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New Tauberian theorems for statistical Cesàro summability of a function of three variables over a locally convex space

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Abstract. In this paper, we prove two new Tauberian theorems via statistical Cesàro summability mean of a continuous function of three variables by using oscillating behavior and De la Vallée Poussin means of a triple integral over a locally convex space. Moreover, some remarks and corollaries are provided here to support our results.

Key Words: Locally convex, triple Cesàro summability, triple improper integral, Tauberian theorems, $(C, 1, 1, 1)$–summability, (C, k, r, t)–summability
Mathematics Subject Classification 2010: 40G05, 40E05

1 Introduction

The notion of statistical convergence was introduced by Fast [5] and Steinhaus [23]. Besides, in this connection, Fridy [6] showed that $k(x_k - x_{k+1}) = O(1)$ is a Tauberian condition for the statistical convergence of (x_k). Subsequently, many researchers have worked in this area under several settings. For more recent works in this direction, one may refer to [11] and [22]. Parida et al. [19] extended the idea for a locally convex Hausdorff topological linear space. Tauber [24] introduced the first theorems for single sequences, according to which an Abel summable sequence is convergent with some suitable conditions. Later, a large number of authors such as Landau [16], Hardy and Littlewood [9], and Schmidt [21] obtained some classical Tauberian theorems for Cessáro and Abel summability methods of single sequences.

Recently, Çanak and Totur [4], and Jena et al. [10] presented and studied several Tauberian theorems for single sequences. On the other hand, Knopp [15] obtained some classical Tauberian theorems for Abel and $(C, 1, 1)$–summability methods of double sequences and proved that methods hold for the set of bounded sequences. Further, Moricz [18] proved some Tauberian theorems for Cesàro summable double sequences and deduced Tauberian theorems of Landau [17] and Hardy [8] type. Çanak and Totur [3] proved Tauberian theorem for Cesáro summability of single integrals and presented new proofs of some classical Tauberian theorems for the Cesáro summability of single integrals. Later, these notions were introduced by Parida et al. [19] for double integrals. On the other hand, the notion of $(C, 1, 1, 1)$–summability of a triple sequence was originally introduced by Çanak and Totur [1] in 2016.

Later, Çanak et al. [2] studied $(C, 1, 1, 1)$–means of statistical convergent of triple sequence and gave classical Tauberian theorems for a triple sequence that P–convergence follows from statistically $(C, 1, 1, 1)$–summability under two-sided boundedness conditions and slowly oscillating conditions in certain senses. Then, in 2020, Totur and Çanak [25] defined Tauberian conditions under which convergence of triple integrals follows from $(C, 1, 1, 1)$–summability.

In the last few years, the study of Tauberian theorems in double or triple summability was of great interest to many mathematicians (see, [12, 13, 20, 7]). In this paper, motivated by the existing results presented in [15], [16] and [20], we prove a statistical versions of Littlewood-Tauberian theorems via $(C, 1, 1, 1)$–summability method for triple integrable functions over a locally convex space under slow oscillation by using the De la Vallée Poussin mean of the triple integral. As a result, we extend a Tauberian theorem due to Parida et al. [19].

2 Definitions and Notations

In this section, we present some notions which are useful for the development of the paper.

Definition 1 *Let $I = [0, \infty) \subset \mathbb{R}$ and let $X(I)$ be the space of all real-valued measurable functions on I. We will say that a functional $\omega : X(I) \to [0, \infty)$ is a modulus on $X(I)$ if the following conditions hold:*

(i) $\omega(f) = 0$ *if and only if* $f = 0$ *for all* $f \in [0, \infty)$,

(ii) $\omega(f + g) = \omega(f) + \omega(g)$ *for all* $f, g \in [0, \infty)$,

(iii) ω *is an increasing function,*

(iv) ω *is continuous on* $[0, \infty)$.

From here on, X denotes a locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous semi-norms q.

Let $f(x, y, z)$ be a function in X. The partial sum of $f(x, y, z)$ is given by

$$s(x, y, z) = \int_0^x \int_0^y \int_0^z f(a, b, c)dadbdc, \qquad 0 < x, y, z < \infty.$$

Definition 2 *The* $(C, 1, 1, 1)$*–mean of* $f(x, y, z)$ *is*

$$\sigma(s(x, y, z)) = \sigma^{(1,1,1)}(s(x, y, z)) = \frac{1}{xyz} \int_0^x \int_0^y \int_0^z s(a, b, c)dadbdc. \tag{1}$$

Definition 3 *Integral*

$$\int_0^x \int_0^y \int_0^z s(a, b, c)dadbdc$$

is $(C, 1, 1, 1)$*–summable to a finite number* $l \in X$ *if for all* $q \in Q$*,*

$$\lim_{x,y,z\to\infty} q(\sigma(x, y, z) - l) \to 0.$$

In this case, we write $\sigma(x, y, z) \to l$ *over* X*.*

Note that the $(C, 1, 0, 0), (C, 0, 1, 0)$ and $(C, 0, 0, 1)$–means of $f(x, y, z) \in X$ are defined by

$$\sigma^{(1,0,0)}(s(x, y, z)) = \frac{1}{x} \int_0^x s(a, y, z)da,$$
$$\sigma^{(0,1,0)}(s(x, y, z)) = \frac{1}{y} \int_0^y s(x, b, z)db,$$
$$\sigma^{(0,0,1)}(s(x, y, z)) = \frac{1}{z} \int_0^z s(x, y, c)dc,$$

respectively.

Definition 4 *Integral*

$$\int_0^x \int_0^y \int_0^z s(a, b, c)dadbdc$$

is statistically $(C, 1, 1, 1)$*–summable to a finite number* $l \in X$ *if for all* $q \in Q$ *and all* $\varepsilon > 0$*,*

$$\lim_{u,v,p\to\infty} \frac{1}{uvp} |\{0 < x, y, z \leqq u, v, p \text{ and } q(\sigma(x, y, z) - l) \geqq \varepsilon\}| = 0.$$

In this case, we write

$$stat \lim_{x,y,z\to\infty} \sigma^{(1,1,1)}(s(x, y, z))$$
$$= stat \int_0^x \int_0^y \int_0^z \left(1 - \frac{a}{x}\right)\left(1 - \frac{b}{y}\right)\left(1 - \frac{c}{z}\right) dadbdc = l.$$

If the integral

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)dxdydz = l,$$

$l \in X$, exists, then the limit of (1) as a, b, c tends to ∞ also exists. Nevertheless, in general, the converse is not true. In order to prove this, we have to use the oscillatory behavior and De la Vallée Poussin mean of the above triple integral over X.

For each non-negative integers k, r, t, we define

$$\sigma^{(k,r,t)}(s(x,y,z)) = \frac{1}{xyz}\int_0^x \int_0^y \int_0^z \sigma^{(k-1,r-1,t-1)} s(a,b,c)dadbdc$$

if $k, r, t \geq 1$, and in the case $k, r, t = 0$, we put

$$\sigma^{(0,0,0)}(s(x,y,z)) = \int_0^x \int_0^y \int_0^z s(a,b,c)dadbdc.$$

Definition 5 *A triple integral*

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)dxdydz$$

is said to be statistically (C, k, r, t)*–summable to* $l \in X$ *if* $\sigma^{(k,r,t)}(s(x,y,z))$ *is summable to* l.

Denote

$$s(x,y,z) - \sigma(s(x,y,z)) = v(f(x,y,z)), \tag{2}$$

where

$$v(f(x,y,z)) = v^{(1,1,1)}(f(x,y,z)) = \frac{1}{xyz}\int_0^x \int_0^y \int_0^z f(a,b,c)dadbdc.$$

The relation (2) is known as Kronecker identity. We can see that

$$\sigma'(s(x,y,z)) = \frac{v(f(x,y,z))}{xyz}. \tag{3}$$

For each non-negative integers k, r and t, put

$$v^{(k,r,t)}(f(x,y,z)) = \frac{1}{xyz}\int_0^x \int_0^y \int_0^z v^{(k-1,r-1,t-1)} abcf(a,b,c)dadbdc$$

if $k, r, t \geq 1$, and let

$$v^{(0,0,0)}(f(x,y,z)) = \int_0^x \int_0^y \int_0^z abcf(a,b,c)dadbdc.$$

Definition 6 *A triple integral*

$$\int_0^\infty \int_0^\infty \int_0^\infty xyzf(x,y,z)dxydz$$

is statistically (C,k,r,t)*–summable to* $l \in X$ *if* $v^{(k,r,t)}(f(x,y,z))$ *is summable to* l.

Definition 7 *The De la Vallée Poussion mean of the triple integral*

$$\int_0^x \int_0^y \int_0^z f(a,b,c)dadbdc$$

is defined by

$$\tau(s(x,y,z)) = \frac{1}{(\lambda x - x)(\lambda y - y)(\lambda z - z)} \int_x^{\lambda x} \int_y^{\lambda y} \int_z^{\lambda z} s(a,b,c)dadbdc$$

for $\lambda \in (1,\infty)$ *and by*

$$\tau(s(x,y,z)) = \frac{1}{(x - \lambda x)(y - \lambda y)(z - \lambda z)} \int_{\lambda x}^{x} \int_{\lambda y}^{y} \int_{\lambda z}^{z} s(a,b,c)dadbdc$$

for $\lambda \in (0,1)$.

Definition 8 *A triple integral*

$$\int_0^x \int_0^y \int_0^z f(x,y,z)dxdydz$$

belonging to X *is oscillating slowly if*

$$\lim_{\lambda \to 1^+} \limsup_{x,y,z\to\infty} \max_{x,y,z \le a,b,c \le \lambda x, \lambda y, \lambda z} |s(a,b,c) - s(x,y,z)| = 0,$$

or, equivalently,

$$\lim_{\lambda \to 1^-} \limsup_{x,y,z\to\infty} \max_{\lambda x, \lambda y, \lambda z \le a,b,c \le x,y,z} |q(s(a,b,c)) - s(x,y,z)| = 0.$$

3 Main Results

We start with some auxiliary lemmas, which are also of independent interest.

Lemma 1 *The sequence of partial sums of a triple integrable functions* $f(x,y,z)$ *over a locally convex space* X *is oscillating slowly if and only if* $v(f(x,y,z)) \in X$ *is bounded and oscillating slowly.*

Proof. Let $s(x,y,z)$ be oscillating slowly. First, we show that $v(f(x,y,z)) = O(1)$ as $x,y,z\to\infty$. We have

$$\int_0^x\int_0^y\int_0^z wzgf(w,z,g)dwdzdg = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\int_{x/3^{i+1}}^{x/3^i}\int_{y/3^{j+1}}^{y/3^j}\int_{z/3^{k+1}}^{z/3^k} wzgf(w,z,g)dwdzdg.$$

From here it follows that

$$\begin{aligned}
&\int_\alpha^\beta\int_\gamma^\delta\int_\theta^\vartheta wzgf(w,z,g)dwdzdg\\
&= \int_\alpha^\beta\int_\gamma^\delta\int_\theta^\vartheta wzgs'f(w,z,g)dwdzdg\\
&= \int_\alpha^\beta\int_\gamma^\delta zg\left(\int_\theta^\vartheta ws'(w,z,g)dw\right)dzdg\\
&= \int_\alpha^\beta\int_\gamma^\delta zg\left[w(s(w,z,g))_\theta^\vartheta - \int_\theta^\vartheta s(w,z,g)dw\right]dzdg\\
&= \int_\alpha^\beta\int_\gamma^\delta\int_\theta^\vartheta zgs(w,z,g)dwdzdg + \vartheta\int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg\\
&= -\theta\int_\alpha^\beta\int_\gamma^\delta zgs(\theta,z,g)dzdg - \theta\int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg\\
&+ \theta\int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg - \int_\alpha^\beta\int_\gamma^\delta\int_\theta^\vartheta zgs(w,z,g)dwdzdg\\
&+ (\vartheta-\theta)\int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg\\
&+ \theta\left(\int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg - \int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg\right)\\
&= \int_\alpha^\beta\int_\gamma^\delta\int_\theta^\vartheta [zg(s(w,z,g)-s(\vartheta,z,g))]dzdgdw\\
&+ \theta\left(\int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg - \int_\alpha^\beta\int_\gamma^\delta zgs(\theta,z,g)dzdg\right)\\
&= (\beta-\alpha)(\delta-\gamma)(\vartheta-\theta)\max_{\alpha,\gamma,\theta\le x,y,z\le\beta,\delta,\vartheta}|s(x,y,z)-s(\beta,\delta,\vartheta)|\\
&+ \theta\left|\int_\alpha^\beta\int_\gamma^\delta zgs(\vartheta,z,g)dzdg - \int_\alpha^\beta\int_\gamma^\delta zgs(\theta,z,g)dzdg)\right|.
\end{aligned}$$

If we take $\beta = x/3^i, \beta/\alpha\le 3, \delta = y/3^j, \delta/\gamma\le 3$ and $\vartheta = z/3^k, \vartheta/\theta\le 3$, we obtain

$$\left|\int_0^x\int_0^y\int_0^z wzgf(w,z,g)dwdzdg\right| \le A\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{z=0}^{\infty}\frac{xyz}{3^{i+j+k}} = O(xyz)$$

as $x,y,z\to\infty$.

Now, we shall show that $\sigma(s(x,y,z))$ is oscillating slowly. Tacking (3) into account, we can write

$$\begin{aligned}|\sigma(s(a,b,c))-\sigma(s(x,y,z))| &= \left|\int_x^a\int_y^b\int_z^c \sigma'(s(w,z,g))dwdzdg\right|\\ &= \left|\int_x^a\int_y^b\int_z^c f(w,z,g)dwdzdg\right|\\ &\le C\int_x^a\int_y^b\left(\int_z^c\frac{dw}{w}\right)\frac{dz}{z}\frac{dg}{g}\\ &= C\log(c/z)\log(b/y)\log(a/x)\end{aligned}$$

for any $x\le\lambda x$, $y\le\lambda y$, $z\le\lambda z$. It is clear that

$$\max_{x,y,z\le a,b,c\le\lambda x,\lambda y,\lambda z}|\sigma(s(a,b,c))-\sigma(s(x,y,z))| \le C\log(\lambda)\log(\lambda)\log(\lambda).$$

Passing to the supremum limit from the both sides as $\lambda\to1^+$, we obtain

$$\lim_{\lambda\to1^+}\limsup_{x,y,z\to\infty}\max_{x,y,z\le a,b,c\le\lambda x,\lambda y,\lambda z}|\sigma(s(a,b,c))-\sigma(s(x,y,z))| = 0.$$

This implies that $v(f(x,y,z))$ is oscillating slowly by Kronecker identity (2).

To prove the converse part, suppose that $v(f(x,y,z))$ is bounded and oscillating slowly. The boundedness of $v(f(x,y,z))$ implies that $\sigma(s(x,y,z))$ is oscillating slowly. Since $v(f(x,y,z))$ is oscillating slowly, so $s(x,y,z)$ is oscillating slowly by Kronecker identity (2). □

Lemma 2 *For $\lambda>1$, it holds*

$$\begin{aligned}&s(x,y,z)-\sigma(s(\lambda x,\lambda y,\lambda z))\\ &=\frac{1}{(\lambda-1)^3}(\sigma(s(\lambda x,\lambda y,\lambda z))-\sigma(s(x,y,z)))+\frac{3}{(\lambda-1)}\sigma(s(\lambda x,\lambda y,\lambda z))\\ &-\frac{1}{(\lambda x-x)(\lambda y-y)(\lambda z-z)}\int_x^{\lambda x}\int_y^{\lambda y}\int_z^{\lambda z}(s(a,b,c)-s(x,y,z))dadbdc,\end{aligned}$$

and for $0<\lambda<1$, one has

$$\begin{aligned}&s(x,y,z)-\sigma(s(\lambda x,\lambda y,\lambda z))\\ &=\frac{1}{(\lambda-1)^3}(\sigma(s(\lambda x,\lambda y,\lambda z))-s(\lambda x,\lambda y,\lambda z))+\frac{3}{(\lambda-1)}\sigma(s(\lambda x,\lambda y,\lambda z))\\ &-\frac{1}{(x-\lambda x)(y-\lambda y)(z-\lambda z)}\int_{\lambda x}^{x}\int_{\lambda y}^{y}\int_{\lambda z}^{z}(s(x,y,z)-s(a,b,c))dadbdc.\end{aligned}$$

Proof. Let $\lambda > 1$. By De la Vallée Poussion mean of $s(x, y, z)$, we can write

$$\tau(s(x,y,z)) = \frac{1}{(\lambda x - x)(\lambda y - y)(\lambda z - z)} \int_x^{\lambda x} \int_y^{\lambda y} \int_z^{\lambda z} s(a,b,c) dadbdc$$
$$= \frac{1}{x(\lambda - 1)y(\lambda - 1)z(\lambda - z)}$$
$$\left(\int_0^{\lambda x} \int_0^{\lambda y} \int_0^{\lambda z} s(a,b,c) dadbdc - \int_0^{x} \int_0^{y} \int_0^{z} s(a,b,c) dadbdc \right).$$

Besides, since

$$\sigma(s(\lambda x, \lambda y, \lambda z)) = \frac{1}{\lambda x \lambda y \lambda z} \int_0^{\lambda x} \int_0^{\lambda y} \int_0^{\lambda z} s(a,b,c) dadbdc$$

and

$$\sigma(s(x,y,z)) = \frac{1}{xyz} \int_0^{x} \int_0^{y} \int_0^{z} s(a,b,c) dadbdc,$$

we have

$$\tau(s(x,y,z)) = \frac{\lambda^3}{(\lambda - 1)^3} \sigma(s(\lambda x, \lambda y, \lambda z)) - \frac{1}{(\lambda - 1)^3} \sigma(s(x,y,z))$$
$$= \left(1 + \frac{1}{(\lambda - 1)}\right)^3 \sigma(s(\lambda x, \lambda y, \lambda z)) - \frac{1}{(\lambda - 1)^3} \sigma(s(x,y,z)).$$

Further,

$$\tau(s(x,y,z)) - \sigma(s(\lambda x, \lambda y, \lambda z)) = \frac{1}{(\lambda - 1)^3} \sigma(s(\lambda x, \lambda y, \lambda z))$$
$$+ \frac{3}{(\lambda - 1)} \sigma(s(\lambda x, \lambda y, \lambda z)) - \frac{1}{(\lambda - 1)^3} \sigma(s(x,y,z)). \tag{4}$$

Subtracting $\sigma(s(\lambda x, \lambda y, \lambda z))$ from the identity

$$s(x,y,z) = \tau(s(x,y,z)) - \frac{1}{(\lambda x - x)(\lambda y - y)(\lambda z - z)}$$
$$\int_x^{\lambda x} \int_y^{\lambda y} \int_z^{\lambda z} (s(a,b,c) - s(x,y,z)) dadbdc,$$

we obtain

$$s(x,y,z) - \sigma(s(\lambda x, \lambda y, \lambda z)) = \tau(s(x,y,z)) - \sigma(s(\lambda x, \lambda y, \lambda z))$$
$$- \frac{1}{(\lambda x - x)(\lambda y - y)(\lambda z - z)} \int_x^{\lambda x} \int_y^{\lambda y} \int_z^{\lambda z} (s(a,b,c) - s(x,y,z)) dadbdc. \tag{5}$$

By (4) and (5), we get

$$s(x,y,z)-\sigma(s(\lambda x,\lambda y,\lambda z))=\frac{1}{(\lambda-1)^3}(\sigma(s(\lambda x,\lambda y,\lambda z))-\sigma(s(x,y,z)))$$
$$+\frac{3}{(\lambda-1)}\sigma(s(\lambda x,\lambda y,\lambda z))$$
$$-\frac{1}{(\lambda x-x)(\lambda y-y)(\lambda z-z)}\int_x^{\lambda x}\int_y^{\lambda y}\int_z^{\lambda z}(s(a,b,c)-s(x,y,z))dadbdc.$$

In the case $0<\lambda<1$, the proof is similar. □

Theorem 1 *If $s(x,y,z)$ is statistically $(C,1,1,1)$–summable to $l\in X$ in a locally convex space X and $s(x,y,z)$ is oscillating slowly, then $s(x,y,z)\to l$ as $x,y,z\to\infty$.*

Proof. Let $s(x,y,z)$ be statistically $(C,1,1,1)$–summable to $l\in X$. This implies that $\sigma(s(x,y,z))$ is $(C,1,1,1)$–summable to l. Now, from (2), we conclude that $v(f(x,y,z))$ is statistically $(C,1,1,1)$–summable to zero. Therefore, by Lemma 1, $v(f(x,y,z))$ is oscillating slowly. Besides, by Lemma 2, we obtain

$$v(f(x,y,z))-\sigma(v(f(\lambda x,\lambda y,\lambda z)))=\frac{1}{(\lambda-1)^3}((\sigma(v(f(\lambda x,\lambda y,\lambda z))))$$
$$-\sigma(v(f(x,y,z)))+\frac{3}{(\lambda-1)}\sigma(v(f(\lambda x,\lambda y,\lambda z)))-\frac{1}{(\lambda x-x)(\lambda y-y)(\lambda z-z)}$$
$$\cdot\int_x^{\lambda x}\int_y^{\lambda y}\int_z^{\lambda z}(v(f(a,b,c))-v(f(x,y,z)))\,dadbdc.$$

Thus, we can write

$$\begin{aligned}&|v(f(x,y,z))-\sigma(v(f(x,y,z)))|\\&\le\frac{1}{(\lambda-1)^3}|(\sigma(v(f(\lambda x,\lambda y,\lambda z))))-\sigma(v(f(x,y,z)))|\\&+\frac{3}{(\lambda-1)}|\sigma(v(f(\lambda x,\lambda y,\lambda z)))|\\&+\max_{x,y,z\le a,b,c\le\lambda x,\lambda y,\lambda z}|v(f(a,b,c))-v(f(x,y,z))|.\end{aligned}$$

Now, passing to the supremum limit from the both sides of the obtained relation as $x,y,z\to\infty$, we obtain

$$\begin{aligned}&\limsup_{x,y,z\to\infty}|v(f(x,y,z))-\sigma(v(f(x,y,z)))|\\&\le\limsup_{x,y,z\to\infty}\frac{1}{(\lambda-1)^3}|(\sigma(v(f(\lambda x,\lambda y,\lambda z))))-\sigma(v(f(x,y,z)))|\\&+\limsup_{x,y,z\to\infty}\frac{3}{(\lambda-1)}|\sigma(v(f(\lambda x,\lambda y,\lambda z)))|\\&+\limsup_{x,y,z\to\infty}\max_{x,y,z\le a,b,c\le\lambda x,\lambda y,\lambda z}|v(f(a,b,c))-v(f(x,y,z))|.\end{aligned}\tag{6}$$

Moreover, as $\sigma(v(f(\lambda x, \lambda y, \lambda z))) \in X$ converges, the first and the second terms in the right hand side of (6) must vanish. Therefore,

$$\limsup_{x,y,z\to\infty} |v(f(x,y,z)) - \sigma(v(f(x,y,z)))| \leq \limsup_{x,y,z\to\infty} \max_{x,y,z\leq a,b,c\leq \lambda x,\lambda y,\lambda z} |v(f(a,b,c)) - v(f(x,y,z))|,$$

and as $\lambda \to 1^+$, we get

$$\limsup_{x,y,z\to\infty} |v(f(x,y,z)) - \sigma(v(f(x,y,z)))| \leq 0.$$

This implies that $v(f(x,y,z)) = O(1)$ as $x,y,z \to \infty$. Since $s(x,y,z)$ is statistically summable to l by Cesáro mean and $v(f(x,y,z)) = O(1)$ as $x,y,z \to \infty$, we conclude that $\lim_{x,y,z\to\infty} s(x,y,z) = l$. □

Corollary 1 *If $s(x,y,z)$ is statistically (C,k,r,t)–summable to $l \in X$ in a locally convex space X and $s(x,y,z)$ is oscillating slowly, then $s(x,y,z) \to l$ as $x,y,z \to \infty$.*

Proof. By Lemma 1, $s(x,y,z)$ and $\sigma^{(k,r,t)}(s(x,y,z))$ are oscillating slowly. Moreover, by Theorem 1, $s(x,y,z)$ is statistically (C,k,r,t)–summable to $l \in X$. Thus

$$stat \lim_{x,y,z\to\infty} \sigma^{(k,r,t)}(s(x,y,z)) = l. \tag{7}$$

Now, by the definition,

$$\sigma^{(k,r,t)}(s(x,y,z)) = \sigma^{(1,1,1)}(s(x,y,z))(\sigma^{(k-1,r-1,t-1)}(s(x,y,z))). \tag{8}$$

It is clear that (7) and (8) imply that $s(x,y,z)$ is statistically $(C,k-1,r-1,t-1)$–summable to $l \in X$. Further, by Lemma 1, $\sigma^{(k-1,r-1,t-1)}(s(x,y,z))$ is oscillating slowly. Thus, Theorem 1 implies

$$\lim_{x,y,z\to\infty} \sigma^{(k-1,r-1,t-1)}(s(x,y,z)) = l.$$

Continuing in this way, we obtain $\lim_{x,y,z\to\infty}(s(x,y,z)) = l$. □

Theorem 2 *If $s(x,y,z)$ is statistically $(C,1,1,1)$–summable to $l \in X$ over a locally convex space X and $v(f(x,y,z))$ is oscillating slowly, then $s(x,y,z) \to l$ as $x,y,z \to \infty$.*

Proof. Since $s(x,y,z)$ is statistically $(C,1,1,1)$–summable to $l \in X$, we conclude that $\sigma^{(1,1,1)}(s(x,y,z))$ is also statistically Cesáro summable to l. Hence, $v(f(x,y,z))$ is statistically Cesáro summable to zero by (2). Using identity (2) to $v(f(x,y,z))$, we obtain that $v(v(f(x,y,z)))$ is statistically

Cesáro summable to zero. Thus, $v(v(f(x,y,z)))$ is oscillating slowly by Lemma 1. Further, by Lemma 2, we have

$$\begin{aligned}
&v(v(f(x,y,z))) - \sigma(v(v(f(\lambda x,\lambda y,\lambda))))\\
&= \frac{1}{(\lambda-1)^3}[\sigma(v(v(f(\lambda x,\lambda y,\lambda)))) - v(v(f(x,y,z)))]\\
&+ \frac{3}{(\lambda-1)}\sigma(v(v(f(\lambda x,\lambda y,\lambda)))) - \frac{1}{(\lambda x - x)(\lambda y - y)(\lambda z - z)}\\
&\cdot \int_x^{\lambda x}\int_y^{\lambda y}\int_z^{\lambda z} (v(v(f(a,b,c))) - v(v(f(x,y,z))))\, dadbdc.
\end{aligned}$$

Now,

$$\begin{aligned}
&|v(v(f(x,y,z))) - \sigma(v(v(f(\lambda x,\lambda y,\lambda))))|\\
&\leq \frac{1}{(\lambda-1)^3}|\sigma(v(v(f(\lambda x,\lambda y,\lambda)))) - v(v(f(x,y,z)))|\\
&+ \frac{3}{(\lambda-1)}|\sigma(v(v(f(\lambda x,\lambda y,\lambda))))|\\
&+ \max_{x,y,z\leq a,b,c\leq \lambda x,\lambda y,\lambda z} |(v(v(f(a,b,c))) - v(v(f(x,y,z)))|.
\end{aligned} \tag{9}$$

Passing to the supremum limit from the both sides of (9) as $x,y,z\to\infty$, we can write

$$\begin{aligned}
&\limsup_{x,y,z\to\infty} |v(v(f(x,y,z))) - \sigma(v(v(f(\lambda x,\lambda y,\lambda))))|\\
&\leq \limsup_{x,y,z\to\infty} \frac{1}{(\lambda-1)^3}|\sigma(v(v(f(\lambda x,\lambda y,\lambda)))) - v(v(f(x,y,z)))|\\
&+ \limsup_{x,y,z\to\infty} \frac{3}{(\lambda-1)}|\sigma(v(v(f(\lambda x,\lambda y,\lambda))))|\\
&+ \limsup_{x,y,z\to\infty} \max_{x,y,z\leq a,b,c\leq \lambda x,\lambda y,\lambda z} |(v(v(f(a,b,c))) - v(v(f(x,y,z)))|.
\end{aligned} \tag{10}$$

Since $\sigma(v(v(f(\lambda x,\lambda y,\lambda z))))\in X$ converges, the first and the second terms in the right hand side of (10) must be zero. Hence,

$$\begin{aligned}
&\limsup_{x,y,z\to\infty} |v(v(f(x,y,z))) - \sigma(v(v(f(\lambda x,\lambda y,\lambda))))|\\
&\leq \limsup_{x,y,z\to\infty} \max_{x,y,z\leq a,b,c\leq \lambda x,\lambda y,\lambda z} |(v(v(f(a,b,c))) - v(v(f(x,y,z)))|,
\end{aligned}$$

and tending $\lambda\to 1^+$, we get

$$\limsup_{x,y,z\to\infty} |v(v(f(x,y,z))) - \sigma(v(v(f(\lambda x,\lambda y,\lambda))))| \leq 0.$$

It is clear that $v(v(f(x,y,z))) = O(1)$ as $x,y,z\to\infty$. Moreover, $s(x,y,z)$ is statistically summable to $l\in X$ by Cesáro mean and $v(v(f(x,y,z))) = O(1)$ as $x,y,z\to\infty$. Thus, $s(x,y,z)\to l$ as $x,y,z\to\infty$. $\square$

Corollary 2 *If $s(x,y,z)$ is statistically (C,k,r,t)–summable to $l \in X$ over a locally convex space X and $v(f(x,y,z))$ is oscillating slowly, then $s(x,y,z) \to l$ as $x,y,z \to \infty$.*

Proof. Since $v(f(x,y,z))$ is oscillating slowly, by Lemma 1, we get that $\sigma^{(k,r,t)}(v(f(x,y,z)))$ is oscillating slowly. Besides, as $v(f(x,y,z))$ is statistically (C,k,r,t)–summable to $l \in X$, by Theorem 2, we have

$$stat \lim_{x,y,z\to\infty} \sigma^{(k,r,t)}(v(f(x,y,z))) = l. \tag{11}$$

Now, by the definition,

$$\lim_{x,y,z\to\infty} \sigma^{(k,r,t)}(v(f(x,y,z))) = \sigma^{(1,1,1)}(v(f(x,y,z)))\sigma^{(k-1,r-1,t-1)}(v(f(x,y,z))). \tag{12}$$

By (11) and (12), $v(f(x,y,z))$ is statistically $(C,k-1,r-1,t-1)$–summable to l. Furthermore, by Lemma 1, since $\sigma^{(k-1,r-1,t-1)}(v(f(x,y,z)))$ is oscillating slowly, Theorem 2 implies that $\lim_{x,y,z\to\infty} v(f(x,y,z)) = l$. □

In the similar way, we can get new theorems and corollaries by using $(C,1,0,0)$, $(C,0,1,0)$ and $(C,0,0,1)$–summability methods.

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