Weighted integral representations of harmonic functions in the unit disc by means of Mittag-Leffler type kernels

F. V. Hayrapetyan

Abstract. For weighted $L^p$-classes of functions harmonic in the unit disc, we obtain a family of weighted integral representations with weight function of the type $|w|^{2p} \cdot (1 - |w|^{2p})^\beta$.

Key Words: Harmonic Functions in the Unit Disc, Weighted Function Space, Weighted Integral Representation

Mathematics Subject Classification 2010: 30F15, 30C40, 30H20, 30E20

1 Introduction

It is well-known that the Cauchy integral formula has numerous applications in complex analysis. This formula makes it possible to reproduce values of holomorphic functions inside of a domain by integration of function along the boundary of the domain. First results are contained in [1, 2], where the values of holomorphic functions inside of a domain were obtained by integration of functions over the whole domain. In [3, 4] for the weighted spaces $H^p(\alpha)$ ($1 \leq p < \infty, \alpha > -1$) of functions $f$ holomorphic in the unit disc $\mathbb{D}$ and satisfying the condition

$$\int_{\mathbb{D}} |f(\zeta)|^p (1 - |\zeta|^2)^\alpha |dudv| < +\infty, \quad \zeta = u + iv,$$

the following result was established:

Theorem 1. Each function $f \in H^p(\alpha)$ has the integral representation

$$f(z) = \frac{\alpha + 1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)(1 - |\zeta|^2)^\alpha}{(1 - z\zeta)^{2+\alpha}} dudv, \quad z \in \mathbb{D}, \quad (1)$$

$$f(0) = \frac{\alpha + 1}{\pi} \int_{\mathbb{D}} \frac{f(\zeta)(1 - |\zeta|^2)^\alpha}{(1 - z\zeta)^{2+\alpha}} dudv, \quad z \in \mathbb{D}. \quad (2)$$
These representations had numerous applications in the theory of factorization of meromorphic functions in the unit disc (see [3, 4] as well as [5]).

Assume that $0 < p < +\infty$, $\rho > 0$, $\alpha > -1$, $\gamma > -1$. Denote by $L^p_{\alpha,\rho,\gamma}(D)$ the set of all complex-valued measurable functions $f(\zeta), \zeta \in D$, for which

$$M^p_{\alpha,\rho,\gamma}(f) \equiv \iint_D |f(\zeta)|^p (1 - |\zeta|^2 \rho)^\alpha |\zeta|^2 \gamma dm(\zeta) < +\infty.$$ 

Also we will use the following notations:

$$H^p_{\alpha,\rho,\gamma}(D) = \{ f \in H(D) : M^p_{\alpha,\rho,\gamma}(f) < +\infty \},$$

$$h^p_{\alpha,\rho,\gamma}(D) = \{ f \in h(D) : M^p_{\alpha,\rho,\gamma}(f) < +\infty \},$$

where $H(D)$ and $h(D)$ are the sets of all holomorphic and harmonic functions in $D$, respectively.

The spaces above were introduced in [6]. Moreover, for these spaces an analogue of representations (1) and (2) were written out by means of special reproducing kernels adapted to new weight functions (see [6] and [7]):

Let $\rho > 0, Re\beta > -1, Re\varphi > -1$ and $\mu = \frac{1 + \varphi}{\rho}$. Then put

$$S_{\beta,\rho,\varphi}(z, \zeta) = \frac{\rho}{\pi \Gamma(\beta + 1)} \sum_{k=0}^{\infty} \frac{\Gamma(\mu + \beta + 1 + \frac{k}{\rho})}{\Gamma(\mu + \frac{k}{\rho})} z^k \zeta^{-k},$$

where $z \in D$ and $\zeta \in \overline{D}$.

The main properties of the kernel $S_{\beta,\rho,\varphi}(z, \zeta)$ can be summarized in the following theorem (see [6], [7]):

**Theorem 2.** 1. For all $z \in D$ and $\zeta \in \overline{D}$, the series $S_{\beta,\rho,\varphi}(z, \zeta)$ is absolutely convergent.

2. For all $z \in D$ and $\zeta \in \overline{D}$,

$$|S_{\beta,\rho,\varphi}(z, \zeta)| \leq \frac{\text{const}(\beta, \rho, \varphi)}{(1 - |z|)^{2 + Re\beta}}.$$ 

3. $S_{\beta,\rho,\varphi}(z, \zeta)$ can be majorated by a positive convergent series uniformly in $z \in K \subset D$ and $\zeta \in \overline{D}$, where $K$ is a compact set.

4. For a fixed $\zeta \in \overline{D}$, $S$ is holomorphic in $z \in D$. For a fixed $z \in D$, $S$ is antiholomorphic in $\zeta \in D$ and continuous in $\zeta \in \overline{D}$. 

5. For all \( z \in \mathbb{D} \) and \( \zeta \in \mathbb{D} \),

\[
S_{\beta,\rho,\varphi}(z,\zeta) = \frac{\rho}{\pi \Gamma(\beta + 1)} \int_0^\infty e^{-t\mu+\beta}E_\rho\left(t\frac{1}{\rho}z\bar{\zeta};\mu\right) dt,
\]

where \( E_\rho(\cdot;\mu) \) is the well-known Mittag-Leffler type entire function. Moreover, the function under the sign of the integral is majorated by a positive integrable function uniformly in \( z \in K \subset \mathbb{D} \) and \( \zeta \in \mathbb{D} \), where \( K \) is a compact set.

The corresponding generalization of (1) and (2) is formulated as follows (see [6], [7]):

**Theorem 3.** Assume that \( 1 \leq p < +\infty, \rho > 0, \alpha > -1, \gamma > -1 \), complex numbers \( \beta \) and \( \varphi \) are satisfying the conditions

\[
Re \beta \geq \alpha, \quad Re \varphi \geq \gamma,
\]

when \( p = 1 \) and the conditions

\[
Re \beta > \frac{\alpha + 1}{p} - 1, \quad Re \varphi > \frac{\gamma + 1}{p} - 1,
\]

when \( p > 1 \), and put \( \mu = (\varphi + 1)/\rho \). Then each function \( f \in H^p_{\alpha,\rho,\gamma}(\mathbb{D}) \) has the following representation:

\[
f(z) = \int\int_{\mathbb{D}} f(\zeta)S_{\beta,\rho,\varphi}(z,\zeta)(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta), \quad z \in \mathbb{D},
\]

and

\[
\overline{f(0)} = \int\int_{\mathbb{D}} \overline{f(\zeta)}S_{\beta,\rho,\varphi}(z,\zeta)(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta), \quad z \in \mathbb{D}.
\]

In the present paper, we prove an analogous result of the theorem above for harmonic functions from the corresponding weighted \( L^p \)-spaces in the unit disc \( \mathbb{D} \).

## 2 Necessary Estimates

First of all, we intend to strengthen some assertions of the Theorem 2.

**Theorem 4.** Assume \( \rho > 0, Re \beta > -1, Re \varphi > -1, \mu = \frac{1 + \varphi}{\rho}, z \in \mathbb{D}, \zeta \in \overline{\mathbb{D}} \) and the kernel \( S_{\beta,\rho,\varphi}(z,\zeta) \) is defined by (3). If

\[
-1 < a_1 \leq Re \beta \leq a_2 < +\infty, \quad |Im \beta| \leq A < +\infty,
\]

\[
-1 < b_1 \leq Re \varphi \leq b_2 < +\infty, \quad |Im \varphi| \leq B < +\infty
\]
and $\zeta \in \mathbb{D}$, $|z| \leq \lambda < 1$, then the expression of $S_{\beta, \rho, \varphi}$ can be uniformly majorated by the convergent series

$$\text{const}(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=0}^{\infty} (k + 1)^{a_2 + 1} \lambda^k. \quad (4)$$

Moreover,

$$|S_{\beta, \rho, \varphi}(z, \zeta)| \leq \frac{\text{const}(a_1, a_2, A, b_1, b_2, B, \rho)}{(1 - |z||\zeta|)^{2+\text{Re}\beta}} \leq \frac{\text{const}(a_1, a_2, A, b_1, b_2, B, \rho)}{(1 - \lambda)^{2+a_2}}. \quad (5)$$

Proof. According to Stirling formula (see, for example, [8], pp. 158-159) there exist numbers $a, b, 0 < a < b < +\infty$, such that

$$a \leq |\Gamma(\mu + R)| \leq b \quad (6)$$

uniformly in $\mu \in K \subset \{\mu \in \mathbb{C} : \text{Re}\mu > 0\}$ and $0 < \delta \leq R < +\infty$. Hence, in view of (6),

$$|S_{\beta, \rho, \varphi}(z, \zeta)| \leq \frac{\rho |\Gamma(\mu + \beta + 1)|}{\pi |\Gamma(\nu + 1)||\Gamma(\mu)|} \sum_{k=1}^{\infty} \frac{|\Gamma(\mu + \beta + 1 + \frac{k}{\rho})|}{|\Gamma(\mu + \frac{k}{\rho})|} |z|^k |\zeta|^k \leq \text{const}(a_1, a_2, A, b_1, b_2, B, \rho)$$

$$+ \text{const}(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=1}^{\infty} \left( \frac{k}{\rho} \right)^{\text{Re}\mu + \text{Re}\beta + 1 + \frac{k}{\rho} - 1} \cdot e^{-\frac{k}{\rho}} |z|^k |\zeta|^k$$

$$= \text{const}(a_1, a_2, A, b_1, b_2, B, \rho) + \text{const}(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=1}^{\infty} k^{\text{Re}\beta + 1} |z|^k |\zeta|^k \leq \text{const}(a_1, a_2, A, b_1, b_2, B, \rho) + \text{const}(a_1, a_2, A, b_1, b_2, B, \rho) \sum_{k=1}^{\infty} k^{a_2 + 1} |z|^k |\zeta|^k.$$

From here (4) follows. As to (5), it directly follows from the estimation

$$k^{\text{Re}\beta + 1} \sim \frac{\Gamma(k + \text{Re}\beta + 2)}{\Gamma(\text{Re}\beta + 2)\Gamma(k + 1)}.$$
and binomial expansion

\[
\frac{1}{(1-x)^s} = \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)\Gamma(k+1)} x^k.
\]

**Corollary.** For a fixed \( z \in \mathbb{D} \) and \( \zeta \in \mathbb{D} \), the kernel \( S_{\beta,\rho,\varphi}(z, \zeta) \) is holomorphic in \( \beta \) and \( \varphi \) with \( \Re\beta > -1 \) and \( \Re\varphi > -1 \).

**Theorem 5.** Assume that \( 1 \leq p < \infty \), \( \rho > 0 \), \( \alpha > -1 \), \( \gamma > -1 \) and \( f \in L_{\alpha,\rho,\gamma}^p(\mathbb{D}) \). Then there exists a positive function \( \Phi(\zeta) \in L^1(\mathbb{D}) \) such that

\[
|f(\zeta)(1-|\zeta|^{2\rho})^\beta \cdot |\zeta|^{2\varphi} \cdot S_{\beta,\rho,\varphi}(z, \zeta)| \leq \Phi(\zeta), \quad \zeta \in \mathbb{D},
\]

uniformly in \( z \in K \subset \mathbb{D} \) and in \( \beta \) and \( \varphi \) satisfying the conditions

\[
\alpha = a_1 \leq \Re\beta \leq a_2, \quad |\Im\beta| \leq A,
\]

\[
\gamma = b_1 \leq \Re\varphi \leq b_2, \quad |\Im\varphi| \leq B,
\]

when \( p = 1 \) and the conditions

\[
\frac{\alpha+1}{p} - 1 < a_1 \leq \Re\beta \leq a_2, \quad |\Im\beta| \leq A,
\]

\[
\frac{\gamma+1}{p} - 1 < b_1 \leq \Re\varphi \leq b_2, \quad |\Im\varphi| \leq B,
\]

when \( p > 1 \).

**Proof.** Note that under the assumptions of the theorem, \( |S_{\beta,\rho,\varphi}(z, \zeta)| \leq \text{const} \cdot |S_{\beta,\rho,\varphi}(z, \zeta)| < +\infty, \zeta \in \overline{\mathbb{D}} \) (see [5]). If \( p = 1 \), then

\[
|f(\zeta) \cdot (1-|\zeta|^{2\rho})^\beta \cdot |\zeta|^{2\varphi} \cdot S_{\beta,\rho,\varphi}(z, \zeta)| \leq \text{const} \cdot |f(\zeta)| \cdot (1-|\zeta|^{2\rho})^{\Re\beta} \cdot |\zeta|^{2\Re\varphi}
\]

\[
\leq \text{const} \cdot |f(\zeta)| \cdot (1-|\zeta|^{2\rho})^{\alpha} \cdot |\zeta|^{\gamma} = \Phi(\zeta) \in L^1(\mathbb{D}),
\]

while for \( p > 1 \) we can write

\[
|f(\zeta) \cdot (1-|\zeta|^{2\rho})^\beta \cdot |\zeta|^{2\varphi} \cdot S_{\beta,\rho,\varphi}(z, \zeta)|
\]

\[
\leq \text{const} \cdot |f(\zeta)| \cdot (1-|\zeta|^{2\rho})^{\Re\beta} \cdot |\zeta|^{2\Re\varphi}
\]

\[
\leq \text{const} \cdot |f(\zeta)| \cdot (1-|\zeta|^{2\rho})^{\alpha_1} \cdot |\zeta|^{\beta_1}
\]

\[
= \text{const} \cdot |f(\zeta)| \cdot (1-|\zeta|^{2\rho})^{\frac{\alpha_1}{2}} \cdot |\zeta|^{\frac{\beta_1}{2}} \cdot (1-|\zeta|^{2\rho})^{\alpha_1 - \frac{\alpha_1}{2}} \cdot |\zeta|^{\beta_1 - \frac{\beta_1}{2}} = \Phi(\zeta).
\]

It remains to show that \( \Phi(\zeta) \in L^1(\mathbb{D}) \). We have
\[ \int \int_{D} \Phi(\zeta) \, dm(\zeta) \leq \text{const} \cdot \left( \int \int_{D} |f(\zeta)|^p (1 - |\zeta|^{2\rho})^\alpha |\zeta|^{2\gamma} \, dm(\zeta) \right)^{1/p} \times \left( \int \int_{D} (1 - |\zeta|^{2\rho})^{q(a_1 - \frac{2}{p})} |\zeta|^{2q(b_1 - \frac{2}{p})} \, dm(\zeta) \right)^{1/q}, \]

and the convergence of the second integral follows from the conditions of the theorem. \[ \square \]

3 Weighted integral representations for the main classes of holomorphic and harmonic functions

**Theorem 6.** Assume \( 1 \leq p < \infty, \rho > 0, \alpha > -1, \gamma > -1, \) complex numbers \( \beta \) and \( \varphi \) are satisfying the conditions
\[ \Re \beta \geq \alpha, \quad \Re \varphi \geq \gamma, \quad (7) \]
when \( p = 1 \) and the conditions
\[ \Re \beta > \frac{\alpha + 1}{p} - 1, \quad \Re \varphi > \frac{\gamma + 1}{p} - 1, \quad (8) \]
when \( p > 1 \), and put \( \mu = \frac{\varphi + 1}{\rho} \). Let \( f = u + iv \in H^p_{\alpha, \rho, \gamma} \). Then the following integral representations hold: for all \( z \in D \)
\[ f(z) = iv(0) + \int \int_{D} u(\zeta) \left( 2S_{\beta, \rho, \varphi}(z, \zeta) - \rho \frac{\Gamma(\mu + \beta + 1)}{\pi \Gamma(\beta + 1) \Gamma(\mu)} \right) (1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} \, dm(\zeta) \quad (9) \]
and
\[ u(z) = \int \int_{D} u(\zeta) \left( S_{\beta, \rho, \varphi}(z, \zeta) + S_{\beta, \rho, \varphi}(\zeta, z) - \rho \frac{\Gamma(\mu + \beta + 1)}{\pi \Gamma(\beta + 1) \Gamma(\mu)} \right) (1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} \, dm(\zeta) \]
\[ (10) \]
Proof. First of all, note that

\[ S_{\beta,\rho,\varphi}(0,\zeta) = S_{\beta,\rho,\varphi}(z,0) = \frac{\rho \Gamma(\mu + \beta + 1)}{\pi \Gamma(\beta + 1) \Gamma(\mu)} \]  
\[ z \in \mathbb{D}, \quad \zeta \in \overline{\mathbb{D}}. \]  
(11)

Further, due to Corollary 1, the expressions under the signs of the integrals are holomorphic in $\beta$ and $\varphi$, $Re\beta > -1, Re\varphi > -1$ for fixed $z$ and $\zeta$. Hence, according to Theorem 5, for a fixed $z \in \mathbb{D}$, the right-hand side of (9) and (10) are also holomorphic in $\beta$ and $\varphi$ when (8) (or (7), depending on $p$) is satisfied. Thus, in view of the uniqueness theorem for holomorphic functions in two complex variables, we can additionally suppose (without loss of generality) that $\beta$ and $\varphi$ are real. According to \[7, Theorem 4.2],

\[ f(z) = \int\int_{\mathbb{D}} f(\zeta) S_{\beta,\rho,\varphi}(z,\zeta)(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta), \quad z \in \mathbb{D}, \]  
(12)

and

\[ f(0) = \int\int_{\mathbb{D}} f(\zeta) S_{\beta,\rho,\varphi}(z,\zeta)(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta), \quad z \in \mathbb{D}. \]  
(13)

Summation of (12) and (13) yields

\[ f(z) + f(0) = \int\int_{\mathbb{D}} 2u(\zeta) S_{\beta,\rho,\varphi}(z,\zeta)(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta). \]  
(14)

According to (12) and (11),

\[ f(0) = \int\int_{\mathbb{D}} f(\zeta) \frac{\Gamma(\mu + \beta + 1)}{\Gamma(\beta + 1) \Gamma(\mu)} \rho \frac{1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta), \quad z \in \mathbb{D}. \]  

Hence,

\[ u(0) = \int\int_{\mathbb{D}} u(\zeta) \frac{\Gamma(\mu + \beta + 1)}{\Gamma(\beta + 1) \Gamma(\mu)} \rho \frac{1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta), \quad z \in \mathbb{D}. \]  
(15)

Combination of (14) and (15) immediately gives (9).

Further, taking the real parts in (14) we get

\[ u(z) = \int\int_{\mathbb{D}} u(\zeta) (S_{\beta,\rho,\varphi}(z,\zeta) + S_{\beta,\rho,\varphi}(\zeta,z))(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta) - u(0). \]  
(16)

The formulas (16) and (15) together establish (10). \qed
Theorem 7. Assume $1 \leq p < \infty$, $\rho > 0$, $\alpha > -1$, $\gamma > -1$, complex numbers $\beta$ and $\varphi$ are satisfying the conditions

\[ \Re \beta \geq \alpha, \quad \Re \varphi \geq \gamma, \]

when $p = 1$ and the conditions

\[ \Re \beta > \frac{\alpha + 1}{p} - 1, \quad \Re \varphi > \frac{\gamma + 1}{p} - 1, \]

when $p > 1$, and put $\mu = (\varphi + 1)/\rho$. For each $u \in h^p_{\alpha, \rho, \gamma}(\mathbb{D})$, the representation (10) holds.

Proof. Repeating the argument from the proof of Theorem 6, we can assume additionally, that $\beta \in \mathbb{R}, \varphi \in \mathbb{R}$ and $\beta > 0$.

Since $u$ is a harmonic function, there exists a holomorphic function $f$ in $\mathbb{D}$ such that $u = Rf$ in $\mathbb{D}$. Fix an arbitrary $z_0 \in \mathbb{D}$ and denote $f_r(\zeta) = f(r \zeta)$. Obviously, $f_r \in H^p_{\alpha, \rho, \gamma}(\mathbb{D})$. As $u(r \zeta) = Rf(r \zeta)$, from (10) we obtain

\[
\begin{align*}
\int \int_{|\zeta| < r} u(\zeta) & \left( S_{\beta, \rho, \varphi}(z_0, \zeta) + S_{\beta, \rho, \varphi}(\zeta, z_0) - \frac{\rho}{\pi} \frac{\Gamma(\mu + \beta + 1)}{\Gamma(\beta + 1)\Gamma(\mu)} \right) \\
& = \int \int_{|\zeta| < r} \left( 1 - |\zeta|^2 \right)^{\beta} |\zeta|^2 \frac{dm(\zeta)}{r^{2\varphi}} \\
& = \int \int_{\mathbb{D}} u(\zeta) \chi_r(\zeta) \left( S_{\beta, \rho, \varphi}(z_0, \zeta) + S_{\beta, \rho, \varphi}(\zeta, z_0) - \frac{\rho}{\pi} \frac{\Gamma(\mu + \beta + 1)}{\Gamma(\beta + 1)\Gamma(\mu)} \right) \\
& = \int \int_{\mathbb{D}} u(\zeta) \chi_r(\zeta) \left( 1 - \frac{|\zeta|^2}{r^{2\varphi}} \right)^{\beta} |\zeta|^2 \frac{dm(\zeta)}{r^{2\varphi}} = I_r,
\end{align*}
\]

where $\chi_r(\zeta)$ is the characteristic function of the disc $\{ \zeta : |\zeta| < r \}$ and $I_r$ stands for the right-most integral in the expression above. Assume also that $0 < r_0 \leq r < 1$ for some $r_0$. We intend to let $r \uparrow 1$ in the both sides of

\[ u(r \cdot z_0) = I_r. \quad (17) \]

As a result, at the left-hand side of (17) we will obtain $u(z_0)$, while the right hand-side of (17) will coincide with the one in (10) for $z = z_0$.

Hence, it remains to show that such passage of the limit is legitimate. To do so, we will use the Lebesgue dominated convergence theorem. Using
and the assumption $\beta > 0$, for the expression under the sign of integral $I_r$, we obtain

$$
\left| u(\zeta) \chi_r(\zeta) \left( S_{\beta,\rho,\varphi} \left( z_0, \frac{\zeta}{r} \right) + S_{\beta,\rho,\varphi} \left( \frac{\zeta}{r}, z_0 \right) - \frac{\rho \Gamma(\mu + \beta + 1)}{\pi \Gamma(\beta + 1) \Gamma(\mu)} \right) \right| 
\leq \frac{|u(\zeta)|}{r^{2\rho + 2\varphi + 2}} |\chi_r(\zeta)|(r^{2\rho} - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} \left( \frac{\text{const}}{(1 - |z_0|)^{\beta + 2}} + \frac{\rho \Gamma(\mu + \beta + 1)}{\pi \Gamma(\beta + 1) \Gamma(\mu)} \right) 
\leq \text{const} |u(\zeta)|(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} = \text{const} |u(\zeta)|(1 - |\zeta|^{2\rho})^\alpha |\zeta|^{2\gamma} \equiv \psi(\zeta).
$$

We want to show that $\psi(\zeta) \in L^1(\mathbb{D})$, which is equivalent to show that $|u(\zeta)|(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} \in L^1(\mathbb{D})$. For $p = 1$, we have

$$
|u(\zeta)|(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} \leq |u(\zeta)|(1 - |\zeta|^{2\rho})^\alpha |\zeta|^{2\gamma} \in L^1(\mathbb{D}).
$$

If $p > 1$, using Holder inequality, we get

$$
\int_\mathbb{D} |u(\zeta)|(1 - |\zeta|^{2\rho})^\beta |\zeta|^{2\varphi} dm(\zeta)
\leq \left( \int_\mathbb{D} |u(\zeta)|^p (1 - |\zeta|^{2\rho})^\alpha |\zeta|^{2\gamma} dm(\zeta) \right)^{1/p}
\leq \left( \int_\mathbb{D} |u(\zeta)|^p (1 - |\zeta|^{2\rho})^\alpha |\zeta|^{2\gamma} dm(\zeta) \right)^{1/p}
\leq \left( \int_\mathbb{D} (1 - |\zeta|^{2\rho})^\alpha |\zeta|^{2\gamma} dm(\zeta) \right)^{1/q}.
$$

The convergence of the last integral follows from the conditions of the theorem. \qed

**Remark.** In [9]-[13], one can find various interesting results relating to the weighted integral representations of harmonic functions.

**References**


Feliks Hayrapetyan
Yerevan State University
1 Alex Manoogian St, 0025 Yerevan, Armenia.
feliks.hayrapetyan1995@gmail.com

Please, cite to this paper as published in
https://doi.org/10.52737/18291163-2021.13.5-1-11