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Balancing in direction $(1,-1)$ in Pascal's Triangle

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Abstract

In this paper, we study the problem of balancing binomial coefficients in Pascal's triangle. We give the complete solution in the direction $(1,-1)$ for the first four cases of the possible rays. Some linked questions have been also examined.

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Introduction

A positive integer x is called *balancing number* if

$$1+2+\cdots+(x-1)=(x+1)+\cdots+(y-1) \qquad (1)$$

holds for some integer $y \geq x+2$. The problem of determining all balancing numbers leads to the resolution of the Pell equation $u^2-2v^2=1$ via introducing new variables. It is known that the solutions x to (1) can be described by the recurrence $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$ with $n \geq 2$ (see Finkelstein [4] or Behera, Panda, [1]). Obviously, the initial

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values $B_0 = 0$ and $B_1 = 1$, as usual, serve only our convenience, the first value satisfies (1) is $B_2 = 6$.

Analogously to (1) one can define the so-called *cobalancing numbers* by

$$1 + 2 + \cdots + x = (x + 1) + \cdots + (y - 1).$$

The cobalancing numbers b_n can be recursively given by the relation $b_n = 6b_{n-1} - b_{n-2} + 2$ and by the initial values $b_0 = 0$, $b_1 = 2$ (see, for example [3]).

One of the most general extensions of balancing numbers is when (1) is being replaced by

$$1^k + 2^k + \cdots + (x - 1)^k = (x + 1)^\ell + \cdots + (y - 1)^\ell, \tag{2}$$

where the exponents k and ℓ are given positive integers. In paper [5] effective and non-effective finiteness theorems on (2) were proved. For small values of k and ℓ all the solutions can be determined. For instance, in the case $(k, \ell) = (2, 1)$ equation (2) can be rewritten as

$$2x^3 + 4x = 3y^2 - 3y.$$

After suitable transformations, using the program package MAGMA, the authors solved the corresponding elliptic equation and found totally three balancing numbers regarded to the pair $(2, 1)$ of the exponents, namely $x = 5,\ 13$ and 36 with $y = 10,\ 39$ and 177, respectively.

In this paper, we pose the balancing and cobalancing problem in Pascal's triangle, and in the direction $\vec{v} = (1, -1)$ we solve the problem for the smallest four cases. Now we explain more precisely the question.

Let $n \in \mathbb{N}$, moreover let $r \in \mathbb{N}^+$, $q \in \mathbb{Z}$. Then the binomial coefficient $\binom{n}{0}$ and the direction (r, q) define a *diagonal ray* in Pascal's triangle which contains the elements

$$T_k^{(n,r,q)} = \binom{n - qk}{rk}, \qquad k = 0, 1, \ldots.$$

Observe, that the condition $q + r > 0$ guarantees that the number of binomial coefficients lying along such a diagonal ray is finite, and in this case the range of k is $0, \ldots, \omega_n = \lfloor \frac{n}{r+q} \rfloor$. It is clear, that the elements $T_k^{(n,r,q)}$ cover not all the elements of the Pascal's triangle if we vary $n \geq 0$ but keep r and q fixed. For example, $\binom{2}{1}$ is not included by any ray if the direction $(r, q) = (2, 1)$ is given. Therefore, for $r \geq 2$, we even define the notion of *intermediate rays* with the positive integer parameter $p \leq r - 1$ by

$$T_k^{(n,r,q,p)} = \binom{n - qk}{p + rk}, \qquad k = 0, \ldots.$$

Obviously, a finite ray ends when k takes the value $\omega_n = \lfloor \frac{n-p}{r+q} \rfloor$. If we allow $p = 0$ in $T_k^{(n,r,q,p)}$ it returns with the elements $T_k^{(n,r,q)}$.

Now, for a fixed direction (r, q) and for a fixed value of p, according to the definition of the balancing numbers, we claim the *balancing binomial coefficient* (in short, BBC)

$$T_x = \binom{n - qx}{p + rx}$$

with a positive integer x for which there exist a positive integer $y \geq x+2$ such that

$$\sum_{k=0}^{x-1} \binom{n-qk}{p+rk} = \sum_{k=x+1}^{y-1} \binom{n-qk}{p+rk}. \tag{3}$$

Applying the symmetry of Pascal's triangle, a trivial solution to (3) can be given linked to the direction direction $(r,q) = (1,0)$ (thus $p=0$) for any $n \geq 2$ even:

$$\binom{n}{0} + \cdots + \binom{n}{\frac{n}{2}-1} = \frac{1}{2}\left(2^n - \binom{n}{\frac{n}{2}}\right) = \binom{n}{\frac{n}{2}+1} + \cdots + \binom{n}{n}.$$

Obviously, the same direction with n odd provides an easy example for the cobalancing version (CBC), that is when the sum is up to x in the left hand side of (3). A non-trivial example for BBC is $\binom{82}{81}$ linked to the direction $(r,q) = (4,-4)$ and parameter $p=1$, see

$$\binom{2}{1} + \binom{6}{5} + \cdots + \binom{78}{77} = 800 = \binom{86}{85} + \cdots + \binom{114}{113}.$$

Another instance is $\binom{7}{2}$ with $(r,q) = (2,1)$, $p=0$:

$$\binom{11}{0} + \binom{9}{1} = 10 = \binom{5}{3}.$$

The proposed problem, in its generality is apparently too hard. The main difficulty is that there exist no summation formula for the truncated sum for the left hand side of (3), although such a formula, even in one generalization of Pascal's triangle, was given for arbitrary full finite rays in [2]. So in the present work we restrict ourselves to examine only the direction $(r,q) = (1,-1)$. Clearly, these rays are infinite and $p=0$. A further specification is to consider only the cases $0 \leq n \leq 3$, where we can describe all balancing and cobalancing binomial coefficients.

To finish the introduction, we recall the useful, well-known identity

$$\sum_{k=0}^{t} \binom{n+k}{k} = \binom{n+t+1}{t}. \tag{4}$$

1 Direction $(r,q) = (1,-1)$

According to the introduction, the BBC problem appears as

$$\sum_{k=0}^{x-1} \binom{n+k}{k} = \sum_{k=x+1}^{y-1} \binom{n+k}{k}, \tag{5}$$

where n is fixed, and the positive integer unknowns x and y satisfy $x+2 \leq y$. In the CBC task the left sum goes up to x (instead of $x-1$), therefore the slightly different inequality $x+1 \leq y$ must hold.

By (4), (5) is equivalent to

$$\binom{n+x}{x-1}+\binom{n+x+1}{x}=\binom{n+y}{y-1}. \tag{6}$$

Observe that in the case $n=0$, we need no (6) since the right leg of Pascal's triangle contains the constant 1 sequence. Therefore, each element $\binom{n}{n}$ is a BBC if $n \geq 1$ (and CBC for $n \geq 0$).

Similarly, when $n=1$, the elements included now construct the sequence of natural numbers, therefore the solution to BBC or CBC task returns with the sequence of balancing numbers or the cobalancing numbers, respectively.

Note that for arbitrary n, the CBC problem

$$2\binom{n+x+1}{x}=\binom{n+y}{y-1}$$

always has at least one solution (in the direction $(1,-1)$), since if $x=n$ and $y=n+2$ then

$$2\binom{2n+1}{n}=\binom{2n+1}{n}+\binom{2n+1}{n+1}=\binom{2n+2}{n+1} \tag{7}$$

holds.

1.1 $n=2$ in BBC problem

In this subsection we prove the following theorem.

Theorem 1 *There is no balancing binomial coefficient with $n=2$ in the direction $(1,-1)$.*

Proof. To prove the theorem, we need to solve the diophantine equation

$$\binom{x+2}{x-1}+\binom{x+3}{x}=\binom{y+2}{y-1},$$

or equivalently

$$(x+1)(x+2)(2x+3)=y(y+1)(y+2).$$

Multiplying it by 4 and putting $K=2x+3$, $L=y+1$, we obtain the diophantine equation $K^3-K=4(L^3-L)$ with $K \geq 5$ and $L \geq 4$.

Assume that the pair (K, L) is a solution. Thus

$$K^3-4L^3=K-4L \tag{8}$$

holds. Let $u=K-4L \in \mathbb{Z}$. Combining it with (8), we get

$$60L^3+48L^2u+12Lu^2+u^3=u. \tag{9}$$

Obviously, for a given u one can easily solves (9). For example, if $u \in \{-1, 0, 1\}$ then only $L = 0$ is possible, but it contradicts to the condition $L \geq 4$. Therefore we may assume $|u| \geq 2$, and suppose that p is a prime factor of u. Clearly, $L \neq 0$ otherwise $|u| \leq 1$ would follow. Put $g = \gcd(u, 60)$ and first suppose

$\mathbf{g = 1}$.

Observe that, by (9), $u \mid 60L^3$. Subsequently, $u \mid L^3$. Since $p \mid u$, we also have $p \mid L$. Let $\alpha \geq 1$ and $\beta \geq 1$ be the p-adic valuation of u and L, respectively. Then there exist integers u_1 and L_1, neither of them is divisible by p, such that

$$u = p^{\alpha} u_1, \qquad L = p^{\beta} L_1. \tag{10}$$

Inserting (10) into (9) it implies $\alpha = 3\beta$. Since it is true for any prime divisor p of u, it follows that $u = v^3$ for a suitable integer v. Then $u \mid L^3$ implies $v \mid L$, that is $L = kv$ for some non-zero integer k. Replacing u by v^3 and L by kv in (9), together with $w = v^2$, it leads to the Thue equation

$$60k^3 + 48k^2 w + 12kw^2 + w^3 = 1. \tag{11}$$

To determine the solutions of (11), we used MAGMA, which provided the only integer pair $(k, w) = (0, 1)$ satisfying (11). Then $v = \pm 1$ contradicts to $|v^3| = |u| \geq 2$. Thus $g > 1$.

The prime factorization $60 = 2^2 \cdot 3 \cdot 5$ yields 11 more possibilities for g. But four choices, when $2 \mid\mid g$, can immediately be excluded among them since the 2-adic valuation $\nu_2(60L^3 + 48L^2 u + 12Lu^2 + u^3) \geq 2$ is larger then $\nu_2(u) = 1$. As far as possible we maintain the remaining seven cases together.

$\mathbf{g = 3, 5, 15, 4, 12, 20, 60}$.

Put $u = gu_1$. Clearly, $60/g$ and $u_1 \neq 0$ are coprime integers. Thus equation (9) leads to

$$\frac{60}{g} L^3 + 48L^2 u_1 + 12gLu_1^2 + g^2 u_1^3 = u_1. \tag{12}$$

From now, one can repeat the treatment of case $g = 1$. That is, first check what happens if $u_1 = \pm 1$. For the eligible values of g, we obtain $L = \pm 1$ or $L = \pm 2$ or there exists no integer L satisfying (12).

Hence $|u_1| \geq 2$, and an analogous way to the case $g = 1$ leads to the Thue equations

$$\frac{60}{g} k^3 + 48k^2 v + 12gkw^2 + g^2 w^3 = 1.$$

The following table shows its solutions determined by MAGMA.

g	3	5	15	4	12	20	60
(k, w)	$(-1, 1)$	$(-2, 1)$, $(-1, 1)$	no	$(-1, 1)$	$(5, -1)$	no	$(1, 0)$

Since either $u_1 = \pm 1$ or $u_1 = 0$ follows from the solutions above, we arrived at a contradiction. Thus the proof of Theorem 1 is complete. $\square$

1.2 $n = 2$ in CBC problem

According to $n = 2$, the specific version of the general

$$\sum_{k=0}^{x} \binom{n+k}{k} = \sum_{k=x+1}^{y-1} \binom{n+k}{k} \tag{13}$$

CBC problem simplifies to

$$2\binom{x+3}{x} = \binom{y+2}{y-1}.$$

By the following result, the only solution to this equation is the trivial one included in (7).

Theorem 2 *In the direction* $(1,-1)$ *the only cobalancing binomial coefficient with* $n = 2$ *is* $\binom{4}{2}$.

Proof. The proof is a clone of proof of Theorem 1. Therefore we do not go into all details, only indicate the steps and the results at certain stages.

The equation (13) is equivalent to $K^3 - 2L^3 = K - 2L$ via $K = y + 1$ and $L = x + 2$. Introducing $u = K - 2L$, we obtain

$$6L^3 + 12L^2u + 6Lu^2 + u^3 = u. \tag{14}$$

There is no non-trivial solution to (14) with $|u| \leq 1$. Now let $g = \gcd(u, 6)$.

When u is coprime to 6 we get the Thue equation

$$6k^3 + 12k^2w + 6kw^2 + w^3 = 1,$$

and it is satisfied only by $(k,w) = (-1,1)$ and $(0,1)$ (we used MAGMA again). These pairs provides no CBC.

If $g = 2,\ 3$ or 6, (14) leads to the equations

$$\begin{aligned}
1 &= 3k^3 + 12k^2w + 12kw^2 + 4w^3,\\
1 &= 2k^3 + 12k^2w + 18kw^2 + 9w^3,\\
1 &= k^3 + 12k^2w + 36kw^2 + 364w^3,
\end{aligned}$$

respectively. The MAGMA results are shown by the following frame.

g	2	3	6
(k,w)	$(-1,1)$	$(-4,1)$, $(-1,1)$	$(1,0)$

These solutions give no new CBC, since previously we checked the possibility $u_1 = \pm 1$ for each eligible g, and found that either $L = 0, \pm 1$ or $L = \pm 5$. Among them only $L = 5$, together with $K = 4$ returns with CBC, namely:

$$\binom{2}{0} + \binom{3}{1} + \binom{4}{2} = \binom{5}{3}.$$

□

1.3 $n = 3$ in BBC and CBC problems

In this part we conclude the following results.

Theorem 3 BBC *There is no balancing binomial coefficient with $n = 3$ in the direction $(1, -1)$.*

CBC *The only cobalancing binomial coefficient in the direction $(1, -1)$ with $n = 3$ is $\binom{6}{3}$.*

Proof. We handle parallel the two problems, which are

$$\binom{x+3}{x-1} + \binom{x+4}{x} = \binom{y+3}{y-1} \quad \text{and} \quad 2\binom{x+4}{x} = \binom{y+3}{y-1}.$$

It is easy to see, that $24\binom{y+3}{y-1} = (y^2 + 3y + 1)^2 - 1$. Put $K = y^2 + 3y + 1$. Consequently, we must solve the equations

$$2(x+1)(x+2)^2(x+3) + 1 = K^2 \quad \text{and} \quad 2(x+1)(x+2)(x+3)(x+4) + 1 = K^2.$$

We use the procedure `IntegralQuarticPoints` of MAGMA. The integral points on the curves are

x	-4	-3	-2	-1	0
K	5	1	1	1	5

and

x	-8	-5	-4	-3	-2	-1	0	3
K	41	7	1	1	1	1	7	41

respectively. Since x must be a positive integer, only $y^2 + 3y + 1 = 41$ remains to check. This case gives to $y = 5$ and the theorem is proved. □

Remark 1 *Keeping $K = y^2 + 3y + 1$ on the right hand sides, the same trick on the left hand sides leads to $2L_1^2 - 2L_1 + 1 = K^2$ (BBC case) and $2L_2^2 - 1 = K^2$ (CBC case) with $L_1 = (x+2)^2$ and $L_2 = x^2 + 5x + 5$, respectively. Although these are Pell equations which can be solved by applying the standard method, the next step is to find quadratic polynomial values among the terms of binary recurrences, and it might be rather difficult.*

1.4 Arbitrary n and fixed difference $y - x$

If one fixes the difference $y - x$, then a new type of equations occurs. Considering (6) with $y \geq x + 2$, it leads to

$$(x+1)\cdots(x+n)(2x+n+1) = y(y+1)\cdots(y+n). \tag{15}$$

Assume that $y = x + 2 + m$ for some non-negative integer m. In the sequel, we examine (15) and its CBC version for small values of m.

2.4.1. BBC with $m = 0$

Now (15) is equivalent to

$$(x+1)(2x+n+1) = (x+n+1)(x+n+2),$$

and then to $x^2 - nx - (n+1)^2 = 0$. The discriminant $5n^2 + 8n + 4$ must be square, let say h^2, hence we obtain the Pell equation $(5n+4)^2 - 5h^2 = -4$. It is well known, that the solutions in h can be described by the Fibonacci sequence with odd subscripts, while $5n + 4$ must be a term with odd subscript of Lucas sequence. Since $L_\nu \equiv 4 \pmod 5$ holds if and only if $\nu \equiv 3 \pmod 4$, it follows that $n = (L_{4k+3} - 4)/5$, $k = 1, 2 \ldots$. Clearly, $x = (n + F_{4k+3})/2$, and the sequence of x's can be given by $x_i = 7x_{i-1} - x_{i-2} + 2$.

2.4.2. BBC with $m = 1$

The corresponding equation is

$$(x+1)(x+2)(2x+n+1) = (x+n+1)(x+n+2)(x+n+3),$$

or equivalently

$$x^3 - (2n-1)x^2 - (3n^2 + 9n + 4)x - (n^3 + 6n^2 + 9n + 4) = 0.$$

A brute force type computer verification found no solution in the interval $n \in [0; 10^9]$. It confirms the following

Conjecture 1 *In the direction* $(1, -1)$ *the BBC problem has no solution with* $y = x + 3$.

2.4.3. CBC with $m = 0, 1$

Now we have

$$2(x+1)\cdots(x+n)(x+n+1) = y(y+1)\cdots(y+n) \tag{16}$$

with $y = x + 1 + m$ $(m \geq 0)$. Trivially, $m = 0$ is a contradiction, so the first possible value is $m = 1$. In this case $x = n$ and $y = n + 2$ easily follow, providing infinitely many solutions, which have already been given by (7).

2.4.3. CBC with $m = 2$

Equation (16) yields $2(x+1)(x+2) = (x+n+2)(x+n+3)$, which can be converted into $x^2 - (2n-1)x - (n^2 + 5n + 2) = 0$. The discriminant $8n^2 + 16n + 9$ is necessarily a square, implying $h^2 - 2(2n+2)^2 = 1$. Without going into details, it leads to every second term of the Pell sequence defined by $P_0 = 0$, $P_1 = 1$ and $P_i = 2P_{i-1} + P_{i-2}$. The solutions in x is determined by the inhomogeneous recurrence $x_0 = 1$, $x_1 = 13$ and $x_i = 6x_{i-1} - x_{i-2} + 6$. Further, we have $y_0 = 4$, $y_1 = 16$ and $y_i = 6y_{i-1} - y_{i-2} - 6$.

2.4.2. CBC with $m = 3$

The appropriate equation appears as

$$2(x+1)(x+2)(x+3) = (x+n+2)(x+n+3)(x+n+4),$$

and collecting x we gain

$$x^3 - (3n-3)x^2 - (3n^2+18n+4)x - (n^3+9n^2+26n+12) = 0.$$

In the interval $n \in [0; 10^9]$ we found no solution to this.

Conjecture 2 *In the direction* $(1, -1)$ *the CBC problem has no solution with* $y = x + 4$.

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