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On the solutions of two third order recursive sequences

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Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to [1, 2, 3, 4, 5, 6] and the references therein.

In this paper, we determine the forbidden set, introduce an explicit formula for the solutions and discuss the global behavior of solutions of the difference equations

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-2}}, \quad n = 0, 1, \dots \tag{1}$$

and

$$x_{n+1} = \frac{x_n x_{n-1}}{-x_n + x_{n-2}}, \quad n = 0, 1, \dots \tag{2}$$

Consider the following subsets of $\mathbb{R}^3$:
$S_1 = \{(u_0, u_{-1}, u_{-2}) : u_{-1} = 0\}$, $S_2 = \{(u_0, u_{-1}, u_{-2}) : u_0 = 0\}$, $S_3 = \{(u_0, u_{-1}, u_{-2}) : u_0 = u_{-2}\}$, $S_4 = \{(u_0, u_{-1}, u_{-2}) : u_{-2} = 0\}$, $B_n = \{(u_0, u_{-1}, u_{-2}) : u_0 = -\frac{1}{n+1}u_{-2}\}$, and $C_n = \{(u_0, u_{-1}, u_{-2}) : u_0 = \frac{1}{n+1}u_{-2}\}$, $n = 0, 1, \dots$.

In [7], H. Sedaghat, discussed the behavior of solutions of the rational difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{x_n + b x_{n-2}}, \qquad n = 0, 1, \dots$$

where $a, b > 0$.
He established the forbidden set and a solution form of it by reducing it to a first order linear difference equation. When $a = b = 1$, the forbidden set is $F = \bigcup_{n=0}^{\infty} B_n \cup S_1 \cup S_2$. Also the solution when $a = b = 1$ is

$$x_n = \begin{cases} x_{-1} \prod_{j=o}^{\frac{n-1}{2}} \frac{1}{\gamma + 2j + 1}, & n = 1, 3, 5, \dots \\ x_0 \prod_{j=o}^{\frac{n-2}{2}} \frac{1}{\gamma + 2j + 2}, & n = 2, 4, 6, \dots \end{cases}$$

where $\gamma = \frac{x_{-2}}{x_0}$.
Finally it was proved, when $a < b + 1$ that every solution converges to 0.

We point out that the AJM Editorial Board suggested that if we apply the substitution $u_n = \frac{x_{n-2}}{x_n}$, the difference equations (1) and (2) are reduced respectively to the first order difference equations $u_{n+1} = -u_n + 1$ and $u_{n+1} = u_n - 1$ where $u_0 = \frac{x_{-2}}{x_0}$, which are easy to deal with.

Theorem 1 *Assume that* $\gamma = \frac{x_{-2}}{x_0}$. *Then we have the following:*

1. *The solution of equation* (1) *is*

$$x_n = \begin{cases} \frac{x_{-1}}{(1-\gamma)^{\frac{n+1}{2}}} & , n = 1, 3, 5, ... \\ \frac{x_0}{\gamma^{\frac{n}{2}}} & , n = 2, 4, 6, ... \end{cases} \tag{3}$$

2. *The solution of equation* (2) *is*

$$x_n = \begin{cases} x_{-1} \prod_{j=o}^{\frac{n-1}{2}} \frac{1}{\gamma - 2j - 1}, & n = 1, 3, 5, ... \\ x_0 \prod_{j=o}^{\frac{n-2}{2}} \frac{1}{\gamma - 2j - 2}, & n = 2, 4, 6, ... \end{cases} \tag{4}$$

Corollary 1 *The following statements are true.*

1. *The forbidden set of equation (1) is* $F_1 = \bigcup_{i=1}^{4} S_i$.
2. *The forbidden set of equation (2) is* $F_2 = \bigcup_{n=0}^{\infty} C_n \cup S_1 \cup S_2$.

Theorem 2 *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (1). *Then*

1. *If* $\gamma \in (-\infty, -1) \cup (2, \infty)$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to zero.*
2. *If* $\gamma \in (0, 1)$, *then both of the subsequences* $\{x_{2n}\}_{n=-1}^{\infty}$ *and* $\{x_{2n+1}\}_{n=-1}^{\infty}$ *are unbounded.*
3. *If* $\gamma \in (-1, 0)$, *then* $\{x_{2n+1}\}_{n=-1}^{\infty}$ *converges to zero and* $\{x_{2n}\}_{n=-1}^{\infty}$ *is unbounded.*
4. *If* $\gamma \in (1, 2)$, *then* $\{x_{2n}\}_{n=-1}^{\infty}$ *converges to zero and* $\{x_{2n+1}\}_{n=-1}^{\infty}$ *is unbounded.*

Theorem 3 *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (1). *Then*

1. *If* $x_{-i} > 0$, $i = 0, 1, 2$ *such that* $x_0 > x_{-2}$, *then the solution* $\{x_n\}_{n=-2}^{\infty}$ *is positive. Moreover, both* $\{x_{2n}\}_{n=-1}^{\infty}$ *and* $\{x_{2n+1}\}_{n=-1}^{\infty}$ *are unbounded.*
2. *If* $x_{-i} < 0$, $i = 0, 1, 2$ *such that* $x_0 < x_{-2}$, *then the solution* $\{x_n\}_{n=-2}^{\infty}$ *is negative. Moreover, both* $\{x_{2n}\}_{n=-1}^{\infty}$ *and* $\{x_{2n+1}\}_{n=-1}^{\infty}$ *are unbounded.*

Example Figure 1 shows that if $\{x_n\}_{n=-2}^{\infty}$ is the solution of equation (1) with initial conditions $x_{-2} = 0.6$, $x_{-1} = 2$, $x_0 = 1.3$, then the solution $\{x_n\}_{n=-2}^{\infty}$ is positive. Moreover, both of the subsequences $\{x_{2n}\}_{n=-1}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are unbounded.

Theorem 4 *Every solution* $\{x_n\}_{n=-2}^{\infty}$ *of equation* (2) *converges to zero.*

We thank the anonymous reviewer for his help to formulate the following theorem.

Theorem 5 *Every solution* $\{x_n\}_{n=-2}^{\infty}$ *of equation* (2) *eventually oscillates about* 0 *with semi-cycles of length* 2.

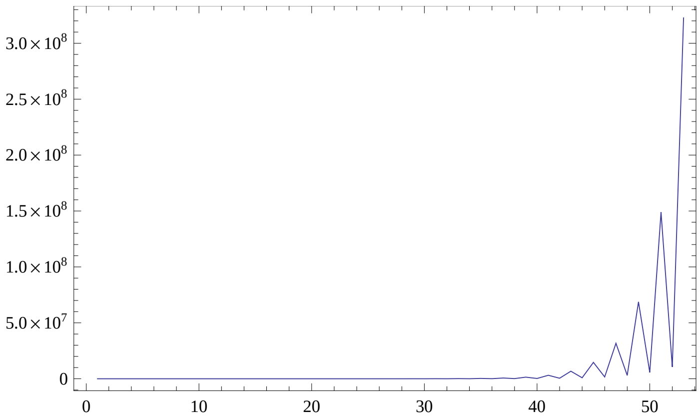

Figure 1: The difference equation $x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-2}}$

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