)}$, $z_0 = x_0 + iy_0$. Denote by ξ_n a measure on $K \times \mathbb{R}$ which is obtained by shifting ξ by $(e_n, -n) \in \mathcal{F}$. Then the measure $\zeta = \sum_{-\infty}^{\infty} \xi_n$ is locally finite on $K \times \mathbb{R}$ and belongs to the space M of measures which are invariant under shifts by the elements of $\mathcal{F}$.

Lemma 11 *Let $\mu_s \in M(G)$ be a representing measure of a point $s = \Phi(\alpha_0, z_0) \in \Omega^0$. Then $\Phi^* \mu_s = \zeta$.*

Proof. Since $\Phi^*\mu_s \in M$ then it suffices to prove that $\zeta|_{F_0} = \Phi^*\mu_s|_{F_0}$, where $F_0 = K \times [0,1)$.

For $a \in \Gamma_+$ we have

$$\int_{F_0} \varphi^a \circ \Phi \, d\Phi^*\mu_s = \int_G \varphi^a \, d\mu_s = \varphi^a \circ \Phi(\alpha_0, z_0) = \alpha_0(a) e^{iaz_0}.$$

On the other hand

$$\int_{F_0} \varphi^a \circ \Phi \, d\zeta = \int_{K\times\mathbb{R}} \varphi^a \circ \Phi \, d \sum_{-\infty}^{\infty} \chi_{F_0} \cdot \xi_n,$$

where χ_{F_0} is a characteristic function of a set F_0. Since $\chi_{F_0} \cdot \xi_n = \chi_{F_n} \cdot \xi$ the last integral above is equal to $\alpha_0(a) \cdot e^{iaz_0}$. To complete the proof it remains to note that as the restriction of A to G is a Dirichlet algebra then each point $s \in \Omega^0$ has only one representing measure concentrated on G (see Section 1.1). □

Proof of Theorem 21. Denote $\tau(s) = \mu_s - \delta_s$, where δ_s is an atomary measure concentrated at the point $s \in \Omega^0$ and μ_s is a representing measure of s concentrated on G. Since for $f \in A$ $\delta_s(f) = f(s) = \int f d\mu_s$ then $\tau(s) \in A^\perp$. Let $s = \Phi(\alpha_0, z_0)$. Since the measure $\xi = \frac{y_0 dx}{\pi((x-x_0)^2+y_0^2)}$, $z_0 = x_0 + iy_0$, concentrated on $\mathbb{R}_{\alpha_0}$, is a representing measure of the point $(\alpha_0, z_0) \in K \times \mathbb{C}_+$ then $\theta = \xi - \delta_{(\alpha_0, z_0)} \in B^\perp$. Therefore, $\theta_n = \xi_n - \delta_{(\alpha_0 \cdot e_n, z_0 - n)} \in B^\perp$. Hence, $\psi(\tau(s)) = 1/(i+z)^2 \sum_{-\infty}^{\infty} \theta_n \in B^\perp$. Let μ be a probability measure on $\Delta^0(r,1) = G \times (r,1)$, $0 < r < 1$. Let us show that there exists a measure $\tilde{\mu}$ on $G = T_1$ such that $\mu - \tilde{\mu} \in A^\perp$. By Krein–Milman theorem there exists a net of measures $\{\sum c(s)\delta_s\}$ with $c(s) > 0, \sum c(s) = 1$, which converges weak* to μ. But A_G is a Dirichlet algebra. Therefore, the net of measures $\{\sum c(s)\mu_s\}$ converges weak* to some probability measure $\tilde{\mu}$ on G. Hence, the net of measures $\{\sum c(s)\tau(s)\} \subset A^\perp$ converges weak* to the measure $\tilde{\mu} - \mu \in A^\perp$. Since $\psi(\sum c(s)\tau(s)) \in B^\perp$ then $\psi(\tilde{\mu} - \mu) \in B^\perp$ (see Lemma 9). Now suppose $\mu \in A^\perp$ with $\operatorname{supp}\mu \subset \Omega(r,1)$ and $\mu_1 = \mu|_{\Omega^0}, \mu_2 = \mu|_G$. Then there exist probability measures $\gamma_1, ..., \gamma_4$ such that $\mu_1 = \sum_{i=1}^4 c_i\gamma_i$. Taking $\tilde{\mu}_1 = \sum_{i=1}^4 c_i\tilde{\gamma}_i$ we get that $\psi(\tilde{\mu}_1 - \mu_1) \in B^\perp$ and since $\mu = \mu_1 - \tilde{\mu}_1 + \mu_2 + \tilde{\mu}_1$ then $\mu_2 + \tilde{\mu}_1 \in A^\perp$ and $\psi(\mu_2 + \tilde{\mu}_1) \in B^\perp$ (see Lemma 10). Therefore, $\psi(\mu) \in B^\perp$.

2.4 Null sets and interpolation sets of the algebra A

In this section we establish a connection between thin sets, zero sets and interpolation sets of the algebra A.

Lemma 12 *Let $F \subset \Omega^0$ be a thin set. Then for any point $s_0 \in \Omega^0, s_0 \neq *$, there exist a neighbourhood $U \subset \Omega^0$ and a function $g \in A, g \not\equiv 0$, such that $F \cap U \subset N(g) \cap U$.*

Proof. Let $U = V \times W$ with $V \subset K, W \subset \mathbb{C}_0$, be a neighbourhood of a point $s_0 \in \Omega^0$ such that some function $f \in \mathcal{O}(U)$ vanishes on $F \cap U$ and satisfies the Definition 9. We can choose U and $f \in \mathcal{O}(U)$ such that the conditions of Lemma 6 will be satisfied as well. Then, by Lemma 6, there exists a function $h \in B$ with $N(h) \subset K \times \mathbb{C}_0$ such that $N(f) \cap U = N(h) \cap U$. By Lemma 7 there exists a function $b \in B$ such that $|b||_{K\times\mathbb{R}} = 1$ and $N(b) = \cup_{-\infty}^{\infty} N(h)_n$. Let $\mathcal{L}$ be a weak*–closed space of measures on G generated by the linear combinations of representing measures $\mu_s, s \in N(h)$, and by the measures from $A^{\perp}$. Since $\frac{b}{(i+z)^2} \cdot \Phi^*(\mu_s) = \sum_{-\infty}^{\infty} \frac{b}{(i+z)^2} \cdot \xi_n$ (see Lemma 11) and $\Phi^{-1}(s) \subset N(b)$ then $\frac{b}{(i+z)^2} \cdot \Phi^*(\mu_s) \in \mathcal{H}_0^1(K \times \mathbb{R}) \subset B^{\perp}$. By Lemma 10 we have that $\psi(M(G) \cap A^{\perp}) \subset \mathcal{H}_0^1(K \times \mathbb{R})$. Therefore, $\frac{b}{(i+z)^2} \cdot \Phi^*(\mathcal{L}) \subset \mathcal{H}_0^1(K \times \mathbb{R})$ (see Lemma 9). Hence, $\mathcal{L} \neq M(G)$ which means that there exists a function $g \in C(G), g \not\equiv 0$, which is orthogonal to $\mathcal{L}$. Since $A^{\perp} \subset \mathcal{L}$ and $\mu_s \in \mathcal{L}, s \in N(h)$, then $g \in A$ and $\int g\, d\mu_s = 0$. Thus, the function $g \in A$ vanishes on $N(h)$ and, in particular, on $F \cap U$. □

Let I be the set of all continuous functions on G that are orthogonal to $\mathcal{L}$. The above arguments imply that $I \subset A$.

Given an ideal J of the uniform algebra A the set $\bigcap_{f \in J} N(f)$ is called the hull of I and is denoted $\operatorname{hull} J$.

Lemma 13 *I is an ideal of the algebra A and $\operatorname{hull} I$ coincides with $N(h)$.*

Proof. From the definition of I we have that $I \subset J = \{f \in A, f(s) = 0, s \in N(h)\}$. But $J^{\perp} \supset \mathcal{L} = I^{\perp}$. Hence, $I = J$. Now suppose $s_0 \notin N(h)$. Since the function $b \in B$ vanishes only on the set $N(b) = \cup_{-\infty}^{\infty} N(h)_n$ (see the proof of the previous lemma) and $\Phi^{-1}(s) \cap N(b) = \varnothing$ then $\frac{b}{(i+z)^2} \cdot \Phi^*(\mu_{s_0}) \notin B^{\perp}$. Therefore, $\mu_{s_0} \notin \mathcal{L}$. Hence, there exists a function $f \in I$ such that $f(s_0) \neq 0$. □

Theorem 22 *Let $F \subset \Omega^0$ be a compact thin set such that $* \notin F$. Then the hull of an ideal $I = \{f \in A, f(s) = 0, s \in F\}$ is F.*

Proof. For a fixed $s \in F$ there exist an open neighbourhood $U \subset \Omega^0$ of s and a family of functions $\{f_\lambda\} \subset \mathcal{O}(U)$ such that $F \cap U = \bigcap_\lambda N(f_\lambda)$ (see Definition 9 and the subsequent remark). Let $\{h_\lambda\} \subset B$ be a family of functions such that $N(h_\lambda) \cap U = N(f_\lambda) \cap U$ (see the proof of Lemma 6). By Corollary 6 the family $\{h_\lambda\}$ can be chosen such that $\bigcap_\lambda N(h_\lambda) = \overline{\bigcap_\lambda N(f_\lambda)}$. If J is an ideal of A generated by the ideals $I_\lambda = \{f \in A, f(s) = 0, s \in N(h_\lambda)\}$ then $\operatorname{hull} J = \bigcap_\lambda N(h_\lambda) = \overline{F \cap U}$. Therefore, since F is compact, there exists a finite family $I_1, ..., I_n$ of ideals of the algebra A such that $F = \bigcup_{i=1}^{n} \operatorname{hull} I_i$. Hence, $\operatorname{hull} J = F$, where $J = I_1 \cdot I_2 \cdot ... \cdot I_n$. But $J \subset I = \{f \in A, f(s) = 0, s \in F\}$. Therefore, $\operatorname{hull} I = F$. □

Theorem 23 *Let $F \subset \Omega^0$ be a thin set such that $* \notin F$. Then there exists a non–zero function $f \in \mathcal{O}(\Omega^0)$ such that $F \subset N(f)$.*

Proof. For an increasing sequence $r_n \to 1$ denote $\Omega_n = \{s \in \Omega, |s| \leq r_n\}$, $\Omega(r_{n-1}, r_n) = G \times [r_{n-1}, r_n]$ and $F_n = F \cap \Omega(r_{n-1}, r_n)$. Each F_n is a compact thin set in Ω^0. Therefore, the hull of an ideal $I_n = \{f \in A, f(s) = 0, s \in F_n\}$ is F_n. Hence, the restriction of I_n

to Ω_{n-2} is dense in $P(\Omega_{n-2})$. Thus, there exists a function $f_n \in I_n, n = 2, 3, ...,$ such that $\sup_{\Omega_{n-2}} |1 - f_n| < 1/2^n$. Last inequality implies that $f = \prod_{n=2}^{\infty} f_n \in \mathcal{O}(\Omega^0)$. Obviously, $F \subset N(f)$. $\square$

Let us now describe the interpolation sets of the algebra A. Recall that a closed set $F \subset \Omega$ is called interpolation set if $A|_F = C(F)$.

Theorem 24 *Let $F \subset G$ be a compact set. Then the following conditions are equivalent:*

1. *F is a peak set for A;*
2. *$A|_F = C(F)$;*
3. *for each $\alpha \in K$ the Lebesgue measure of the set $F_\alpha = \{t \in [0, 1], \Phi(\alpha, t) \in F\}$ is zero.*

Proof. 1) $\Rightarrow$ 2) If F is a peak set for A then $\mu|_F = \mu_F \in A^\perp$ for any measure $\mu \in A^\perp$ (see Section 1.1). But $\operatorname{supp} \mu = G$. Therefore, $\mu_F = 0$. Hence, $A|_F = C(F)$. 2) $\Rightarrow$ 3) Let $\psi : A \to C(F)$ be the operator of restriction. Clearly, ψ is continuous. As $F \neq G$, then for any $\varepsilon > 0$ there exists a function $f \in A$ with $||f|| = \sup_G |f| = 1$ and $|f|_F| < \varepsilon$. Hence, from 2), using Banach–Steinhaus theorem, we get that the ideal $I = \{f \in A, f(s) = 0, s \in F\}$ of the algebra A is non–trivial. Since $\varphi^a \circ \Phi(\alpha, z) = \alpha(a)e^{iaz} \in B$ and A is generated by the family of functions $\{\varphi^a\}_{a \in \Gamma_+}$ then $A \circ \Phi \subset B$ and, therefore, $I \circ \Phi \subset B$. Now if for some $\alpha \in K$ the Lebesgue measure of a set F_α is not zero then, by Fatou's theorem (see [7], p. 127), $I \circ \Phi|_{\mathbb{R}_\alpha} \equiv 0$. But $\overline{\Phi(\mathbb{R}_\alpha)} = G$, and, therefore, $I = 0$, a contradiction. 3) $\Rightarrow$ 1) Suppose $\mu \in A^\perp$. Then the theorem of F. and M. Riesz (see [7], p. 127) and Lemma 10 imply that the restriction of the measure $\Phi \circ \mu$ to the set $E = \Phi^{-1}(F) \cap (K \times [0, 1))$ is a null measure. And since $K \times [0, 1)$ is a fundamental region of Φ we get that $\mu_F = 0$, i.e., $A|_F = C(F)$. $\square$

Let $F \subset K \times \mathbb{C}_+$ be a closed set. We say that F is an interpolation set for the algebra B if the restriction of B to F is closed in the uniform norm on F. The interpolation sets of the uniform algebras are described by the orthogonal measures. In particular, a closed set $F \subset K \times \mathbb{C}_+$ is an interpolation set for B if and only if there exists $d < 1$ such that $||\mu|_F|| < d||\mu||$ for any $\mu \in B^\perp$. The interpolation sets for an algebra A are defined similarly.

Theorem 25 *Suppose $F \subset \Omega(r, 1)$ is a compact set. The set F is an interpolation set for A if and only if $\Phi^{-1}(F)$ is an interpolation set for B.*

Proof. If F is not an interpolation set for A then for any $n > 0$ there exists $\mu \in A^\perp$ such that $||\mu_F|| \geq n||\mu_{\Omega \setminus F}||$. Hence, if we denote $E = \Phi^{-1}(F)$ and $E^0 = (K \times \mathbb{C}_+) \setminus E$, then, applying Lemma 9, we get $||\psi(\mu)|_E|| \geq n/2(1 - \log r)^2||\psi(\mu)|_{E^0}||$ and as $\psi(\mu) \in B^\perp$ then E is not an interpolation set for B.

Conversely, assume that the set $E = \Phi^{-1}(F)$ is not an interpolation set for B. Then for any $m > 0$ there exists a measure $\mu \in B^{\perp}$ such that $\operatorname{supp}\mu \subset \mathbb{C}_\alpha = \{\alpha\} \times \mathbb{C}, \alpha \in K$, and $||\mu_{E_\alpha}|| \geq m||\mu_{E^0_\alpha}||$, where $E_\alpha = \mathbb{C}_\alpha \cap E$ and $E^0_\alpha = \mathbb{C}_\alpha \cap E^0$. Let μ_n be a measure on $K \times \mathbb{C}_+$ obtained by shifting the measure μ by an element $(e_n, -n) \in \mathcal{F}$ and let $\xi = \cup_{-\infty}^{\infty} \mu_n$. Obviously, $\xi \in M$ and by Lemma 8 there exists $\zeta \in M_*$ such that $\xi = \Phi^*(\zeta)$. Since $\int_G f d\zeta = \int_{K\times\mathbb{C}_0} f \circ \Phi(\alpha, z) d\zeta = \int_{K\times\mathbb{C}_0} f \circ \Phi(\alpha, z) d\mu = 0$ then $\zeta \in A^{\perp}$. Furthermore,

$$||\zeta|_F|| = ||\xi|_{E\cap K\times\mathbb{C}_0}|| = ||\mu|_{E_\alpha}|| > m||\mu|_{E^0_\alpha}|| = m||\xi|_{(K\times\mathbb{C}_0)\setminus E}|| = m||\zeta|_{\Omega\setminus F}||$$

which means that F is not an interpolation set for A. □

2.5 Generalized meromorphic functions

Let $D \subset \Omega$ be an open set. For each point $s \in D$ the notion of an order of a function $f \in \mathcal{O}(D)$ at s is introduced in the following way: $\operatorname{ord} f(s) = \inf\{n \in \mathbb{Z}_+, f^{(n)}(s) = 0, f^{(n+1)}(s) \neq 0\}$, where $f^0 = 1$ and $f^{(n)} = \mathbb{D}^n(f)$ is the n-th derivative of f (see Section 1.5). From the definition of derivative it follows that if $U = V \times W \subset D, V \subset K, W \subset \mathbb{C}$, is a neighbourhood of a point $s = (\alpha_0, z_0)$ then $\operatorname{ord} f(s_0) = \operatorname{ord} f_{\alpha_0}(z_0)$, where $f_{\alpha_0}(z) = f(\alpha_0, z)$.

Definition 10 *An integer–valued, non–negative, continuous from above function $\partial(s)$ defined on an open neighbourhood D is called a divisor if there exists a thin set $F \subset D$ such that $\partial(s) = 0$ on $D \setminus F$.*

For example, the function $\operatorname{ord} f$ with $f \in \mathcal{O}(\Omega^0), f \not\equiv 0$, is a divisor.

Theorem 26 *Suppose $\partial(s)$ is a divisor on Ω^0 with $\partial(*) = 0$. Then there exists $f \in \mathcal{O}(\Omega^0), f \not\equiv 0$, such that* $\operatorname{ord} f(s) \geq \partial(s)$*.*

Proof. ¿From the Definition 10 and the condition of theorem we have that the set $F = \{s \in \Omega^0, \partial(s) > 0\}$ is a thin set in Ω^0 and $* \notin F$. For an increasing sequence $r_n \to 1$ denote $\Omega_n = \{s \in \Omega, |s| < r_n\}$, $\Omega(r_{n-1}, r_n) = G \times [r_{n-1}, r_n]$ and $F_n = F \cap \Omega(r_{n-1}, r_n)$. As F is a thin set in Ω^0 then F_n is a compact thin set in Ω^0 and the hull of an ideal $I_n = \{f \in A, f(s) = 0, s \in F_n\}$ is F_n. Denote $k_n = \sup_{F_n} |\partial(s)|$ and let $\{\varepsilon_n\}_1^\infty$ be a sequence of positive numbers such that $\prod_1^\infty (1+\varepsilon_n)^{k_n} < \infty$. Since for $n \in \mathbb{Z}_+$ the restriction of A to Ω_n is dense in $P(\Omega_n)$ and $F_{n+2} \cap \Omega_n = \varnothing$ then the restriction of I_{n+2} to Ω_n is also dense in $P(\Omega_n)$. Therefore, for any $n \in \mathbb{Z}_+$ there exists a function $f_n \in I_{n+2}$ such that $\sup_{\Omega_n} |1 - f_n| < \varepsilon_n$. Hence, the function $f = g \cdot \prod_1^\infty f_n^{k_n}, g \in I_1$, belongs to $\mathcal{O}(\Omega^0)$ and $\operatorname{ord} f(s) \geq \partial(s)$. □

We now give the definition of generalized meromorphic function.

Definition 11 *Let $D \subset \Omega^0$ be an open set. A function f will be called generalized meromorphic function (or just meromorphic function) on D if*

1. f is a generalized analytic function on $D \setminus F$, where F is a thin set in D,

2. f cannot be continued analytically to any point of F,

3. for any point $s \in F$ there exist a neighbourhood U of s and a function $g \in \mathcal{O}(\Omega^0)$ such that $N(g)$ is a thin set in U and $f \cdot g$ can be continued to some function from $\mathcal{O}(U)$.

Let f be a meromorphic function on Ω^0. For $s \in \Omega^0$ let $\mathcal{O}(s)$ be a germ of generalized analytic functions at s, i.e., $\mathcal{O}(s)$ consists of the functions that are analytic on some neighbourhood of s, and let $\mathcal{O}(s,f)$ be the set of all functions $g \in \mathcal{O}(s)$ such that $f \cdot g$ can be extended to a generalized analytic function from $\mathcal{O}(s)$. Define a divisor $\partial_f^-(s) = \inf_{g \in \mathcal{O}(s,f)} \operatorname{ord} g(s)$ which represents the poles of a function f. If $\partial_f^-(s) = 0$ then, obviously, $f \in \mathcal{O}(s)$.

Theorem 27 *Let f be a generalized meromorphic function on Ω^0 such that $* \in \Omega^0$ is either a removable singularity or an isolated pole of f. Then there exists a function $g \in \mathcal{O}(\Omega^0), g \not\equiv 0$, such that $f \cdot g$ can be extended to some function from $\mathcal{O}(\Omega^0)$.*

Proof. Assume first that $* \in \Omega^0$ is a removable singularity for a meromorphic function f, i.e., $\partial_f^-(*) = 0$. By Theorem 26 there exists a function $g \in \mathcal{O}(\Omega^0)$ such that $\partial_f^-(s) \leq \operatorname{ord} g(s), s \in \Omega^0$. Let us show that $g \in \mathcal{O}(s,f)$ for all $s \in \Omega^0$. Indeed, let $U = V \times W \subset \Omega^0$, with $V \subset K$ and $W \subset \mathbb{C}_0$ be a neighbourhood of a point $s_0 \in F = \{s \in \Omega^0, \partial_f^-(s) = 0\}$. For each fixed $\alpha \in V$ the restriction of a function f to $W_\alpha = \{\alpha\} \times W$ is a meromorphic function which orders of poles do not exceed $\partial_f^-(s)$. Therefore, $f \cdot g$ can be analytically continued to W_α. Denote by ψ the function on U obtained by such continuations. Obviously, $\psi \in \mathcal{O}(U \setminus F)$ and, by the maximum principle for the functions of one complex variable, we have that $\sup_{V \times W} |\psi| = \sup_{V \times \partial W} |\psi|$. The sets V and W can be chosen such that the functions f and g would be bounded. Hence, ψ is a bounded function on $U \setminus F$. By Theorem 18 $\psi \in \mathcal{O}(U)$. Therefore, $g \in \mathcal{O}(s,f), s \in \Omega^0$, and $f \cdot g$ can be extended to some function from $\mathcal{O}(\Omega^0)$.

Suppose now that $* \in \Omega^0$ is an isolated pole of f. Then the function $1/f$ is bounded on the set $\Omega_r \setminus *$ where $\Omega_r = \{s \in \Omega, |s| < r\}$. By Theorem 17 the function $1/f$ can be extended to a function from $\mathcal{O}(\Omega_r^0)$ which vanishes at $*$. By Theorem 4 there exist $a \in \Gamma_+$ and $g \in \mathcal{O}(\Omega_r), g(*) \neq 0$, such that $1/f = \varphi^a \cdot g$. Therefore, $* \in \Omega^0$ is a removable singularity for a meromorphic function $f \cdot \varphi^a$.

Since each function $f \in \mathcal{O}(\Omega^0(r_1,r_2))$, $\Omega^0(r_1,r_2) = G \times (r_1,r_2)$, can be represented in the form of formal series $\sum_{a \in \Gamma} c_a(f)\varphi^a$ then the Frechet algebra $\mathcal{O}(\Omega^0(r_1,r_2))$ contains two following subalgebras: the algebra of functions whose spectrum lies in Γ_+ and the algebra of functions whose spectrum lies in $-\Gamma_+$. First subalgebra is implemented by the algebra $\mathcal{O}(\Omega_{r_2})$ and the second is implemented by $\widetilde{\mathcal{O}}(\widetilde{\Omega}_{r_1})$ which is the algebra of functions $f \in \mathcal{O}(\widetilde{\Omega}_{r_1})$, with $\widetilde{\Omega}_{r_1} = \Delta \setminus \Omega_{r_1}$, for which $\lim_{|s| \to \infty} |f(s)|$ is finite. Obviously, the Frechet algebras $\mathcal{O}(\Omega_{r_2})$ and $\widetilde{\mathcal{O}}(\widetilde{\Omega}_{r_1})$ are isomorphic. □

Theorem 28 *Let f be a generalized meromorphic function on $\Omega^0(r_1, r_2)$. Then there exists a generalized analytic function $g \in \mathcal{O}(\Omega^0(r_1, r_2)), g \not\equiv 0$, such that $f \cdot g$ can be extended to a function from $\mathcal{O}(\Omega^0(r_1, r_2))$.*

Proof. As the function $\partial_f^-(s)$ is a divisor on $\Omega^0(r_1, r_2)$ then the functions

$$\psi_1(s) = \begin{cases} 0 & \text{if } s \in \Omega^0_{r_0}, r_1 < r_0 < r_2, \\ \partial_f^-(s) & \text{if } s \in \Omega^0(r_1, r_2) \setminus \Omega^0_{r_0} \end{cases} \quad \text{and} \quad \psi_2(s) = \begin{cases} 0 & \text{if } s \in \Omega^0(r_1, r_2) \setminus \Omega^0_{r_0}, \\ \partial_f^-(s) & \text{if } s \in \Omega^0_{r_0} \end{cases}$$

are divisors on $\Omega^0_{r_2}$ and $\widetilde{\Omega}_{r_1}$ respectively. Therefore, there exist the functions $\varphi_1 \in \mathcal{O}(\Omega^0_{r_2})$ and $\varphi_2 \in \widetilde{\mathcal{O}}(\widetilde{\Omega}^0_{r_1})$ such that $\psi_1 \leq \operatorname{ord} \varphi_i, i = 1, 2$. Hence, the function $f \cdot \varphi_1 \cdot \varphi_2$ can be extended to a function from $\mathcal{O}(\Omega^0(r_1, r_2))$. $\square$

Theorem 29 *Let $f, g \in \mathcal{O}(\Omega^0(r_1, r_2))$ be such that $\operatorname{ord} g(s) \leq \operatorname{ord} f(s), s \in \Omega^0(r_1, r_2)$. Then the meromorphic function f/g can be uniquely extended to some function from $\mathcal{O}(\Omega^0(r_1, r_2))$.*

Proof. ¿From the condition of the theorem we have that the set $F = \{s \in \Omega^0(r_1, r_2), \operatorname{ord} g(s) \geq 1\}$ is a thin set in $\Omega^0(r_1, r_2)$ and each point $s \in F$ has a neighbourhood $U = V \times W$ such that f/g is bounded on $U \setminus F$. Therefore, f/g can be extended to some function from $\mathcal{O}(U)$. Hence, f/g can be extended to a function from $\mathcal{O}(\Omega^0(r_1, r_2))$. $\square$

3 Bohr-Riemann surfaces

3.1 Unbranched coverings of the generalized plane

In this part of the work we develop the theory of Bohr–Riemann surfaces. Recall that a mapping $\pi : Y \to X$ between two topological spaces Y and X is called (in general, *branched*) *covering* if it is continuous, open and discrete, i.e. for each $x \in X$ the set $\pi^{-1}(x)$ is a discrete set in Y (see [17], p. 25). A mapping $\pi : Y \to X$ between topological spaces Y and X is called *unbranched covering* if each point $x \in X$ has a (so called evenly–covered) neighbourhood $U \ni x$ such that

$$\pi^{-1}(U) = \bigcup_{i \in \mathcal{A}} U_i$$

is a disjoint union of open sets in Y, and each restriction $\pi|_{U_i} : U_i \to U, i \in \mathcal{A}$ is a homeomorphism. If the set $\mathcal{A}$ is finite then π is called an unbranched, finite-sheeted (or n-fold, where $n = \operatorname{card} \mathcal{A}$) covering.

Definition 12 *Topological space X is called a Bohr-Riemann surface over the generalized plane Δ if there exist a thin set $K \subset \Delta$ and a covering $\pi : X \to \Delta$ such that the restriction of π to the set $X^* = X \backslash \pi^{-1}(K)$ is an unbranched, finite-sheeted covering of the set $\Delta^* = \Delta \backslash K$.*

Note that as the set $K = \{*\}$ is obviously a thin set then the existence of a covering $\pi : X \to \Delta$ such that the restriction of π to the set $X^* = X \backslash \pi^{-1}(*)$ is an unbranched, finite-sheeted covering of Δ^0 implies that the space X is a Bohr-Riemann surface over Δ.

The above definition can be extended to open subsets of Δ as follows.

Definition 13 *Let D be an open subset of Δ. Topological space X is called a Bohr-Riemann surface over D if there exist a thin set $K \subset D$ and a covering $\pi : X \to D$ such that the restriction of π to the set $X^* = X \backslash \pi^{-1}(K)$ is an unbranched, finite-sheeted covering of the set $D^* = D \backslash K$.*

We now turn to a study of main properties of unbranched coverings of Δ^0. Let us first define the notion of a cylindrical neighbourhood of a continuous path in Δ^0.

Definition 14 *Let $\gamma : I \to \Delta^0$ be a continuous mapping that determines a continuous path $\gamma(I)$ in Δ^0 and let U be an arbitrary neighbourhood of the unit element $e_0 \in \Delta^0$. Then an open set $W = U \cdot \gamma(I)$ will be called cylindrical neighbourhood of a path $\gamma(I)$ or just a cylinder. The sets $U \cdot \gamma(0)$ and $U \cdot \gamma(1)$ will be called, respectively, the beginning and the ending of the cylinder W.*

The above definition implies that if W is a cylindrical neighbourhood of a path $\gamma(I)$ then for any $s \in U$, W is a cylindrical neighbourhood of the path $\gamma_s(I)$ as well, where $\gamma_s(t) = s \cdot \gamma(t), t \in I$.

Now suppose that $\pi : X^0 \to \Delta^0$ is an unbranched, n–fold covering and $\gamma(I) \subset \Delta^0$ is some non– self–intersecting path with $\gamma(0) \neq \gamma(1)$. By path lifting theorem there exist n non–intersecting paths

$$\hat{\gamma}_i(I) \subset X^0, i = 1, ..., n,$$

which cover the path $\gamma(I)$, that is, $\gamma_i(I) = \pi \circ \hat{\gamma}_i(I), i = 1, ..., n$ (see [17], §4). Moreover, let us show that each path $\hat{\gamma}_i(I), i = 1, ..., n$, has a neighbourhood $W_i \supset \hat{\gamma}_i(I)$ such that

$$\pi(W_i) = \pi(W_j), 1 \leq i, j \leq n,$$

and the restriction of π to W_i is a homeomorphism between W_i and the set $V = \pi(W_i)$. Indeed, as X^0 is a Hausdorff space then for compact sets $\hat{\gamma}_i(I)$ there exist mutually disjoint open sets $\hat{W}_i \subset X^0$ such that $\hat{\gamma}_i(I) \subset \hat{W}_i$, $i = 1, ..., n$ (see [9], p. 197). Then, obviously, the path $\gamma(I)$ lies in each $\pi(\hat{W}_i)$ and, therefore, $\gamma(I)$ lies in the intersection $\cap_{i=1}^{n} \pi(\hat{W}_i) := V$. Denote $W_i = \hat{W}_i \cap \pi^{-1}(V)$. Then for $i = 1, ..., n$, we have that $\pi(W_i) = \pi(\hat{W}_i \cap \pi^{-1}(V)) \subset \pi(\pi^{-1}(V)) = V$. Let us now show the converse inclusion. Assume $g \in V$ then from the construction of V it follows that there exists $x_i \in \hat{W}_i$ such that $\pi(x_i) = g$, that is, $x_i \in \pi^{-1}(g) \subset \pi^{-1}(V)$ and $g = \pi(x_i) \in \pi(\hat{W}_i \cap \pi^{-1}(V)) = \pi(W_i)$. Hence, $\pi(W_i) = V, i = 1, .., n$. Since the sets $\hat{W}_i$ (and, therefore, the sets W_i as well) are disjoint, then again from the construction of V it follows that for each $i, 1 \leq i \leq n$, the mapping $\pi|_{W_i} : W_i \to V$ is a bijection (surjectivity and injectivity of the restriction of an n–fold covering π to each of the sets $W_i, i = 1,..,n$, immediately follow from the established equalities $\pi(W_i) = V, i = 1,..,n$, and $\cap_{i=1}^{n} W_i = \varnothing$) and since π is open then it is a homeomorphism (see [9], pp. 64-65).

Using the compactness of $\gamma(I)$ it can be shown that each neighbourhood of the path $\gamma(I)$ contains cylindrical neighbourhood of $\gamma(I)$. Clearly, if $U \subset V$ is a cylindrical neighbourhood of the path $\gamma(I)$ then $W^i := W_i \cap \pi^{-1}(U)$ is a neighbourhood of the path $\hat{\gamma}_i(I)$, which will be called cylindrical neighbourhood as well. Thus, the following lemma holds.

Lemma 14 *Let $\gamma(I) \subset \Delta^0$ be a non-self-intersecting path with $\gamma(0) \neq \gamma(1)$. Then $\gamma(I)$ has a cylindrical neighbourhood U such that the set $W = \pi^{-1}(U)$ is representable as a disjoint union*

$$W = \bigsqcup_{i=1}^{n} W_i$$

of cylindrical neighbourhoods W_i of the paths $\hat{\gamma}_i(I)$, where each W_i is homeomorphic to the set U.

The above lemma and the continuity of π imply that if $\gamma'(I)$ is a path in U then each lifting of $\gamma'(I)$ is contained in some W_i, $1 \leq i \leq n$.

Next theorem presents a method of defining a group structure on the covering spaces of the group Δ^0 and describes the structures of the obtained groups up to a topological isomorphism.

Theorem 30 *Let $\pi : X^0 \to \Delta^0$ be an n-fold, unbranched covering of the punctured generalized plane Δ^0 by a connected topological space X^0. Then there can be defined a group structure on X^0 turning π into a group homomorphsim between X^0 and Δ^0. The group X^0 is then topologically isomorphic to the Cartesian product $G_1 \times (0, +\infty)$ where G_1 is a compact subgroup of X^0.*

Proof. Assume $(x, t) \in X^0 \times (0, +\infty)$ with $t > 1$. Consider a path $\pi(x) \cdot \xi$, $\xi \in [1, t]$, in Δ^0. Let $\widehat{\pi(x) \cdot \xi}$, $\xi \in [1, t]$, be a path in X^0 starting at the point x and lifting $\pi(x) \cdot \xi$, i.e., $\widehat{\pi(x)} = x$ and $\pi(x) \cdot \xi = \pi \circ \widehat{\pi(x) \cdot \xi}$, $\xi \in [1, t]$. Define $T_1(x) = \widehat{\pi(x)} = x$ and $T_t(x) = \widehat{\pi(x)t}$ – the end–point of the path $\widehat{\pi(x) \cdot \xi}$, $\xi \in [1, t]$. Then

$$\pi(x)t = \pi \circ T_t(x). \tag{8}$$

For $t \in (0, 1)$ the path $\widehat{\pi(x) \cdot \xi}$ and, therefore, the function $T_t(x)$ is defined similarly by considering the liftings of a path $\pi(x) \cdot \xi$, $\xi \in [t, 1]$. Let us consider the mapping

$$T : X^0 \times (0, +\infty) \to X^0 : (x, t) \mapsto T_t(x).$$

Later on we will indicate the sheet of the pre–image of the neighbourhood of a point $\pi(x)t \in \Delta^0$ under the n–fold covering $\pi : X^0 \to \Delta^0$ that we are interested in, and, therefore, from

(8) using the homeomorphism which is locally inverse to π we will get more explicit form of the function $T_t(x) = \widehat{\pi(x)t}$.

Define $G_1 = \pi^{-1}(G) \subset X^0$ and consider the restriction $p = T|_{G_1\times(0,+\infty)}$, i.e., the mapping

$$p : G_1 \times (0, +\infty) \to X^0 : p(u, r) = T_r(u).$$

Note that from the definition of the set $G_1 \ni u$ we have $\pi(u) \in G$ which means that $|\pi(u)| = 1$, and, therefore, using (8) we get that $|\pi(T_r(u))| = |\pi(u)r| = r$. It is easy to check that last equality implies that the inverse of the mapping p has the following form

$$q : X^0 \to G_1 \times (0, +\infty) : q(x) = (T_{|\pi(x)|^{-1}}(x), |\pi(x)|).$$

Let us show that p is a homeomorphism from $G_1 \times (0, +\infty)$ to X^0. Fix an arbitrary $(x, t) \in G_1 \times (0, +\infty)$ with $t > 1$. Then, by Lemma 14, the path $\pi(x) \cdot \xi$, $\xi \in [1, t]$, has a cylindrical neighbourhood $U = U_0 \cdot \pi(x) \cdot \xi$, $\xi \in [1, t]$, such that $\pi^{-1}(U) = \bigsqcup_{i=1}^{n} W_i$, where each cylindrical neighbourhood W_i is homeomorphic to U (U_0 is a neighbourhood of the unit element $e_0 \in \Delta^0$ which determines the cylindrical neighbourhood U). Denote the set W_i that contains the path $\widehat{\pi(x)\cdot\xi}$, $\xi \in [1, t]$, by W. Then $\widehat{\pi(x)\cdot\xi} \subset W$ and $\pi : W \to U$ is a homeomorphism. Also, denote $\phi : U \to W$ the inverse of $\pi : W \to U$: $\phi \circ \pi = \mathrm{id}_W$. By definition of the topology $\tau_{\Delta^0} \cong k \times \tau_{(0,+\infty)}$ on Δ^0 there exist a neighbourhood $\tilde{U} \in k$ of the unit element e_0 in G and a number $\delta > 0$ such that the neighbourhood $\rho(\tilde{U} \times (e^{-\delta}, e^{\delta})) = \tilde{U}(e^{-\delta}, e^{\delta})$ of e_0 in Δ^0 lies in U_0, where ρ is the natural topological isomorphism between Δ^0 and the space $\{\alpha \cdot r : \alpha \in G, r \in (0, \infty)\}$.

Therefore, $\rho(\pi(x) \cdot \tilde{U} \times (te^{-\delta}, te^{\delta})) = \pi(x) \cdot \tilde{U} \cdot (te^{-\delta}, te^{\delta}) \subset \pi(x) \cdot U_0 \cdot t \subset U$, i.e., the set $\pi(x) \cdot \tilde{U} \cdot (te^{-\delta}, te^{\delta})$ lies in a domain of the homeomorphism $\phi : U \to W$. The constructions above imply that $(x, t) \in \phi(\pi(x)\tilde{U}) \times (te^{-\delta}, te^{\delta})$, that is, $\phi(\pi(x)\tilde{U}) \times (te^{-\delta}, te^{\delta})$ is a neighbourhood of the point (x, t) in $X^0 \times (0, +\infty)$. Let us now investigate the mapping p on that neighbourhood. Suppose that $\tilde{u} \in \tilde{U}$ and $r \in (te^{-\delta}, te^{\delta})$. Then, from the definition of a mapping p, $p(\phi(\pi(x)\tilde{u}), r) = T_r(\phi(\pi(x)\tilde{u}))$. From (8) we have that $\pi \circ T_r(\phi(\pi(x)\tilde{u})) = \pi(\phi(\pi(x)\tilde{u}))r = \pi(x)\tilde{u}r$. Since $\pi(x)\tilde{u}r \in \pi(x) \cdot \tilde{U} \cdot (te^{-\delta}, te^{\delta}) \subset U$ then we can apply $\phi = \pi^{-1}$ to the equation derived above, and, thus, we get: $T_r(\phi(\pi(x)\tilde{u})) = \phi(\pi(x)\tilde{u}r)$. Therefore, p maps the set $\phi(\pi(x) \cdot \tilde{U}) \times (te^{-\delta}, te^{\delta})$, which is a neighbourhood of the point (x, t), to $\phi(\pi(x)\cdot\tilde{U}\cdot(te^{-\delta}, te^{\delta}))$. Since (x, t) was arbitrary and p is bijective then the desired conclusion is established by the following local factorization of p:

$$\begin{array}{ccc} \phi(\pi(x)\tilde{U}) \times (te^{-\delta}, te^{\delta}) & \xrightarrow{p} & \phi(\pi(x)\tilde{U}(te^{-\delta}, te^{\delta})) \\ \Big\downarrow{\scriptstyle \pi \times id_{\mathbb{R}}} & & \Big\uparrow{\scriptstyle \phi} \\ (\pi(x)\tilde{U}) \times (te^{-\delta}, te^{\delta}) & \xrightarrow{\rho} & \pi(x)\tilde{U}(te^{-\delta}, te^{\delta}) \end{array}$$

where all the mappings composing p are continuous and open. The case $t \in (0, 1]$ is considered similarly.

Thus, p is a homeomorphism. Since G is compact and π is open and continuous then G_1 is also compact. Let us show that G_1 is connected. Indeed, as X^0 is connected and p is a homeomoprhsm (and, therefore, so is q), then $G_1 \times (0, +\infty) = q(X^0)$ is connected which means that G_1 is connected as well. Thus, for the restriction $\varphi = \pi|_{G_1} : G_1 \to G$ the conditions of the Theorem from [18] are satisfied. By that theorem there can be defined a group structure on G_1 turning $\varphi = \pi|_{G_1}$ into a group homomorphism from G_1 to G. Clearly, having a group structure on G_1 we can define a group structure on $G_1 \times (0, +\infty)$ as well. Denote by '$\odot$' the operation of multiplication in the group $G_1 \times (0, +\infty)$. Now define a multiplication in X^0 as follows:

$$x_1 x_2 := p(q(x_1) \odot q(x_2)),$$

for $x_1, x_2 \in X^0$. Then it is easy to check that X^0 becomes a group with unit element being $p(e, 1)$, where e is the unit element of G_1, and $x^{-1} := p(q(x)^{-1})$ being an inverse of the element $x \in X^0$. It follows from definition of multiplication that

$$p(q(x_1))p(q(x_2)) = x_1 x_2 = p(q(x_1) \odot q(x_2)),$$

for $x_1, x_2 \in X^0$, that is, p is a homomorphism, which means that X^0 is topologically isomorphic to $G_1 \times (0, +\infty)$. Finally, let us show that π is a homomorphism. Let $x \in X^0$ and $x = p(\xi, r), \xi \in G_1, r \in (0, +\infty)$. Then from (8) we get $\pi(x) = \pi(p(\xi, r)) = \pi(T_r(\xi)) = \pi(\xi)r$, hence, for $x_1, x_2 \in X^0$ with $x_1 = p(\xi_1, r_1), x_2 = p(\xi_2, r_2)$, since p and the restriction $\pi|_{G_1}$ are homomorphisms, we get that

$$\pi(x_1 x_2) = \pi(p(\xi_1, r_1) \cdot p(\xi_2, r_2)) = \pi(p(\xi_1 \xi_2, r_1 r_2)) = \pi(\xi_1 \xi_2) r_1 r_2 =$$

$$= \pi(\xi_1) r_1 \pi(\xi_2) r_2 = \pi(x_1)\pi(x_2).$$

Thus π is a group homomorphism. The theorem is proved.

□

3.2 Algebraic coverings

Definition 15 *Bohr–Riemann surface X is called algebraic over an open subset $D \subset \Delta$ if there exist polynomials $P_j(x, s) = x^{k_j} + f_{1,j}(s)x^{k_j - 1} + ... + f_{k_j,j}(s)$ with $f_{m,j} \in \mathcal{O}(D)$, $1 \le m \le k_j$, $j = \overline{1, N}$, such that X is homeomorphic to the subspace $X_N = \{(s, z_1, ..., z_N) \in D \times \mathbb{C}^N : P_j(z_j, s) = 0, 1 \le j \le N\}$.*

We don't know whether each Bohr–Riemann surface is algebraic but we claim that the unbranched coverings over the punctured generalized plane Δ^0 are algebraic.

Lemma 15 *Suppose that a group homomorphism $\varphi : G_1 \to G$ implements an unbranched n-fold covering of a (compact solenoidal) group G by a connected group G_1. Then the group G_1*

is commutative and there exist $a_1, ..., a_m \in \Gamma_+$ *and* $n_1, ..., n_m \in \mathbb{N}$*, such that* G_1 *is isomorphic to the algebraic covering of a group* G *by the algebraic equations*

$$y_i^{n_i} - \chi^{a_i} = 0, i = \overline{1, m.}$$

Proof. By definition of an unbranched covering there exist the neighbourhoods $V \ni e_0$ and $W \ni e$ of the unit elements of the groups G and G_1 respectively, such that on W φ is a homeomorphism between W and V. Let $W_0 \subset G_1$ be a neighbourhood of e such that $W_0^2 \subset W$. Then, as $G \subset \Delta$ is a commutative group, for any $\alpha, \beta \in W_0$ we have $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta) = \varphi(\beta)\varphi(\alpha) = \varphi(\beta\alpha)$, therefore, since φ is homeomorphic and, hence, is injective on the set $W \supset W_0^2 \ni \alpha\beta, \beta\alpha$ then $\alpha\beta = \beta\alpha$. The connectivity of G_1 implies the representation $G_1 = \cup_{n=1}^{\infty} W_0^n$ which allows to extend the commutativity to the whole group G_1. In particular, this means that the group $K = \ker\varphi$ is a finite abelian group and since φ is an n–fold covering we have $|K| = n$. Denote by $\tilde{\varphi} : \hat{G} \to \hat{G}_1$ the dual mapping of φ, where $\hat{G}$ and $\hat{G}_1$ are the groups of characters of G and G_1 respectively. From definition of dual mapping we have that $\tilde{\varphi}(\hat{g}) = \hat{g} \circ \varphi \in \hat{G}_1$ for each $\hat{g} \in \hat{G}$. For $k \in K$ we have that $\hat{g} \circ \varphi(k) = \hat{g}(e_0) = 1$, which implies that the image $\tilde{\varphi}(\hat{G})$ under the dual mapping $\tilde{\varphi} : \hat{G} \to \hat{G}_1$ has the following form

$$\tilde{\varphi}(\hat{G}) = \{\chi \in \hat{G}_1 : \chi(K) = 1\}. \tag{9}$$

The surjectivity of φ implies that the mapping $\tilde{\varphi}$ is injective and, therefore, $\tilde{\varphi}$ is a topological isomorphism from $\hat{G}$ to a subgroup $\{\chi \in \hat{G}_1 : \chi(K) = 1\}$ of a group $\hat{G}_1$ ($\tilde{\varphi}$ inherits continuity and openness from φ (see [20], p.498)). Consider the homomorphisms $e : \mathbb{R} \to G : e(t) = e_t$ and $\sigma : e(\mathbb{R}) \to G_1 : \sigma(e(t)) = T_t(e)$, where $T_t(e) = \hat{e}(t)$ is a path in G_1 which starts at the point e and lifts the path $e(t) = e_t, t \in \mathbb{R}$, in G. Denote $\mathbb{R}_e = \{T_t(e) : t \in \mathbb{R}\} = \sigma(e(\mathbb{R})) \subset G_1$. Since $\overline{\mathbb{R}_e} = G_1$ (see [18]), then the image of the homomorphism $\kappa = \sigma \circ e : \mathbb{R} \to G_1$ is dense in G_1: $\overline{\kappa(\mathbb{R})} = G_1$, and, therefore, dual mapping $\tilde{\kappa} : \hat{G}_1 \to \hat{\mathbb{R}} \cong \mathbb{R}$ is injective. Denote further $\Gamma_1 := \tilde{\kappa}(\hat{G}_1)$. Then Γ_1 is an algebraic subgroup of $\mathbb{R}$ and $\Gamma_1 \cong \hat{G}_1$ is an algebraic isomorphism and, hence, it is a topological isomorphism of discrete groups which means that $\hat{\Gamma}_1 \cong G_1$ is a topological isomorphism of compact groups. The above arguments together with (9) and the fact that G is a group of characters of the group Γ imply the following equality

$$\Gamma = \{b \in \Gamma_1 : \chi^b(K) = 1\}, \tag{10}$$

where $b \mapsto \chi^b$ is the standard parametrization of the groups $\hat{G} \tilde{\subset} \hat{G}_1$ by the groups $\Gamma \subset \Gamma_1$ which follows from the Pontryagin duality theorem. Let us find the explicit form of Γ_1. Finite abelian group K is isomorphic to the direct product $K_1 ... K_m$ where each group K_i is a cyclic group of order $n_i, i = \overline{1, m}$ (obviously, $n_1 n_2 ... n_m = n$). It is known that the group of characters $\hat{K}_i$ is also cyclic group of the same order: $\hat{K}_i = \{\hat{e}, \gamma_i, ..., \gamma_i^{n_i - 1}\}$, $i = \overline{1, m}$, where $\hat{e}$

is the unit element of a group $\hat{G}_1$. Since K is compact, each character from $\hat{K} \cong \hat{K}_1 ... \hat{K}_m$ can be extended to a character of a group G_1 (see [8], p. 56). Therefore, for every $i, 1 \leq i \leq m$, there exists $c_i \in \Gamma_{1+}$ such that $\chi^{c_i}|_{K_i} = \gamma_i$ and $\chi^{c_i}|_{K_j} = \hat{e}_1, j \neq i, 1 \leq i, j \leq m$. Then for $0 < k < n_i$ we have that $\chi^{kc_i}(K) \neq 1$, hence, from (10) we get that $kc_i \notin \Gamma$, $0 < k < n_i$, and for $a_i := n_i c_i$ we have $\chi^{a_i}(K) = 1$ which means that $a_i \in \Gamma_+$. Thus, presented arguments imply that the group Γ_1 has the following form:

$$\Gamma_1 = \{k_1 c_1 + ... + k_m c_m + d : d \in \Gamma, 0 \leq k_i < n_i, i = 1, ..., m\}, \tag{11}$$

and, again by definition of $c_i, i = \overline{1, m}$, we have $k_1 c_1 + ... + k_m c_m \notin \Gamma$. To construct the required algebraic covering isomorphic to G_1 let us consider the set

$$G_0 = \{(\alpha, z_1, ..., z_m) \in G \times \mathbb{T}^m : z_i^{n_i} = \chi^{a_i}(\alpha), i = 1, ..., m\},$$

where $\mathbb{T}$ is the unit circle of the complex plane. Then G_0 is a compact abelian group under the coordinate–wise multiplication

$$(\alpha, z_1, ..., z_m)(\beta, w_1, ..., w_m) = (\alpha\beta, z_1 w_1, ..., z_m w_m).$$

A mapping $\psi : G_0 \to G : (\alpha, z_1, ..., z_m) \mapsto \alpha$ is a group homomorphism which implements an n–fold algebraic covering of a group G. Indeed, the homomorphity of ψ directly follows from the definition of multiplication on G_0, and since for given α the equation $z_i^{n_i} = \chi^{a_i}$ has precisely n_i solutions, $i = 1, ..., m$, and $n_1 n_2 ... n_m = n$ we get that ψ is an n–fold covering. Thus, it remains to prove that $G_1 \cong G_0$. Let us find a dual group of G_0. The group G_0 is a subgroup of the Cartesian product $G \times \mathbb{T}^m$. By Theorem 54 from [6] (p.283), we have

$$\hat{G}_0 \cong \widehat{G \times \mathbb{T}^m} / A(\widehat{G \times \mathbb{T}^m}, G_0), \tag{12}$$

where $A(\widehat{G \times \mathbb{T}^m}, G_0) = \{\chi \in \widehat{G \times \mathbb{T}^m} : \chi(G_0) = 1\}$ is an annihilator of G_0 in a group $\widehat{G \times \mathbb{T}^m} \cong \Gamma \times \mathbb{Z}^m$. Note that since G_0 is compact each character from $\hat{G}_0$ can be continued to a character from $\widehat{G \times \mathbb{T}^m}$, hence, we may assume that $\hat{G}_0 \subset \Gamma \times \mathbb{Z}^m$. Using the topological isomorphism of duality $\Gamma \times \mathbb{Z}^m \cong \widehat{G \times \mathbb{T}^m}$ the action of a group $\Gamma \times \mathbb{Z}^m$ on $\widehat{G \times \mathbb{T}^m}$ can be described in the following way:

$$\Gamma \times \mathbb{Z}^m \ni (c, q_1, ..., q_m) \mapsto [\chi^{(c, q_1, ..., q_m)} : (\alpha, z_1, ..., z_m) \mapsto \chi^c(\alpha) z_m^{q_m} ... z_1^{q_1}] \in \widehat{G \times \mathbb{T}^m}.$$

Using this description, since $z_i^{n_i} = \chi^{a_i}(\alpha) \Leftrightarrow \chi^{-a_i}(\alpha) z_i^{n_i} = 1, i = \overline{1, m}$, the group G_0 is represented in the following form:

$$G_0 = \{(\alpha, z_1, ..., z_m) \in G \times \mathbb{T}^m : \chi_i(\alpha, z_1, ..., z_m) = 1, i = 1, ..., m\}, \tag{13}$$

where $\chi_i := \chi^{(-a_i, 0, ..., 0, n_i, 0, ..., 0)}, i = \overline{1, m}$. We use this representation to describe the annihilator $A(\widehat{G \times \mathbb{T}^m}, G_0)$. We claim that the characters $\chi_1, ..., \chi_m$ form the system of generators of the annihilator, that is,

$$A(\widehat{G \times \mathbb{T}^m}, G_0) = \{\chi_1^{p_1} \cdots \chi_m^{p_m} : p_1, ..., p_m \in \mathbb{Z}\}. \tag{14}$$

Indeed, the inclusion "$\supset$" follows from (13), and to prove the inclusion "$\subset$" suppose that

$$\chi^{(c,q_1,...,q_m)}(G_0) = 1. \tag{15}$$

Representing $q_i = p_i n_i + k_i, 0 \le k_i < n_i, i = \overline{1,m}$, we get

$$\chi^{(c,q_1,...,q_m)} = \chi^{(c,p_1n_1+k_1,...,p_mn_m+k_m)} = \chi^{(c+p_1a_1-p_1a_1+...+p_ma_m-p_ma_m,p_1n_1+k_1,...,p_mn_m+k_m)} =$$

$$= \chi^{(c+p_1a_1+...+p_ma_m,k_1,...,k_m)}\chi^{(-p_1a_1,p_1n_1,0...,0)} \cdots \chi^{(-p_ma_m,0...,0,p_mn_m)} =$$

$$= \chi^{(c+p_1a_1+...+p_ma_m,k_1,...,k_m)}\chi_1^{p_1} \cdots \chi_m^{p_m}. \tag{16}$$

Consider $g_j = (e_0, 1, ..., 1, e^{2\pi i/n_j}, 1, ..., 1), j = \overline{1,m}$ (e_0 is the unit element of G). Obviously, $\chi_i(g_j) = 1, i = 1, ..., m$, and, therefore, all $g_j, j = \overline{1,m}$, belong to G_0. We have

$$\chi^{(c+p_1a_1+...+p_ma_m,k_1,...,k_m)} = e^{2\pi i k_j/n_j},$$

which must be equal to 1 by (15), hence $k_j = 0, j = \overline{1,m}$. Then, by (15) and (16) (using the fact that $\chi_i(G_0) = 1, i = \overline{1,m}$) we get

$$\chi^{(c+p_1a_1+...+p_ma_m,0,...,0)}(G_0) = 1,$$

which means that

$$\chi^{(c+p_1a_1+...+p_ma_m)}(G) = 1, .$$

Therefore, $c + p_1a_1 + ... + p_ma_m = 0$, and (16) gives us (14). Thus, by (12), we get

$$\hat{G}_0 \cong \widehat{G \times \mathbb{T}^m}/A(\widehat{G \times \mathbb{T}^m}, G_0) =$$

$$= \{\chi^{(c,q_1,...,q_m)}A(\widehat{G \times \mathbb{T}^m}, G_0) : c \in \Gamma, q_i = p_in_i + k_i, 0 \le k_i < n_i, i = \overline{1,m}\} =$$

$$= \{\chi^{(c+p_1a_1+...+p_ma_m,k_1,...,k_m)}\chi_1^{p_1} \cdots \chi_m^{p_m}A(\widehat{G \times \mathbb{T}^m}, G_0) : c \in \Gamma, 0 \le k_i < n_i, i = \overline{1,m}\},$$

hence, since $a_i \in \Gamma, i = \overline{1,m}$ (and, therefore, $p_ia_i \in \Gamma, i = \overline{1,m}$) and $\chi_i \in A(\widehat{G \times \mathbb{T}^m}, G_0), i = \overline{1,m}$, we get that

$$\hat{G}_0 \cong \{\chi^{(d,k_1,...,k_m)}A(\widehat{G \times \mathbb{T}^m}, G_0) : d \in \Gamma, 0 \le k_i < n_i, i = \overline{1,m}\}$$

From (11) we get that the right hand side of the last expression is isomorphic to the group Γ_1, hence, $\hat{G}_0 \cong \Gamma_1$ and, therefore, $G_0 \cong \hat{\Gamma}_1 \cong G_1$.

Lemma 15 is proved. □

Definition 16 *Let $\pi : Y \to X$ be a covering of a topological space X by a topological space Y. A homeomorphism $f : Y \to Y$ such that $\pi \circ f = \pi$ is called a covering transformation of π. An unbranched covering $\pi : Y \to X$ is called a Galois covering if for any $y_1, y_2 \in Y$ with $\pi(y_1) = \pi(y_2)$ there exists a covering transformation $f : Y \to Y$ such that $f(y_1) = y_2$.*

Using the theorem 30 and the lemma 15 we get the following statement.

Theorem 31 *Each n-fold covering* $\pi : X^0 \to \Delta^0$ *is an algebraic Galois covering.*

Proof. Let us consider the space

$$Y^0 = \{(s, w_1, ..., w_m) \in \Delta^0 \times \mathbb{C}^m : w_i^{n_i} = \varphi^{a_i}(s), i = \overline{1, m}\}$$

and the corresponding algebraic covering $\sigma : Y^0 \to \Delta^0 : (s, w_1, ..., w_m) \mapsto s$ of the space Δ^0. From lemma 15 we have $G_0 \cong G_1$, therefore $G_0 \times (0, +\infty) \cong G_1 \times (0, +\infty) \cong X^0$ (Theorem 30). Now the isomorphity of the covering $\pi : X^0 \to \Delta^0$ to the algebraic covering $\sigma : Y^0 \to \Delta^0$ follows from the following topological group isomorphism:

$$\eta : G_0 \times (0, +\infty) \to Y^0 : (\alpha, z_1, ..., z_m, r) \mapsto (\alpha r, r^{c_1} z_1, ..., r^{c_m} z_m, r)$$

with inverse being

$$\eta^{-1} : Y^0 \to G_0 \times (0, +\infty) : (s, w_1, ..., w_m) \mapsto (s|s|^{-1}, w_1|s|^{-c_1}, ..., w_m|s|^{-c_m}, |s|),$$

where the numbers c_i are determined by n_i and a_i: $c_i n_i = a_i, i = \overline{1, m}$. Finally, let us show that

$$\pi_1 : G_1 \times (0, +\infty) \to G \times (0, +\infty) : \pi_1(\xi, r) = \pi(\xi) r$$

is a Galois covering and, since $G_1 \times (0, +\infty) \cong X^0$ and $G \times (0, +\infty) \cong \Delta^0$, this will imply that the covering $\pi : X^0 \to \Delta^0$ is also a Galois covering. For $\theta \in \operatorname{Ker} \pi|_{G_1}$ define a mapping $f_\theta : G_1 \times (0, +\infty) \to G_1 \times (0, +\infty) : f_\theta(\xi, r) = (\theta\xi, r)$. As $\pi(\theta) = e_0$, the unit element of a group G, then $\pi_1 \circ f_\theta(\xi, r) = \pi_1(\theta\xi, r) = \pi(\theta\xi) r = \pi(\theta)\pi(\xi) r = \pi(\xi) r = \pi_1(\xi, r)$, that is, $\pi_1 \circ f_\theta = \pi_1$. Therefore, f_θ is a covering transformation. Finally, if $\pi_1(\xi, r) = \pi_1(\omega, r)$ then $\pi(\xi) = \pi(\omega)$ and, therefore, $\theta = \omega\xi^{-1} \in \ker \pi|_{G_1}$, and for a covering transformation f_θ we have that $f_\theta(\xi, r) = (\theta\xi, r) = (\omega, r)$, hence π_1 is a Galois covering, and, therefore, so is π. Theorem is proved. □

3.3 Analytic paths

Let $\mathbb{C}_0$ be a plane in Δ^0 passing through the unit element e_0 of a group Δ^0 defined in the Section 1.2. As we have already seen in the Section 1.2, the set $\mathbb{C}_0$, which is the image of the additive group of complex numbers $\mathbb{C}$ under the group homomorphism $\varphi : \mathbb{C} \to \Delta^0$, is a dense subgroup of the group Δ^0. As $e_0 \in \mathbb{C}_0$ then for any $s \in \Delta^0$ the set $s \cdot \mathbb{C}_0 = \mathbb{C}_s$ is a plane in Δ^0 passing through s. The set of all planes of this form break up into the cosets of a subgroup $\mathbb{C}_0$ in a group Δ^0.

Consider a path in Δ^0, that is, a mapping $\gamma : I = [0, 1] \to \Delta^0$ which is continuous with respect to the topology τ_{Δ^0} in Δ^0.

Definition 17 *A path $\gamma(I) \subset \Delta^0$ is called analytic if it is entirely contained in some plane $\mathbb{C}_{s_0}, s_0 \in \Delta^0$.*

Let $\gamma(I)$ be an analytic path in Δ^0 which lies in a plane $\mathbb{C}_{s_0}$, $s_0 \in \Delta^0$. Then for any $s \in \Delta^0$ the path $\gamma_s(I)$ with $\gamma_s(t) = s\gamma(t), t \in I$, lies in a plane $\mathbb{C}_{ss_0}$ and, therefore, $\gamma_s(I)$ is an analytic path as well.

Let X be a Bohr–Riemann surface over Δ and let K be a thin set of "critical points" of a covering $\pi : X \to \Delta$ in Δ. Let us now define the notion of an analytic path on the set $X^* = \pi^{-1}(\Delta^*) \subset X$, where $\Delta^* = \Delta \backslash K$ (we assume that $* \in K$ and consider the initial covering $\pi : X \to \Delta$ over the punctured generalized plane $\Delta^0 = \Delta \backslash \{*\}$).

Recall that by the path lifting property for each analytic path $\gamma(I)$ in Δ^* and for each point $w \in \pi^{-1}(\gamma(0))$ there is a unique path $\hat{\gamma}(I) \subset X^*$ starting at the point w and lifting $\gamma(I)$, i.e., $\hat{\gamma}(0) = w$ and $\gamma(t) = \pi \circ \hat{\gamma}(t), t \in I$.

Definition 18 *A path in X^* is called analytic if it is a lifting of some analytic path from Δ^*.*

Thus, the path $\hat{\gamma}(I) \subset X^*$ is an analytic path if it is a lifting of some analytic path $\gamma(I) \subset \mathbb{C}_s^*, s \in \Delta^0$, where $\mathbb{C}_s^* = \mathbb{C}_s \backslash K$.

We now introduce the notion of equivalent points on the sets $\pi^{-1}(s), s \in \Delta^*$.

Definition 19 *Two points $w_1, w_2 \in \pi^{-1}(s)$ will be called equivalent if there exists an analytic path $\hat{\gamma}(I) \subset X^*$ such that $w_1 = \hat{\gamma}(0)$ and $w_2 = \hat{\gamma}(1)$.*

Equivalence of the points w_1 and w_2 will be denoted as $w_1 \sim w_2$. It is easy to check that if $w_1 \sim w_2$ and $w_2 \sim w_3$ then $w_1 \sim w_3$. Thus, the set $\pi^{-1}(s) = \{w_1, ..., w_n\}$ breaks up into the finite number of equivalence classes. Define a function $\nu : X^* \to \mathbb{Z}_+$ on X^* as

$$\nu(w_0) = \operatorname{card}\{w \in \pi^{-1}(\pi(w_0)) : w \sim w_0\}$$

for $w_0 \in X^*$. Thus, the function ν acts on the set X^* and assigns to each point $w_0 \in X^*$ the number of its equivalent points.

3.4 Local constantness of a function ν

The main result of this section is a proof of local constantness of a function $\nu : X^* \to \mathbb{Z}_+$.

We first prove this result on the sets $\pi^{-1}(\mathbb{C}_s^*), s \in \Delta^*$. Let $s \in \Delta^*$. Denote by $\mu(s)$ the number of equivalence classes (in terms of definition 19) over s, i.e., the number of equivalence classes on the set $\pi^{-1}(s)$:

$$\mu(s) = \operatorname{card}\{C(w) : w \in \pi^{-1}(s)\},$$

where

$$C(w) = \{u \in \pi^{-1}(\pi(w)) : u \sim w\}.$$

Thus, the mapping C acts on X^* and assigns to each point $w \in X^*$ the set of its equivalent points, and, therefore, $\operatorname{card} C(w) = \nu(w)$. We now pass to the proof of local constantness of a function ν on $\pi^{-1}(\mathbb{C}_s^*), s \in \Delta^*$.

Lemma 16 *Let $s \in \Delta^*$. Then the function $\mu : \Delta^* \to \mathbb{Z}_+$ is constant on $\mathbb{C}_s^*$ and the function $\nu : X^* \to \mathbb{Z}_+$ is constant on the connected components of a pre–image $\pi^{-1}(\mathbb{C}_s^*)$.*

Proof. Note first that since a continuous mapping $\pi_s := \pi|_{\pi^{-1}(\mathbb{C}_s^*)} : \pi^{-1}(\mathbb{C}_s^*) \to \mathbb{C}_s^*$ is a covering of a path connected space $\mathbb{C}_s^*$ by a non–connected, in general, Riemann surface $\pi^{-1}(\mathbb{C}_s^*)$, then its restriction $\pi_s|_L$ to any path connected component L of a surface $\pi^{-1}(\mathbb{C}_s^*)$ is a covering of a space $\mathbb{C}_s^*$ as well. In particular, since π is a finite–sheeted covering, the number of such components is finite and for any $\sigma \in \mathbb{C}_s^*$ the quantity $m(L) = \operatorname{card}(\pi^{-1}(\sigma) \cap L)$ is constant which does not depend on σ and which is equal to the number of sheets of a covering $\pi_s|_L$ of the space $\mathbb{C}_s^*$. Obviously, the sum of all numbers $m(L)$ by all connected components L gives n – the number of sheets of a covering π.

Fix an arbitrary $\sigma \in \mathbb{C}_s^*$ and consider a partition $\pi^{-1}(\sigma) = C(w_1) \cup ... \cup C(w_m)$ of a set $\pi^{-1}(\sigma)$ into the disjoint union of equivalence classes over σ. By definition of equivalent points we have that for any $i, 1 \leq i \leq m$, all the points of the class $C(w_i)$ are connected by analytic paths and, therefore, for any $i, 1 \leq i \leq m$, the class $C(w_i)$ lies in some connected component L_i of a space $\pi^{-1}(\mathbb{C}_s^*)$ containing a point $w_i \in L_i$, and $\pi^{-1}(\sigma) \cap L_i = C(w_i)$ because all the points from $\pi^{-1}(\sigma)$, which are contained in the same connected component with w_i, are obviously equivalent to w_i. As the restriction of a covering π_s to each connected component of a surface $\pi^{-1}(\mathbb{C}_s^*)$ is a covering of a space $\mathbb{C}_s^*$ then $\pi^{-1}(\mathbb{C}_s^*)$ does not have any other connected components besides $L_i, i = \overline{1, m}$, because the existence of one more such component would mean that there are points in $\mathbb{C}_s^*$ which are covered more times than σ which contradicts the fact that π is a covering. Therefore, m coincides with the number of connected components of $\pi^{-1}(\mathbb{C}_s^*)$, and, hence, does not depend on σ. Thus, for any $\sigma \in \mathbb{C}_s^*$ we have that $\mu(\sigma) = m$.

Now suppose that $w \in \pi^{-1}(\mathbb{C}_s^*)$ and L is a connected component of $\pi^{-1}(\mathbb{C}_s^*)$ containing $w \in L$. As we have shown, $C(w) = \pi^{-1}(\pi(w)) \cap L$. Therefore, $\nu(w) = \operatorname{card} C(w) = m(L)$.

Lemma is proved. $\square$

To prove the local constantness of ν on X^* we need the following lemma which can be considered as a version of the homotopy lifting property for a covering $\pi : X^* \to \Delta^*$.

Lemma 17 *Suppose that we are given a point $s \in \Delta^*$ and a closed path $\gamma \subset \Delta^*$ which begins and ends at s: $\gamma(0) = \gamma(1) = s$. Assume that $\pi^{-1}(s) = \{x_1, x_2, ..., x_n\}$ and let $\hat{\gamma} : I \to X^*$ be* *a lifting of γ in X^*, $\gamma(t) = \pi \circ \hat{\gamma}(t), t \in I$, with initial point $\hat{\gamma}(0) = x_1$ and endpoint $\hat{\gamma}(1) = x_2$, where $x_1, x_2 \in \pi^{-1}(s), x_1 \neq x_2$. Assume further that there is a fixed decomposition*

$$\pi^{-1}(U) = \bigcup_{i=1}^{n} V_i, \tag{17}$$

of the pre–image $\pi^{-1}(U)$ of an open evenly–covered neighbourhood U of the point s into a disjoint union of open sets V_i which are homeomorphic to U under the mappings $\pi|_{V_i} : V_i \to U$ with inverses $\varphi_i = (\pi|_{V_i})^{-1} : U \to V_i$, and $\varphi_i(s) = x_i, i = \overline{1,n}$, i.e., the numeration in (17) is chosen so that $x_1 \in V_1$ and $x_2 \in V_2$. Then there exists an open neighbourhood W_0 of the unit element e_0 of a group Δ^0 such that $sW_0 \subset U$ and for any $\sigma \in W_0$ the lifting $\hat{\gamma}_\sigma : I \to X^*$ of a path $\gamma_\sigma(t) = \sigma\gamma(t), t \in I$, with initial point $\varphi_1(\sigma s) \in V_1$ has an endpoint at $\varphi_2(\sigma s) \in V_2$. In other words, if there is a lifting of the path γ with initial point and endpoint lying on the sheets V_1 and V_2 respectively, then the lifting of "perturbed" path γ_σ with initial point lying on the sheet V_1 also terminates on the sheet V_2.

Proof. Let us use the standard scheme of a construction of the path $\hat{\gamma}$ with $\hat{\gamma}(0) = x_1$, which will be adapted to the case we consider.

First of all, let us cover the compact set $\gamma(I)$ by evenly–covered sets of special form. Specifically, let us establish the existence of an open neighbourhood $W \subset s^{-1}U$ of the unit element e_0 of a group Δ^0 such that for any $t \in I$ the set $\gamma(t)W$ is evenly–covered.

Since the open evenly–covered sets form a base for a space Δ^* then there exists a finite covering of a compact set $\gamma(I)$ by such sets:

$$\gamma(I) \subset \bigcup_{i=1}^{l} U_i.$$

Let $\{W_j\}_{j \in J}$ be an open base of locally compact space Δ^0 at the point e_0 such that for any $j \in J$ the closure $\overline{W}_j$ is compact. For every $j \in J$ define a set

$$K_j = \{t \in I : \gamma(t)\overline{W}_j \subset U_i \,\text{for some}\, i, 1 \leq i \leq l\}.$$

Since all $\overline{W}_j$ are closed and each set $U_i, i = \overline{1,l}$, is open then K_j is also open, $j \in J$. Furthermore, we have that $\gamma(I) \subset \bigcup\limits_{i=1}^{l} U_i$, therefore for any $t \in I$ there exists i such that $\gamma(t) \subset U_i$ and as $\{W_j\}_{j \in J}$ is a base at the point e_0 then there exists $j \in J$ such that $\gamma(t)\overline{W}_j \subset U_i$, and, hence, $t \in K_j$. Thus, the family $\{K_j\}_{j \in J}$ forms an open cover of the compact set I, therefore, we can choose a finite number of indices $j_1, .., j_d$, such that $I \subset \bigcup\limits_{k=1}^{d} K_{j_k}$. Let us now consider the set $W = \bigcap\limits_{k=1}^{d} W_{j_k} \cap s^{-1}U \subset s^{-1}U$. Since the sets $\{W_{j_k}\}_{k=1}^{d}$ and $s^{-1}U$ are open neighbourhoods of the unit element e_0, then the set W is non–empty and is an open neighbourhood of e_0 as well. Now choose an arbitrary point $t \in I$. As $I \subset \bigcup\limits_{k=1}^{d} K_{j_k}$ then there exists j_m such that $t \in K_{j_m}$, which by definition of the set K_{j_m} implies the existence of $i, 1 \leq i \leq m$ such that $\gamma(t)\overline{W}_{j_m} \subset U_i$, hence, since all U_i are evenly–covered, we get that the set $\gamma(t)W \subset \gamma(t)\overline{W}_{j_m}$ is also evenly–covered. We have thus established the existence of a set W with desired properties.

Let us now recall main concepts of the construction of path–lifting.

Open sets $\gamma(t)W, t \in I$, obviously cover the compact set $\gamma(I)$, therefore, there exist finite number of points $\{t'_k\}_{k=1}^m$ such that the sets $\gamma(t'_k)W, k = \overline{1, m}$ cover $\gamma(I)$ and the intersections of "adjoining" sets $\gamma(t'_k)W \cap \gamma(t'_{k+1})W, k = \overline{1, m-1}$ are non–empty. Moreover, the points $\{t'_k\}_{k=1}^m$ can be chosen such that $t'_1 = 0$ and $t'_m = 1$. Then there exists a partition of the interval $I = [0, 1]$ by the points $0 = t_0 < t_1 < ... < t_m = 1$, such that for each $k, k \in \overline{1, m}$, the image $\gamma([t_{k-1}, t_k])$ is entirely contained in open, evenly–covered set $\gamma(t'_k)W$. Clearly, $\gamma(t_k) \in \gamma(t'_k)W \cap \gamma(t'_{k+1})W, k \in \overline{1, m-1}$. Denoting $\gamma_k := \gamma(t'_k), k = \overline{1, m}$, for the pre–image of an open, evenly–covered set $\gamma_k W$ we get the following representation:

$$\pi^{-1}(\gamma_k W) = \bigcup_{i=1}^{n} V_i^k,$$

where for each $i, i = \overline{1, n}$, the restriction $\pi|_{V_i^k} : V_i^k \to \gamma_k W$ is a homeomorphism with inverse $\varphi_i^k := (\pi|_{V_i^k})^{-1} : \gamma_k W \to V_i^k, i = \overline{1, n}, k = \overline{1, m}$. Let us now pass to stepwise construction of a path $\hat{\gamma}$. We have that $\gamma = \pi \circ \hat{\gamma}$, therefore, on the initial interval $[t_0, t_1] = [0, t_1] \subset I$ there are n possible ways to construct the initial part of the path $\hat{\gamma}$, namely: $\hat{\gamma}([0, t_1]) = \varphi_i^1 \circ \gamma([0, t_1]), i = \overline{1, n}$. As for the lifting $\hat{\gamma}$ we have $\hat{\gamma}(0) = x_1$, we choose that i for which $\varphi_i^1(\gamma(0)) = x_1$. Denote the chosen i by i_1. The construction of a continuous path $\hat{\gamma}$ goes on by cohesion of continuous on $[t_{k-1}, t_k]$ parts $\hat{\gamma} = \varphi_{i_k}^k \circ \gamma, k = \overline{1, m}$ at the points t_k due to the choice of the following $\varphi_{i_k}^k$ by preceding $\varphi_{i_{k-1}}^{k-1}$ such that $\varphi_{i_k}^k(b_{k-1}) = \varphi_{i_{k-1}}^{k-1}(b_{k-1})$, where $b_{k-1} = \gamma(t_{k-1}) \in \gamma_{k-1}W \cap \gamma_k W$. A chain of homeomorphisms

$$\varphi_{i_k}^k : \gamma_k W \to V_{i_k}^k$$

provides the continuity of a path $\hat{\gamma}$ on the sequence of sheets $V_{i_k}^k, k = \overline{1, m}$ on which it lies. Since the path $\hat{\gamma}$ with $\hat{\gamma}(0) = x_1$ is uniquely determined by γ (uniqueness of path–lifting), then it does not depend on the presented construction which has been chosen to be in line with the conditions of lemma.

Furthermore, we have that $\gamma_1 = \gamma(t'_1) = \gamma(0) = s = \gamma(1) = \gamma(t'_m) = \gamma_m$ and, therefore, $\gamma_1 W = sW \subset U$ and the first obtained homemorphism $\varphi_{i_1}^1 : sW \to V_{i_1}^1$ satisfies the condition $\varphi_{i_1}^1(\gamma(0)) = x_1 \in V_1$. Hence, $\varphi_{i_1}^1$ is a restriction of a mapping $\varphi_1 : U \to V_1$ to the set sW: $\varphi_{i_1}^1 = \varphi_1|_{sW}$ (because both homeomorphisms φ_1 and $\varphi_{i_1}^1$ are local inverses of π). Then, using the ordering of numeration given in the condition of the lemma ($\hat{\gamma}(1) = x_2 \in V_2$), we similarly get that $\varphi_{i_m}^m = \varphi_2|_{sW}$.

So we have presented the construction of lifted path in our case. The problem is to show that small perturbation of an initial point $x_1 \in V_1$ does not divert the lifted path from the given sheets and, therefore, its endpoint is again in V_2. To solve this problem let us prove the existence of the sets U_k and $\tilde{U}_k$ with following properties.

First, for any $k, 1 \le k \le m$, the compactness of $\gamma([t_{k-1}, t_k]) \subset \gamma_k W$ and openness of $\gamma_k W$ imply that there exists an open neighbourhood U_k of the unit element e_0 such that $U_k\gamma([t_{k-1}, t_k]) \subset \gamma_k W$. Secondly, for any $k, 2 \le k \le m$, there exists a neighbourhood $\tilde{U}_k$ of e_0 such that $\varphi^k_{i_k}(\beta) = \varphi^{k-1}_{i_{k-1}}(\beta)$ for all $\beta \in b_{k-1}\tilde{U}_k$. Indeed, we have that $b_{k-1} = \gamma(t_{k-1}) \in \gamma_{k-1}W \cap \gamma_k W$ and $\varphi^{k-1}_{i_{k-1}}(b_{k-1}) = \varphi^k_{i_k}(b_{k-1})$. As $\gamma_{k-1}W$ and $\gamma_k W$ are open then the set $\gamma_{k-1}W \cap \gamma_k W \ni b_{k-1}$ is also open, therefore, there exists a neighbourhood $\tilde{U}_k$ of e_0 such that $b_{k-1}\tilde{U}_k \subset \gamma_{k-1}W \cap \gamma_k W$, hence, $\varphi^{k-1}_{i_{k-1}}(\beta) = \varphi^k_{i_k}(\beta), \beta \in b_{k-1}\tilde{U}_k$, as both homeomorphisms $\varphi^{k-1}_{i_{k-1}}$ and $\varphi^k_{i_k}$ are local inverses of π.

Finally, let us prove that if the above conditions are satisfied then for any σ from an open neighbourhood $W_0 = \bigcap\limits_{k=2}^{m} (U_k \cap \tilde{U}_k) \cap U_1$ of e_0 the lifting $\hat{\gamma}_\sigma$ of a path γ_σ with initial point $\varphi_1(\sigma s)$ has an endpoint at $\varphi_2(\sigma s)$. To this purpose let us consider a mapping

$$v(t) = \varphi^k_{i_k}(\sigma\gamma(t)), t \in [t_{k-1}, t_k], k = \overline{1, m},$$

and let us show that v is a continuous path that coincides with $\hat{\gamma}_\sigma$. Clearly, it is sufficient to prove the continuity of v at the points $t_k, k = \overline{1, m-1}$. We have

$$v(t) = \begin{cases} \varphi^k_{i_k}(\sigma\gamma(t)), t \in [t_{k-1}, t_k], \\ \varphi^{k+1}_{i_{k+1}}(\sigma\gamma(t)), t \in [t_k, t_{k+1}] \end{cases}$$

Since $\sigma \in W_0 \subset \tilde{U}_{k+1}$ then $b_k\sigma \in b_k\tilde{U}_{k+1}$, therefore, $\varphi^k_{i_k}(b_k\sigma) = \varphi^{k+1}_{i_{k+1}}(b_k\sigma)$, that is, $\varphi^k_{i_k}(\gamma(t_k)\sigma)) = \varphi^{k+1}_{i_{k+1}}(\gamma(t_k)\sigma)$ and the continuity of v at t_k is proved. Thus, $v(t), t \in I$, is a continuous path. As every $\varphi^k_{i_k}, k = \overline{1, m}$, on its domain is an inverse of π then, from definition of the mapping v, we get $\pi \circ v(t) = \sigma\gamma(t) = \gamma_\sigma(t), t \in I$, hence v is a lifting of the path γ_σ. Furthermore, $v(0) = \varphi^1_{i_1}(\sigma\gamma(0)) = \varphi^1_{i_1}(\sigma s)$. As $\sigma \in W_0 \subset U_m$ then from definition of the set U_m we get that $\sigma\gamma([t_{m-1}, t_m]) \subset \gamma_m W = sW$, and, in particular, $\sigma s = \sigma\gamma(t_m) \in sW$ and since $\varphi^1_{i_1} = \varphi_1|_{sW}$ we get $v(0) = \varphi^1_{i_1}(\sigma s) = \varphi_1(\sigma s)$. Thus, v actually coincides with lifted path $\hat{\gamma}_\sigma$ from the condition of the Lemma. Let us now show that the endpoint of the path $\hat{\gamma}_\sigma$ lies on the sheet V_2. We have $\hat{\gamma}_\sigma(1) = v(1) = \varphi^m_{i_m}(\sigma\gamma(1)) = \varphi^m_{i_m}(\sigma s)$, and, since $\sigma s \in sW$ and $\varphi^m_{i_m} = \varphi_2|_{sW}$, we get that $\hat{\gamma}_\sigma(1) = \varphi^m_{i_m}(\sigma s) = \varphi_2(\sigma s) \in V_2$. Lemma is proved. □

Corollary 7 *Each element $w \in X^*$ has a neighbourhood V such that $\nu(z) \ge \nu(w)$ for any $z \in V$.*

Proof. Suppose $w \in X^*$ and $\pi(w) = s \in \Delta^*$. Let U be an evenly–covered neighbourhood of a point s such that $\pi^{-1}(U) = \bigcup\limits_{i=1}^{n} V_i$ and each restriction $\pi|_{V_i} : V_i \to U$ is a homeomorphism whose inverse is $\varphi_i : U \to V_i$. Suppose that $w \in V_1$. Choose some $u \ne w$ from $C(w)$. Then $\pi(u) = s$ and the homeomorphity of π on each V_i implies that $u \notin V_1$. Let us say $u \in V_2$. Since $u \in C(w)$ then by definition of the set $C(w)$ there exists an analytic path starting at w and ending at u, that is, there exists an analytic path $\gamma \subset \Delta^*$ with $\gamma(0) = \gamma(1) = s$, such that for its lifting $\hat{\gamma} \subset X^*$ we have $\hat{\gamma}(0) = w, \hat{\gamma}(1) = u$. Let $W_0^{(2)}$ be the set W_0 from the previous lemma for the case we now consider (we add an index 2 because we assume that $u \in V_2$). Denote $V_1^{(2)} = V_1 \cap \pi^{-1}(sW_0^{(2)}) = \varphi_1(sW_0^{(2)})$. Then, by the previous lemma, for any $x \in V_1^{(2)}$ there exists an analytic path which starts at x and ends at a point from the set $\varphi_2(sW_0^{(2)}) \subset V_2$. Therefore, on a sheet V_2 the points from $V_1^{(2)}$ have as many equivalent points as w (namely, each of those points has precisely one equivalent point on V_2). Then considering one after another the sets $V_3, ..., V_n$ and taking into account the fact that w can have equivalent points only on the sheets $V_i, i = \overline{2, n}$ (with no more than one equivalent point on each sheet), we similarly get the sets $V_1^{(3)}, ..., V_1^{(n)}$. Then the set $V = \bigcap_{i=2}^{n} V_1^{(i)}$ obviously satisfies the condition of the corollary. □

We are now ready to prove the main result of this section.

Theorem 32 *The function $\nu : X^* \to \mathbb{Z}_+$ is locally constant on X^*.*

Proof. Let us first prove that the function $\mu : \Delta^* \to \mathbb{Z}_+$ is constant on Δ^*. We have $\mu(\sigma) = \text{card}\,\{C(w), w \in \pi^{-1}(\sigma)\}$. By Corollary 7 for $w_1 \in \pi^{-1}(\sigma)$ there exists a neighbourhood V_1 such that $\nu(z) \geq \nu(w_1), z \in V_1$, i.e., each point $z \in V_1$ has at least as many equivalent points as w_1. Suppose $\pi^{-1}(\sigma) = (w_1, ..., w_n)$ and let $V_1, ..., V_n$ be the corresponding neighbourhoods of those points. Define $U = \bigcap_{i=1}^{n} \pi(V_i)$. Assume that $\xi \in U$ and consider $\mu(\xi) = \text{card}\,\{C(z), z \in \pi^{-1}(\xi)\}$. Choose an arbitrary $z \in \pi^{-1}(\xi)$ and assume that $z \in V_i$ for some $i, 1 \leq i \leq n$. Then by definition of the set V_i we have that the number of equivalent points of a point $z \in \pi^{-1}(\xi)$ is not less than the number of equivalent points of $w_i \in \pi^{-1}(\sigma)$: $\nu(z) \geq \nu(w_i)$, and, therefore, the number of equivalence classes of the points from $\pi^{-1}(\xi)$ is not greater than the number of equivalence classes of the points from $\pi^{-1}(\sigma)$, that is, $\mu(\xi) \leq \mu(\sigma)$.

Thus, for any $\sigma \in \Delta^*$ there exists a neighbourhood U of a point σ such that

$$\mu(\xi) \leq \mu(\sigma),\ \xi \in U. \tag{18}$$

Denote $\mu = \min_{\sigma \in \Delta^*} \mu(\sigma)$ and $D = \{\sigma \in \Delta^* : \mu(\sigma) = \mu\}$. Since the function μ takes values from $\mathbb{Z}_+$ then, obviously, $D \neq \varnothing$. Let us show that $D = \Delta^*$, i.e., $\mu(s) = \mu$ on Δ^*.

Fix an arbitrary $s \in \Delta^*$ and any $\sigma \in D$. Then by (18) there exists a neighbourhood $U \ni \sigma$ such that $\mu|_U \leq \mu(\sigma) = \mu \leq \mu(s)$. Since the set $\mathbb{C}_s^*$ is dense in Δ^* then $U \cap \mathbb{C}_s^* \neq \varnothing$ and, from Lemma 16 we get that

$$\mu|_{\mathbb{C}_s^*} = \mu|_{U \cap \mathbb{C}_s^*} \leq \mu(\sigma) = \mu \leq \mu(s) = \mu|_{\mathbb{C}_s^*},$$

i.e., $\mu(s) = \mu(\sigma) = \mu$ and, therefore, $s \in D$. Thus, $D = \Delta^*$ and the function μ is constant Δ^*.

The constantness of μ on Δ^* immediately implies the equality $\nu(z) = \nu(w)$ for any z from a neighbourhood V of a point w (see Corollary 7), because otherwise there would exist

$z \in V$ with $\nu(z) > \nu(w)$ and, by the first part of the proof, this leads to a strict inequality $\mu(\pi(z)) < \mu(\pi(w))$ which is a contradiction. Thus, ν is locally constant on X^*.

Theorem is proved. □

3.5 Local constantness of a function ν: algebraic version

In the previous section it was shown that the Corollary 7 implies the local constantness of a function ν on X^* (see Theorem 32), and the Corollary 7 has been obtained by constructive lifting of the path from Δ^* (see Lemma 17). In this section we present an algebraic proof of the result stated in the Corollary 7 for an algebraic version of the theory.

Let us first prove one technical result.

Lemma 18 *Let K be a compact set and let $p(t,x) = x^n + g_1(t)x^{n-1} + ... + g_{n-1}(t)x + g_n(t), t \in K$, be a polynomial with continuous coefficients: $g_i \in C(K), i = \overline{1,n}$. Suppose further that a function $f \in C(K)$ satisfies the condition $p(t, f(t)) = 0, t \in K$, and let $C = \max_{1 \le i \le n}\{||g_i||\}$. Then $||f|| := \sup_{t \in K} |f(t)| < 1 + C$.*

Proof. If $C = 0$ then all the functions $g_i, i = \overline{1,n}$, are 0 which means that $p(x,t) = x^n$ and, hence, $f = 0$. Thus, $||f|| = 0 < 1 + 0 = 1 + C$.

In case $C > 0$ and $||f|| \le 1$ the conclusion is obvious: $||f|| < 1 + C$.

Suppose now that $C > 0$ and $||f|| > 1$. Then there exists $t_0 \in K$ such that $|f(t_0)| = ||f|| > 1$. Since $f(t_0)^n = -g_1(t_0)f(t_0)^{n-1} - ... - g_n(t_0)$ then

$$|f(t_0)| \le C(1 + \frac{1}{|f(t_0)|} + ... + \frac{1}{|f(t_0)|^{n-1}}) < C\frac{|f(t_0)|}{|f(t_0)| - 1},$$

hence, $||f|| = |f(t_0)| < 1 + C$.

Lemma is proved. □

Note that as the example of a polynomial $q(x) = x^2 - C$ shows for sufficiently small C ($C < 1/4$) the obtained estimate can not be improved to $||f|| \le 2C$.

The next lemma apparently belongs to mathematical folklore.

Lemma 19 *Let $K = [0,1]$ and let $p(t,x) = x^n + g_1(t)x^{n-1} + ... + g_n(t)$ be a polynomial with continuous coefficients $g_i \in C(K), i = \overline{1,n}$, and with discriminant not equal to zero on K: $d_p(t) \ne 0, t \in K$. Then there exist exactly n functions $h_i \in C(K), i = \overline{1,n}$, which mutually does not coincide at any point of K and represent the set of solutions of the equation $p(t,x) = 0$ on K:*

$$p(t, h_i(t)) = 0, t \in K, i = \overline{1,n}.$$

Remark. Note that as for every point $t_0 \in K$ the equation $p(t_0, x) = 0$ has exactly n solutions then the mutually distinct values $h_i(t_0), i = \overline{1,n}$, represent *all* the solutions of the equation $p(t_0, x) = 0$, i.e., the values $\{h_i(t)\}_{i=1}^n, t \in K$, represent the *whole* set of solutions of the equations $p(t,x) = 0, t \in K$.

Proof. Define a set
$$K_p := \{(t,x) \in K \times \mathbb{C} : p(t,x) = 0\}.$$
We want to find continuous functions $h_i \in C(K), i = \overline{1,n}$, which mutually does not coincide at any point of K such that
$$K_p = \{(t,x) \in K \times \mathbb{C} : p(t,x) = 0\} = \bigcup_{i=1}^{n} \{(t, h_i(t)) : t \in K\}.$$
By Hurwitz–Rouche's theorem the projection to fist coordinate $\pi : K_p \to K, \pi(t,x) = t$, is an unbranched n–fold covering and continuity of the functions $g_i \in C(K), i = \overline{1,n}$ implies that the projection to second coordinate $\eta : K_p \to \mathbb{C}, \eta(t,x) = x$ is a continuous mapping.

Consider a path $u : I \to K$, $u(t) = t, t \in I(= K)$ and a set $\pi^{-1}(0) = \{(0,x_1), ..., (0,x_n)\}$ of the pre–images of a point $0 \in K$. By path–lifting theorem there exist n liftings $\hat{u}_i : I \to K_p, i = \overline{1,n}$, of a path u such that $u = \pi \circ \hat{u}_i$ and $\hat{u}_i(0) = (0, x_i), i = \overline{1,n}$.

Define $h_i = \eta \circ \hat{u}_i, i = \overline{1,n}$, and let us show that the family $h_i, i = \overline{1,n}$, satisfies the desired conditions.

Note first that the functions h_i are continuous as they are compositions of continuous functions η and $\hat{u}_i$, $i = \overline{1,n}$. Furthermore, by definition of the mapping η we have that $h_i(t)$ is the second "coordinate" of a point $\hat{u}_i(t)$. From $\pi \circ \hat{u}_i(t) = u(t) = t$ we get that the first "coordinate" of a point $\hat{u}_i(t)$ is t. Thus,
$$\hat{u}_i(t) = (t, h_i(t)), t \in K, \tag{19}$$
i.e., $(t, h_i(t)) \in K_p, t \in K, i = \overline{1,n}$.

Let us now show that for any $t \in K$ the points $h_i(t), i = \overline{1,n}$, are mutually distinct. Assume to the contrary that there exist a point $t_0 \in K$ and the indices $i \neq j$ such that $h_i(t_0) = h_j(t_0)$.

Consider the set $T = \{t \in K : h_i(t) = h_j(t)\}$. By our assumption T is not empty. From continuity of functions h_i and h_j it follows that T is a closed set. Let us show that T is also open in K.

Let $t' \in T$. Then by (19) we have that $\hat{u}_i(t') = \hat{u}_j(t')$. Since π is a covering then there exist an open set $U \ni \pi(\hat{u}_i(t')) = t'$ in K and an open set $V \ni \hat{u}_i(t') = \hat{u}_j(t')$ in K_p such that $\pi : V \to U$ is a homeomorphism and, hence, a bijection on V. On the other hand, as $\hat{u}_i$ and $\hat{u}_j$ are continuous then there exists $\delta > 0$ such that for $t \in K, |t - t'| < \delta$ we have that $\hat{u}_i(t)$ and $\hat{u}_j(t)$ belong to V. Since on the set V π is a bijection, then the equalities $\pi(\hat{u}_i(t)) = t = \pi(\hat{u}_j(t))$ imply that for $t \in K, |t - t'| < \delta$ we have the following equality of the liftings:
$$\hat{u}_i(t) = \hat{u}_j(t), \tag{20}$$
that is, $h_i(t) = h_j(t)$, and, therefore, $K \cap (t' - \delta, t' + \delta) \subset T$ which means that the set T is open. As K is connected we get that $T = K$. This means that the equality 20 holds on the

whole K which is impossible since $\hat{u}_i(0) = (0, x_i) \neq (0, x_j) = \hat{u}_j(0)$. Thus, we have shown that the values $h_i(t), i = \overline{1, n}$ are mutually different on $t \in K$. Lemma is proved. $\square$

Note that the last statement proved in the lemma can be reformulated as follows: there is no continuous function $g \neq h_i, i = \overline{1, n}$ on K which at any point of K coincides with one of the functions h_i.

Our main tool in this part of the work is the following lemma.

Lemma 20 *Under the conditions of the previous lemma, for any $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that for every collection of the functions $\varepsilon_i \in C(K), \varepsilon_i : K \to \mathbb{C}$ with $||\varepsilon_i|| < \varepsilon, i = \overline{1, n}$ there are n functions $\tilde{h}_i \in B_\delta(h_i), i = \overline{1, n}$ such that for each $t \in K$ the points $\tilde{h}_i(t), i = \overline{1, n}$ represent n distinct zeros of "perturbed" polynomial*

$$p_\varepsilon(t, x) := x^n + \sum_{i=1}^{n} (g_i(t) + \varepsilon_i(t)) x^{n-k}. \tag{21}$$

Here $B_\delta(h) = \{f \in C(K) : ||f - h|| < \delta\}$.

Proof. Let us first prove the existence of $\varepsilon_0 > 0$ such that for every $\varepsilon \leq \varepsilon_0$ each polynomial of the form (21) with $||\varepsilon_i|| < \varepsilon, i = \overline{1, n}$, satisfies the conditions of Lemma 19, i.e., has a non–zero discriminant everywhere on K. For this purpose we use the well–known interpretation of $\mathbb{C}^n$ as a space of the coefficients of the polynomials over the field $\mathbb{C}$. Let $D = \{w \in \mathbb{C}^n : d(w) = 0\}$ be a set of zeros of a discriminant mapping $d : \mathbb{C}^n \to \mathbb{C}$ which assigns to each vector $w \in \mathbb{C}^n$ of the coefficients of a polynomial the value $d(w)$ of its discriminant.

Consider a mapping

$$G : K \to \mathbb{C}^n : t \mapsto (g_1(t), ..., g_n(t)) \cong x^n + g_1(t) x^{n-1} + ... + g_n(t) = p(t, x).$$

Then the image $G(K) = g_1(K) \times ... \times g_n(K)$ is a compact set, and, by the condition of the lemma, $G(K) \cap D = \varnothing$, as the discriminant of a polynomial $p(t, x)$ does not vanish on K. Denote by $d_0 = d(G(K), D)$ the distance between the sets $G(K)$ and D. Since those sets are closed and, moreover, $G(K)$ is compact then $d_0 > 0$. Let us show that ε_0 may be taken to be the constant number $d_0/2\sqrt{n}$. Indeed, for any collection of functions $\tilde{G} = (\tilde{g}_1, ..., \tilde{g}_n)$ with $||\tilde{g}_i - g_i|| < \varepsilon \leq \varepsilon_0, i = \overline{1, n}$, we have $d(\tilde{G}(t), G(t)) < \varepsilon_0 \sqrt{n} = d_0/2$ for any $t \in K$. The inequality $|d(G(t), D) - d(\tilde{G}(t), D)| < d(G(t), \tilde{G}(t)), t \in K$ (see, e.g., [9], p. 377), then implies that for any $t \in K$ the following inequalities hold: $d(\tilde{G}(t), D) \geq d(G(t), D) - d(\tilde{G}(t), G(t)) > d_0 - d_0/2 > 0$, which ensure that the desired condition $\tilde{G}(K) \cap D = \varnothing$ is fulfilled.

Thus, under the established conditions for any polynomial of the form (21) by Lemma 19 there exist n functions $\tilde{h}_i, i = \overline{1, n}$, which (for each fixed $t \in K$) represent the zeros of that polynomial with $\varepsilon_i = \tilde{g}_i - g_i$. Let us now show the existence of $\varepsilon > 0$ such that for $||\varepsilon_i|| < \varepsilon, i = \overline{1, n}$ continuous solutions of the equations $p_\varepsilon(t, x) = 0$ are contained in $B_\delta(h_i), i = \overline{1, n}$.

First, for any choice of $\tilde{G}$ with $||\tilde{g}_i - g_i|| < \varepsilon_0$ we have $||\tilde{g}_i|| < ||g_i|| + \varepsilon_0, i = \overline{1,n}$. Then $\tilde{C} := \max_i ||\tilde{g}_i|| < C + \varepsilon_0$, where $C = \max_i ||g_i||$. By Lemma 18 we have that $||\tilde{h}_i|| < 1 + \tilde{C} < 1 + C + \varepsilon_0$ for all $i = 1, .., n$.

Define further $\delta_0 = \min\limits_{1\le i<j\le n} \inf\limits_{t\in K} |h_i(t) - h_j(t)|$. By Lemma 19 we have that $\delta_0 > 0$. Since for $\delta_1 < \delta_2$ obviously $B_{\delta_1}(h) \subset B_{\delta_2}(h)$, then, without loss of generality, we may assume that our arbitrary chosen δ satisfies the condition $\delta < \delta_0/2$. Then, by Hurwitz–Rouche's theorem, there exists a constant $\varepsilon_1 > 0$ such that if $|b_i - g_i(0)| < \varepsilon_1, i = \overline{1,n}$ the polynomial $P(x) = x^n + b_1x^{n-1} + .. + b_n$ has exactly one zero (of order 1) in each disk $|x - h_i(0)| < \delta, i = \overline{1,n}$.

Now fix an arbitrary $\varepsilon > 0$ with following conditions:

a) $\varepsilon < \varepsilon_0$; then, by definition of ε_0, the inequality $||\tilde{g}_i - g_i|| < \varepsilon$ implies the existence of mutually non–coinciding functions $\tilde{h}_i \in C(K), i = \overline{1,n}$, which represent the zeros of a polynomial (21),

b) $\varepsilon < \varepsilon_1$; then, by definition of ε_1, if $|\tilde{g}_i(0)-g_i(0)| < \varepsilon$ the functions $\tilde{h}_i$ can be enumerated so that $|\tilde{h}_i(0) - h_i(0)| < \delta, i = \overline{1,n}$, and, finally,

c) $\varepsilon[(1 + C + \varepsilon_0)^n - 1]/(C + \varepsilon_0) < \delta^n$; then for each $i \in \{1, .., n\}$, using the equality $p_\varepsilon(t, \tilde{h}_i(t)) = 0$ we get that

$|p(t, \tilde{h}_i(t))| = |p(t, \tilde{h}_i(t)) - p_\varepsilon(t, \tilde{h}_i(t))| =$

$$= |(\tilde{h}_i(t)^n + g_i(t)\tilde{h}_i(t)^{n-1} + .. + g_n(t)) - (\tilde{h}_i(t)^n + \tilde{g}_i(t)\tilde{h}_i(t)^{n-1} + .. + \tilde{g}_n(t))| =$$

$$= |\varepsilon_1(t)\tilde{h}_i(t)^{n-1} + ... + \varepsilon_n(t)| < \delta^n \tag{22}$$

on K for $||\varepsilon_i|| < \varepsilon$, where $\varepsilon_i = \tilde{g}_i - g_i$.

Let us now show that for such ε the following implication holds:

$$||\varepsilon_i|| < \varepsilon \Rightarrow \tilde{h}_i \in B_\delta(h_i), i = 1, ..., n.$$

Choose an arbitrary $i_0 \in \{1, ..., n\}$ and consider the number

$$t_0 := \sup\{\tau \in [0, 1] : |h_{i_0}(t) - \tilde{h}_{i_0}(t)| < \delta \textit{ for } t \in [0, \tau]\}.$$

Then $t_0 > 0$ as the modulus $|h_{i_0}(t) - \tilde{h}_{i_0}(t)| := r(t)$ is continuous and is strictly less than δ when $t = 0$ (see b)). Obviously, $t_0 \le 1$. Assume that $t_0 < 1$. By definition of t_0 we have $r(t) < \delta$ for $t \in [0, t_0)$. Furthermore, we have that $r(t_0) = \delta$. Indeed, an assumption $r(t_0) < \delta$ contradicts the fact that $t_0 < 1$ is a supremum defined above and the assumption $r(t_0) > \delta$ contradicts the continuity of a function $r(t)$. Finally, for any $j \ne i_0$ using the definition of δ_0 we get

$$|h_j(t_0) - \tilde{h}_{i_0}(t_0)| = |h_j(t_0) - h_{i_0}(t_0) + h_{i_0}(t_0) - \tilde{h}_{i_0}(t_0)| \ge$$

$$\ge |h_j(t_0) - h_{i_0}(t_0)| - r(t_0) \ge \delta_0 - \delta > 2\delta - \delta = \delta. \tag{23}$$

As the pairs $(t_0, h_j(t_0)), j = \overline{1,n}$, are roots of a polynomial $p(t,x)$ then we can write $p(t_0,x) = \prod_{j=1}^{n}(x - h_j(t_0))$, hence, by (22) and (23) we get

$$\delta^n > |p(t_0, \tilde{h}_{i_0}(t_0))| = r(t_0) \prod_{j=1, j\neq i_0}^{n} |\tilde{h}_{i_0}(t_0) - h_j(t_0)| > \delta\delta^{n-1} = \delta^n. \tag{24}$$

The obtained contradiction shows that t_0 must be 1.

However, taking $t_0 = 1$ in (23) and in (24) we see that the assumption $r(1) = \delta$ also leads to a contradiction. Thus, $|h_{i_0}(t) - \tilde{h}_{i_0}(t)| < \delta$ for $t \in K$, and since the functions h_{i_0} and $\tilde{h}_{i_0}$ are continuous then $||\tilde{h}_{i_0} - h_{i_0}|| < \delta$, that is, $\tilde{h}_{i_0} \in B_\delta(h_{i_0})$. Since i_0 was arbitrary the Lemma 20 is proved. □

We now pass to the algebraic version of this theory.

Let

$$p(s,x) = x^n + f_1(s)x^{n-1} + ... + f_n(s)$$

be a polynomial with generalized analytic coefficients $f_i \in \mathcal{O}(\Delta^0), i = \overline{1,n}$ and with discriminant d_p. Then, obviously, d_p is also a generalized analytic function: $d_p \in \mathcal{O}(\Delta^0)$. Let us denote $N_p = N(d_p)$ – the set of zeros of discriminant d_p. Then either N_p is a nowhere dense (discrete) set in Δ^0 or $N_p = \Delta^0$. We assume that N_p is nowhere dense in Δ^0 in which case the zero set N_p will play a role of a thin set. Consider a space

$$\Delta^0_p = \{(s,x) \in \Delta^0 \times \mathbb{C} : p(s,x) = 0\},$$

and a covering

$$\pi : \Delta^0_p \to \Delta^0 : (s,x) \mapsto s.$$

The restriction $\pi|_{\Delta^*_p} : \Delta^*_p = \pi^{-1}(\Delta^*) \to \Delta^*$ is then an unbranched covering over $\Delta^* = \Delta^0 \setminus N_p$ which will also be denoted by π. Thus, Δ^0_p becomes a Bohr–Riemann surface. Denote $\mathbb{C}^*_s = \mathbb{C}_s \cap \Delta^* = \mathbb{C}_s \setminus N_p$, $\mathbb{C}^*_{p,s} = \pi^{-1}(\mathbb{C}^*_s)$ and $\mathbb{C}_{p,s} = \pi^{-1}(\mathbb{C}_s)$.

Recall that a path $u : I \to \Delta^0$ is called analytic if $u(I) \subset \mathbb{C}_s$ for some $s \in \Delta^0$ (we can take s equal to $u(0)$).

Definition 20 *A path $\hat{u} : I \to \Delta^*_p$ in Δ^*_p is called analytic if its projection $u = \pi \circ \hat{u}$ under a covering π is analytic.*

The following lemma immediately follows from the above definitions.

Lemma 21 *The following conditions are equivalent:*

1. *$\hat{u} : I \to \Delta^*_p$ is an analytic path*

2. *there exists $s \in \Delta^0$ such that $\hat{u}(I) \subset \mathbb{C}^*_{p,s}$.*

As we have already seen, the structure of a locally compact abelian group on Δ^0 enables for each $s \in \Delta^0$ and each analytic path $u : I \to \Delta^0$ to define an analytic path $u_s : I \to \Delta^0$ setting $u_s(t) = s \cdot u(t), t \in I$.

Lemma 22 *Let $u : I \to \Delta^*$ be an (analytic) path. Then there exists a neighbourhood U of the unit element of a group Δ^0 such that for each $s \in U$ the (analytic) path $u_s(I)$ is contained in Δ^*.*

Proof. We have that $u(I) \subset \Delta^*$, therefore, $u(I)$ does not contain points from N_p. As the set N_p is discrete then there exists a neighbourhood of a path $u(I)$ which does not intersect N_p, that is, there exists a neighbourhood U of the unit element e_0 such that $u(I) \cdot U \cap N_p = \varnothing$. Then, obviously, for any $s \in U$ the path $u_s(I) = s \cdot u(I)$ does not intersect N_p, i.e., $u_s(I) \subset \Delta^*$. Lemma is proved. □

As before, two points $w, w' \in \Delta^*_p$ will be called *equivalent* ($w \sim w'$), if $\pi(w) = \pi(w')$ and there exists an analytic path $\hat{u} : I \to \Delta^*_p$ such that $\hat{u}(0) = w$ and $\hat{u}(1) = w'$. Again, if $w \sim w'$ and $w' \sim w''$ then $w \sim w''$. Suppose, as before, that $C(w)$ is the set of all points (including w) which are equivalent to w. As we have an n–fold covering then, clearly, $\operatorname{card} C(w) \leq n$. Also, transitivity of the equivalence relation implies that for any $w \in \Delta^*_p$ there exists an analytic path $\hat{u}(I)$ such that $\hat{u}(0) = w$ and $C(w) \subset \hat{u}(I)$. Let us now pass to investigation of local behaviour of a function $\nu : \Delta^*_p \to \mathbb{Z}_+$, $\nu(w) = \operatorname{card} C(w)$ on Δ^*_p. As we have already noted, the Corollary 7 implies the proof of local constantness of a function ν on a Bohr–Riemann surface (see Theorem 32). In the next theorem we give an algebraic proof of the statement of Corollary 7 for our case, which will imply the local constantness of a function ν on Δ^*_p.

Theorem 33 *Each element $w \in \Delta^*_p$ has a neighbourhood V such that $\nu(z) \geq \nu(w)$ for any $z \in V$.*

Proof. Let us fix an arbitrary $w_0 \in \Delta^*_p$ with $\pi(w_0) = s_0 \in \Delta^*$. Suppose $\nu(w_0) = k$. Suppose further that $C(w_0) = (w_0, w_1, ..., w_{k-1})$ and $\hat{u} : I \to \Delta^*_p$ is an analytic path with $\hat{u}(0) = w_0$ and $C(w_0) \subset \hat{u}(I)$. Then there exists $0 = t_0 < t_1 < ... < t_{k-1} \leq 1$ such that $\hat{u}(t_i) = w_i$ and $\pi \circ \hat{u}(t_i) = \pi(w_i) = s_0, i = \overline{0, k-1}$. Consider a path $u(t) = \pi \circ \hat{u}(t), t \in I$ which is a projection of an analytic path $\hat{u} \subset \Delta^*_p$. We have that $u(I) \subset \Delta^*$ and $u(t_i) = \pi \circ \hat{u}(t_i) = s_0, i = \overline{0, k-1}$.

Clearly, to prove the theorem it is sufficient to show that for any sequence $w_\lambda \to w_0$ there exists λ_0 such that if $\lambda > \lambda_0$ then $\nu(w_\lambda) \geq \nu(w_0) = k$.

From $w_\lambda \to w_0$ it follows that $s_\lambda := \pi(w_\lambda) \to s_0$. Denote $s^0_\lambda = s_0^{-1} \cdot s_\lambda$. Then $s^0_\lambda \to e_0$, where e_0 is the unit element of Δ^0. Define the paths $u_\lambda : I \to \Delta^0$ as $u_\lambda(t) = s^0_\lambda \cdot u(t), t \in I$. Then, by Lemma 22, there exists λ_1 such that for $\lambda > \lambda_1$ the paths $u_\lambda(I)$ are contained in Δ^*.

Let us now consider the polynomials

$$p(u(t),x) = x^n + f_1(u(t))x^{n-1} + ... + f_n(u(t))$$

and

$$p(u_\lambda(t),x) = x^n + f_1(u_\lambda(t))x^{n-1} + ... + f_n(u_\lambda(t)).$$

Since the path $u(t), t \in I$, is contained in the set Δ^* then, by Lemma 19, the equation $p(u(t),x) = 0, t \in I$, has exactly n continuous mutually non–coinciding solutions. Obviously, for any $\varepsilon > 0$ there exists λ_ε such that for $\lambda \geq \lambda_\varepsilon$ the following inequality holds

$$\max_{1\leq i\leq n} ||f_i(u(t)) - f_i(u_\lambda(t))||_{C(I)} < \varepsilon.$$

Applying Lemma 20 we get that for $\lambda > \max\{\lambda_1, \lambda_\varepsilon\}$ the equation $p(u_\lambda(t),x) = 0, t \in I$, also has exactly n distinct continuous solutions, which are (uniformly on [0,1]) close to the solutions of the equations $p(u(t),x) = 0, t \in I$.

Let $\hat{u}(t) = (\hat{s}(t), \hat{x}(t)), t \in I$. From definition of the covering π we have $u(t) = \pi \circ \hat{u}(t) = \hat{s}(t), t \in I$, that is, $\hat{u}(t) = (u(t), \hat{x}(t)), t \in I$, and, in particular, $w_i = \hat{u}(t_i) = (u(t_i), \hat{x}(t_i)) = (s_0, \hat{x}(t_i)), i = \overline{0, k-1}$. Since $\hat{u}(t) \subset \Delta_p^*, t \in I$, then from definition of a set Δ_p^* we get that

$$\hat{x}^n(t) + f_1(u(t))\hat{x}^{n-1}(t) + ... + f_n(u(t)) = 0, t \in I,$$

i.e., the function $\hat{x}(t)$ is one of the solutions of the equation $p(u(t),x) = 0$. Therefore, by Lemma 20, for $\lambda > \lambda_{\varepsilon(\delta)}$ among the solutions of an equation $p(u_\lambda(t),x) = 0$ there exists $\hat{x}_\lambda(t)$ such that

$$||\hat{x}_\lambda - \hat{x}||_{C(I)} < \delta, \tag{25}$$

where

$$\delta < \min_{1\leq i<j\leq k-1} |\hat{x}(t_i) - \hat{x}(t_j)|/2, \tag{26}$$

with $w_i = (s_0, \hat{x}(t_i)), i = \overline{0, k-1}$. As the path u is analytic then a path u_λ is also analytic, hence, from $u_\lambda = \pi(u_\lambda, \hat{x}_\lambda)$ we get that a path $\hat{u}_\lambda : I \to \Delta_p^*$ with $\hat{u}_\lambda(t) = (u_\lambda(t), \hat{x}_\lambda(t)), t \in I$, is analytic as well. By construction we have that $u_\lambda(t_i) = s_\lambda^0 \cdot u(t_i) = s_0^{-1} \cdot s_\lambda \cdot s_0 = s_\lambda, i = \overline{0, k-1}$. Thus, the points $\hat{u}_\lambda(t_i) = (s_\lambda, \hat{x}_\lambda(t_i)), i = \overline{0, k-1}$, lie on a path $\hat{u}_\lambda(I)$. Since $\pi(\hat{u}_\lambda(t_0)) = s_\lambda = \pi(w_\lambda)$ and $w_\lambda \to w_0 = (s_0, \hat{x}(t_0))$, $s_\lambda \to s_0$, then choosing δ in (25) sufficiently small and λ sufficiently large ($\lambda > \lambda_0 > \max\{\lambda_1, \lambda_{\varepsilon(\delta)}\}$), we get that $w_\lambda = \hat{u}_\lambda(t_0)$. Moreover, using (25) and (26), for $i \neq j$ we get $|\hat{x}_\lambda(t_i) - \hat{x}_\lambda(t_j)| = |(\hat{x}(t_i) - \hat{x}(t_j)) - (\hat{x}(t_i) - \hat{x}_\lambda(t_i)) - (\hat{x}_\lambda(t_j) - \hat{x}(t_j))| \geq |(\hat{x}(t_i) - \hat{x}(t_j))| - |(\hat{x}(t_i) - \hat{x}_\lambda(t_i))| - |\hat{x}_\lambda(t_j) - \hat{x}(t_j)| > 2\delta - \delta - \delta = 0$, that is, $\hat{x}_\lambda(t_i) \neq \hat{x}_\lambda(t_j)$, and, therefore, $\hat{u}_\lambda(t_i) \neq \hat{u}_\lambda(t_j), i \neq j$. Thus, we have constructed an analytic path $\hat{u}_\lambda$ in Δ_p^* such that $\hat{u}_\lambda(0) = \hat{u}_\lambda(t_0) = w_\lambda$, $\pi(\hat{u}_\lambda(t_i)) = s_\lambda, i = \overline{0, k-1}$, and $\hat{u}_\lambda(t_i) \neq \hat{u}_\lambda(t_j), i \neq j$. This means that w_λ has at least k equivalent points $\hat{u}_\lambda(t_i), i = \overline{0, k-1}$, and, therefore, $\nu(w_\lambda) \geq k = \nu(w_0)$. Theorem is proved. □

Corollary 8 *The function* $\nu : \Delta_p^* \to \mathbb{Z}_+$ *is locally constant on* Δ_p^*.

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