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A representation for convex bodies

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Abstract

In this paper we extend the representation for the support function of centrally symmetric convex bodies to a larger class of arbitrary convex bodies. We discuss some questions on unique determination of convex bodies and consider some classes of convex bodies in terms of support functions.

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Basic definitions and properties

We denote by $\mathbf{R}^n$ ($d \geq 2$) the Euclidean n dimensional space and $\mathbf{S}^{n-1}$ the unit sphere in $\mathbf{R}^n$. The class of convex bodies (nonempty compact convex sets) $\mathbf{K}$ in $\mathbf{R}^n$ we denote by $\mathcal{K}$ and the class of centrally symmetric convex bodies (so called the *centred* bodies) denote by $\mathcal{K}_o$.

The most useful analytic description of compact convex sets is by the support function. The support function of $\mathbf{K}$ is defined as

$$H(\mathbf{K}, x) = \sup_{y \in \mathbf{K}} \langle y, x \rangle, \quad x \in \mathbf{R}^n.$$

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbf{R}^n$. The support function of $\mathbf{K}$ is positively homogeneous and convex. Below, we consider the support function $H(\mathbf{K}, \cdot)$ of a convex body as a function on the unit sphere $\mathbf{S}^{n-1}$ (because of the positive homogeneity of $H(\mathbf{K}, \cdot)$, the values on $\mathbf{S}^{n-1}$ determine $H(\mathbf{K}, \cdot)$ completely).

It is well known (Leichtweiz [10]) that a convex body $\mathbf{K} \in \mathcal{K}$ is determined uniquely by its support function.

It is known that (Blaschke [5], Weil [6], Bolker [4], Schneider [7], Ambartzumian [1]) *the support function $H(\boldsymbol{K},\cdot)$ of a sufficiently smooth centrally symmetric convex body $\boldsymbol{K}\in\mathcal{K}_o$ has the following representation (we assume that the center of $\boldsymbol{K}$ is the origin of $\mathbf{R}^n$)*

$$H(\boldsymbol{K},u)=\int_{\mathbf{S}^{n-1}} \mid \langle u,\nu\rangle \mid\, m(d\nu) \;\text{ for all }\quad u\in\mathbf{S}^{n-1}. \tag{1}$$

with a signed even Borel measure m on $\mathbf{S}^{n-1}$. Such bodies (whose support functions have the integral representation (1) *with a signed even measure) are called* generalised zonoids . *In the case then m is a positive even Borel measure $\boldsymbol{K}$ is called a zonoid .*
The class of zonoid in $\mathbf{R}^n$ we denote by $\mathcal{Z}$ and the class of generalised zonoids we denote by $\mathcal{GZ}$. We have $\mathcal{Z}\subset\mathcal{GZ}\subset\mathcal{K}_o$.
Also it is known that (Schneider [8]):
The generalised zonoids are dense in the class of centrally symmetric convex bodies.
In this paper we try to extend the representation (1) to a larger class of convex bodies which we call overall zonoids.

Analytic characterization of convex bodies

To represent the support function of a non-centrally symmetric convex body $\mathbf{K}\in\mathcal{K}$ we transform the integral (1) into the following integral over the hemisphere involving a non-even measure on $\mathbf{S}^{n-1}$

$$H(u)=\int_{\langle u,\nu\rangle\geq 0}\langle u,\nu\rangle\, m(d\nu) \;\textit{ for all }\quad u\in\mathbf{S}^{n-1}. \tag{2}$$

We have the following theorems.

Theorem 1 *Let m be a positive Borel measure on $\mathbf{S}^{n-1}$. Then the function*

$$H(x)=\int_{\langle x,\nu\rangle\geq 0}\langle x,\nu\rangle\, m(d\nu),\quad \text{for }\; x\in\mathbf{R}^n \tag{3}$$

is positively homogeneous and convex.

Proof. It is easy to verify that for $\nu\in\mathbf{S}^{n-1}$ the function

$$F(x,\nu)=\langle x,\nu\rangle I_{\langle x,\nu\rangle\geq 0}\;\; \textit{for}\;\; x\in\mathbf{R}^n$$

where I - the indicator function is positively homogeneous and convex, which implies Theorem 1. $\square$

The next theorem shows that the measure m in (3) can not be uniquely determined by H. This result can be formulated in terms of signed measures. We remind that μ is an even measure on $\mathbf{S}^2$ if $\mu(A)=\mu(-A)$ for $A\subset\mathbf{S}^{n-1}$ and μ is an odd measure on $\mathbf{S}^2$ if $\mu(A)=-\mu(-A)$ for $A\subset\mathbf{S}^{n-1}$.

Theorem 2 *If μ is a signed measure on $\mathbf{S}^{n-1}$ with*

$$\int_{\langle x,\nu\rangle\geq 0} \langle x,\nu\rangle\,\mu(d\nu)=0, \quad \text{for all} \;\; x\in\mathbf{R}^n \tag{4}$$

then either $\mu\equiv 0$ or μ is odd.

Proof. The result of Aleksandrov [2] states (which also has been rediscovered a number of times, e.g., by Matheron [3], Blaschke [5](1916, for $d=3$ and smooth measures)) that if μ is an even signed measure on $\mathbf{S}^{n-1}$ with

$$\int_{\mathbf{S}^{n-1}} |\langle x,\nu\rangle|\,\mu(d\nu)=0, \quad \textit{for all} \;\; x\in\mathbf{R}^n \tag{5}$$

then $\mu\equiv 0$.
Now let $\mu\neq 0$ be a non-odd signed measure on $\mathbf{S}^{n-1}$ satisfying (4). We distinguish two cases.
a) μ is an even measure on $\mathbf{S}^2$. We have

$$\int_{\mathbf{S}^{n-1}} |\langle x,\nu\rangle|\,\mu(d\nu)=2\int_{\langle x,\nu\rangle\geq 0} \langle x,\nu\rangle\,\mu(d\nu)=0, \quad \textit{for all} \;\; x\in\mathbf{R}^n \tag{6}$$

contrary to the uniqueness result of Aleksandrov.
b) μ is a non- even measure on $\mathbf{S}^{n-1}$. We consider the measure $\mu+\mu^*$, where $\mu^*(A)=\mu(-A)$ for $A\subset\mathbf{S}^{n-1}$. For the even measure $\mu+\mu^*$ on $\mathbf{S}^{n-1}$ we have

$$\begin{aligned}\int_{\mathbf{S}^{n-1}} |\langle x,\nu\rangle|\,(\mu+\mu^*)(d\nu) &= 2\int_{\langle x,\nu\rangle\geq 0} \langle x,\nu\rangle\,(\mu+\mu^*)(d\nu)= \\ 2\int_{\langle x,\nu\rangle\geq 0} \langle x,\nu\rangle\,\mu(d\nu)&+2\int_{\langle x,\nu\rangle\geq 0} \langle x,\nu\rangle\,\mu^*(d\nu)= \\ 2\int_{\langle x,\nu\rangle\geq 0} \langle x,\nu\rangle\,\mu(d\nu)+2\int_{\langle (-x),\nu\rangle\geq 0} &\langle (-x),\nu\rangle\,\mu(d\nu)=0+0=0, \quad \textit{for all} \;\; x\in\mathbf{R}^n\end{aligned} \tag{7}$$

hence $\mu+\mu*\equiv 0$ and μ is odd. $\square$

Note, that there is an odd measure on $\mathbf{S}^{n-1}$ satisfying (4). For example, for $n=2$ the singed measure $\mu(d\varphi)=\sin 3\varphi d\varphi$ satisfies (4).
The following result is well known (see [9]). *Let H be a positively homogeneous and convex function defined on $\mathbf{R}^n$. Then there exists a unique nonempty, convex body $\boldsymbol{K}\in\mathcal{K}$ with support function H .*
Hence to each positive Borel measure m (not necessary even) on $\mathbf{S}^{n-1}$ via the equation (2) corresponds convex body $\mathbf{K}$. On the other hand the definition of the support function depends on the origin. To create correspondence between measures defined on the sphere and the class of convex bodies we have to fix a point in the body. In this paper we will consider the support function of convex body $\mathbf{K}$ with respect to its Steiner point $s(\mathbf{K})$.

The Steiner point $s(\mathbf{K})$ of a convex body $\mathbf{K} \in \mathcal{K}$, known for over a century, is defined by the following integral

$$s(\mathbf{K}) = \frac{n}{\lambda_{n-1}(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} u\, H(\mathbf{K}, u)\, \lambda_{n-1}(du).$$

Here λ_{n-1} is the Lebesque measure on $\mathbf{S}^{n-1}$.
Let the support function of $\mathbf{K}$ with respect to its Steiner point $s(\mathbf{K})$ satisfies to (2) for a positive Borel measure m (not necessary even) on $\mathbf{S}^{n-1}$. Below we always assume that $s(\mathbf{K})$ is the origin of $\mathbf{R}^n$. Since the Steiner point $s(\mathbf{K})$ is the origin we have

$$s(\mathbf{K}) = \int_{\mathbf{S}^{n-1}} u\, H(\mathbf{K}, u)\, \lambda_{n-1}(du) = 0. \tag{8}$$

Substituting (3) into (8) and using Fubini's theorem, we get

$$s(\mathbf{K}) = \int_{\mathbf{S}^{n-1}} u \int_{\langle u,\nu\rangle \geq 0} \langle u, \nu\rangle\, m(d\nu)\, \lambda_{n-1}(du) =$$
$$\int_{\mathbf{S}^{n-1}} m(d\nu) \int_{\langle u,\nu\rangle \geq 0} u\, \langle u, \nu\rangle\, \lambda_{n-1}(du) \; = \frac{2\pi}{3} \int_{\mathbf{S}^{n-1}} \nu\, m(d\nu) = 0. \tag{9}$$

Definition 1. *We call a convex body $\boldsymbol{K} \in \mathcal{K}$ an overall zonoid if its support function has the integral representation* (2) *with a positive Borel measure m (not necessary even) on $\mathbf{S}^{n-1}$ satisfying the condition*

$$\int_{\mathbf{S}^{n-1}} u\, m(du) = 0. \tag{10}$$

We have the following result.

Theorem 3 *The set of overall zonoids in $\mathbf{R}^n$ is closed.*

This follows from a standard compactness argument for measures on the sphere.
Note that a zonoid $\mathbf{K}$ is a overall zonoid since an even Borel measure m on $\mathbf{S}^{n-1}$ satisfies the condition (10).
Since a generalised zonoid is not a overall zonoid it is reasonable to expect that overall zonoids are not dense in the class of convex bodies $\mathcal{K}$.

Definition 2. *A convex body $\boldsymbol{K} \in \mathcal{K}$ is a generalised overall zonoid if its support function has the integral representation* (2) *with a signed Borel measure m (not necessary even) on $\mathbf{S}^{n-1}$ satisfying the condition* (10).

It seems that the generalised overall zonoids are not dense in the class of convex bodies $\mathcal{K}$.
Let $\mathcal{C}_s(\mathbf{S}^{n-1})$ be the space of continuous real functions f defined on $\mathbf{S}^{n-1}$ satisfying the following condition

$$\int_{\mathbf{S}^{n-1}} u\, f(u)\, \lambda_{n-1}(du) = 0. \tag{11}$$

We have the following result.

Lemma 1 *There is $f \in \mathcal{C}_s^\infty(\mathbf{S}^{n-1})$ such that the equation*

$$f(u) = \int_{\langle u,\nu\rangle \geq 0} \langle u, \nu\rangle \, g(\nu) \, \lambda_{n-1}(d\nu), \tag{12}$$

where g is the unknown function has no solution in $\mathcal{C}^\infty(\mathbf{S}^{n-1})$.

For example, for $n = 2$ we consider $f(\varphi) = \sin 3\varphi + \cos 3\varphi$ satisfies (11), where the points on $\mathbf{S}^1$ identified by the corresponding angle φ. It is known that for $n = 2$ the spherical harmonics of order k are exactly the functions $a \, \cos k\varphi + b \, \sin k\varphi$. Also it is known that the harmonic expansion of a given function f on $\mathbf{S}^{n-1}$ converges uniformly to f if this function is sufficiently often differentiable. This follows, for example, from estimates given by Schneider [8]. We are looking for

$$g(\psi) = \sum_{k=0}^{\infty} a(k) \, \cos k\psi + b(k) \, \sin k\psi. \tag{13}$$

Substituting (13) into (12) we get

$$\sin 3\varphi + \cos 3\varphi = \int_{\cos(\varphi-\psi)\geq 0} \cos(\varphi - \psi) \, \sum_{k=0}^{\infty} [a(k) \, \cos k\psi + b(k) \, \sin k\psi] \, d\psi =$$
$$\sum_{k=0}^{\infty} a(k) \int_{\cos(\varphi-\psi)\geq 0} \cos(\varphi - \psi) \, \cos k\psi \, d\psi + b(k) \int_{\cos(\varphi-\psi)\geq 0} \cos(\varphi - \psi) \, \sin k\psi \, d\psi =$$
$$\sum_{k=0}^{\infty} a(k) \int_{\mathbf{S}^1} z(\cos(\varphi - \psi)) \, \cos k\psi \, d\varphi + b(k) \int_{\mathbf{S}^1} z(\cos(\varphi - \psi)) \, \sin k\psi \, d\psi, \tag{14}$$

where z is the following function on $[-1, 1]$

$$z(t) = \begin{cases} 1, & t \in [0, 1] \\ 0, & t \in [-1, 0) \end{cases} \tag{15}$$

If S_k is any spherical harmonics of order k on $\mathbf{S}^{n-1}$ and z a bounded integrable function on $[-1, 1]$, the Funk-Hecke theorem [11] states that

$$\int_{\mathbf{S}^{n-1}} z(\langle u, \nu\rangle) \, S_k(\nu) \, \lambda_{n-1}(d\nu) = d_k(z) S_k(u) \tag{16}$$

where $d_k(z)$ depends only on n, k and z. We have

$$\int_{\cos(\varphi-\psi)\geq 0} \cos(\varphi-\psi) \, \cos 3\psi \, d\psi = 0 \cdot \cos 3\varphi \;\; \textit{and} \;\; \int_{\cos(\varphi-\psi)\geq 0} \cos(\varphi-\psi) \, \sin 3\psi \, d\psi = 0 \cdot \sin 3\varphi. \tag{17}$$

Hence (12) for $f(\varphi) = \sin 3\varphi + \cos 3\varphi$ does not have a solution.

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