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Viscosity approximation method for solving variational inequality problem in real Banach spaces

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Abstract. In this paper, we study the implicit and inertial-type viscosity approximation method for approximating a solution to the hierarchical variational inequality problem. Under some mild conditions on the parameters, we prove that the sequence generated by the proposed methods converges strongly to a solution of the above-mentioned problem in q-uniformly smooth Banach spaces. The results obtained in this paper generalize and improve many recent results in this direction.

Key Words: Fixed Point, Hierarchical Fixed Point Problems, Strongly Accretive Mapping, Lipschitzian Mapping, Nonexpansive Mapping
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1 Introduction

Let E be a real Banach space and K be a nonempty, closed and convex subset of E. We denote by $J_q, 1 < q < \infty$ the generalized duality mapping from E to 2^{E^*} (E^* is the dual space of E) defined by

$$J_q = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^q, ||f^*|| = ||x||^{q-1}\},\ x \in E,$$

where $\langle ., . \rangle$ denotes the duality pairing between element of E and that of E^*. If $q = 2$, J_2 simply denoted by J, is called the normal duality mapping. The space E is said to have weakly (sequentially) continuous duality map if J_q is singled valued and (sequentially) continuous. It is known (see, for example, [32]) that $J_q(x) = ||x||^{q-2}J(x)$ if $x \neq 0$, and if E^* is strictly convex then J_q is single valued. Also all ℓ_p spaces, $(1 < p < \infty)$ have weakly sequentially continuous duality mappings.

Let $T : D(T) \subset E \to E$ be a nonlinear mapping, where $D(T)$ denotes the domain of T. A point $x \in D(T)$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in E : Tx = x\}$. The mapping T is said to be

(i) *accretive* if for all $x, y \in D(T)$, there exists $j_q(x-y) \in \ J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \ \geq 0; \tag{1}$$

(ii) η-*strongly accretive* if for all $x, y \in D(T)$, there exists $j_q(x-y) \in J_q(x-y)$ and $\eta \in (0,1)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \geq \eta||x-y||^q; \tag{2}$$

(iii) κ-*Lipschitzian*, if for some $\kappa > 0$,

$$||Tx - Ty|| \leq \kappa||x-y||, \quad x, y \in D(T).$$

If $\kappa \in (0,1)$, then T is a *contraction* mapping, while T is called *nonexpansive* if $\kappa = 1$.

In a Hilbert space H, accretive mapping is called monotone mapping, while inequalities (1) and (2) holds by replacing j_q with identity map on H.

Let $A : K \to E$ be a nonlinear mapping. The variational inequality problem defined on K and A is the following problem:

$$\text{Find } u \in K \text{ such that } \langle Au, j_q(v-u)\rangle \geq 0 \text{ for any } v \in K. \tag{3}$$

We denote the solution set of variational inequality problem by $VI(K, A)$. The variational inequality problem (VIP) (3) was first introduced by Stampacchia [24] and have been used as a strong methodology in studying broad range of problems in both science and applied science (see, for example, [4, 9, 10, 13, 17, 23, 29, 34, 35, 36] and references therein). In particular, viscosity approximation techniques have shown to be an efficient and implementable iterative method to an find approximate solution to VIPs.

In 2006, Marino and Xu [18] used concept of viscosity method introduced by Moudafi [19] to study the following general iterative method for approximating fixed points of nonexpansive mapping in a real Hilbert space:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)Tx_n, \ n \geq 0, \tag{4}$$

where A is a strongly positive mapping, f is contraction on nonempty closed and convex subset C of a Hilbert space H and T is a nonexpansive mapping. Under some appropriate conditions on $\{\alpha_n\}$, they proved that the sequence $\{x_n\}$ generated by (4) converges strongly to a unique solution $x^* \in F(T)$ of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \ x \in F(T),$$

which is optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

On the other hand, Yamada [33] introduced the following hybrid iteration process for approximating solutions of the variational inequality:

$$x_{n+1} = Tx_n - \mu\lambda_n G(Tx_n),\ n \geq 0, \tag{5}$$

where G is κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0,\ \eta > 0,\ 0 < \mu < 2\eta/k^2$. Under some appropriate conditions on $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (5) converges strongly to a unique solution $x^* \in F(T)$ of the variational inequality

$$\langle Gx^*, x - x^*\rangle \geq 0,\ x \in F(T).$$

By combining (4) and (5), Tian [26] introduced the following general viscosity method for approximating fixed point of nonexpansive mapping in real Hilbert spaces:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (1 - \mu\alpha_n G)Tx_n,\ n \geq 0, \tag{6}$$

where G is a κ-Lipschitzian and η-strongly monotone operator, f is a contraction on C and T is a nonexpansive mapping. He proved that (6) converges strongly to a unique solution $x^* \in F(T)$ of the variational inequality

$$\langle(\mu G - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in F(T),$$

in the frame work of a real Hilbert space.

On the other hand, Moudafi and Mainge [20] introduced the following hirarchical fixed point problem (HFPP) for a nonexpansive mapping T with respect to a nonexpansive mapping S on C:

Find $z \in F(T)$ such that $\langle z - Sz, z - y\rangle \leq 0$ for any $y \in F(T)$.

In 2011, Ceng et al. [8] generalized the iterative method (6) of Tian [26] by replacing the contraction mapping f with Lipschitzian mapping U. They studied the following iterative method:

$$x_{n+1} = P_C[\alpha_n\gamma U(x_n) + (I - \alpha_n\mu G)Tx_n], \quad n \geq 0, \tag{7}$$

where P_C is a metric projection onto C and T is a nonexpansive mapping, and proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a unique solution of the variational inequality

$$\langle(\mu G - \gamma U)x^*, x - x^*\rangle \geq 0, \quad x \in F(T). \tag{8}$$

Wang and Xu [28] introduced the following iterative method to solve HFPP

$$\begin{cases} y_n = \beta_n Sx_n + (1 - \beta_n)x_n, \\ x_{n+1} = P_C[\alpha_n\gamma U(x_n) + (I - \alpha_n\mu G)Ty_n], \quad n \in \mathbb{N}, \end{cases} \tag{9}$$

where S and T are nonexpansive mappings on C, while U and G are L_1-Lipschitzian and L_2-Lipschitzian, $\eta-$strongly monotone mappings, respectively. Under some assumptions on the parameters, they proved that the sequence $\{x_n\}$ generated by (9) converges strongly to the hierarchical fixed point of T with respect to the mapping S which is a unique solution to the variational inequality (8).

In 2017, supposing that the operator G is $\eta-$inverse strongly monotone, Tian and Jiang [27] studied zero points of inverse strongly monotone mapping and fixed points of nonexpansive mapping in Hilbert space using the following method:

$$x_{n+1} = (1 - \mu\lambda_n G)((1 - \alpha_n)x_n + \alpha_n Tx_n),\ n \geq 0, \tag{10}$$

where T is a nonexpansive mapping. Under some conditions on the parameters $\{\mu\lambda_n\}, \{\alpha_n\}$, they proved that the sequence $\{x_n\}$ generated by (10) converges weakly to a point $w \in \Gamma := F(T) \cap G^{-1}0$, where $G^{-1}0$ is zero point of G and $w = \lim_{n\to\infty} P_\Gamma x_n$ which is also a special point in $VI(F(T), G)$.

Furthermore, to speed up convergence rate of algorithms, Polyak [21] studied the heavy ball method, an inertial extrapolation process for minimizing a smooth convex function. Since then, many authors have introduced this technique in different methods for solving variational inequality problems (see, for example, [1, 2, 3, 5, 6] for more details).

Recently, Tan and Li [25] introduced inertial Mann algorithms to find solutions to HFPP of nonexpansive mappings in a real Hilbert space as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \beta_n S w_n + (1 - \beta_n)w_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, \quad n \in \mathbb{N}, \end{cases} \tag{11}$$

where S and T are nonexpansive mappings on C and f is a contraction mapping. Under some assumptions on the parameters, they proved that the sequence $\{x_n\}$ generated by (11) converges strongly to the hierarchical fixed point u of T with respect to nonexpansive mapping S, where $u = P_{F(T)}f(u)$.

The following question naturally arises: Can the results of Ceng et al. [8], Wang and Xu [28], Tian and Jiang [27] and Tan and Li [25] be extended from Hilbert spaces to Banach spaces?

Motivated and inspired by ongoing research in this direction, our purpose in this study is to provide an affirmative answer to the question mentioned above by introducing an inertial-type viscosity approximation method for solving hierarchical variational inequality problem in $q-$uniformly smooth Banach spaces. Under suitable conditions on the parameters, we prove a strong convergence theorem for our proposed methods for solving some variational inequality problems. Our result extends and generalizes the results of Ceng et al. [8], Wang and Xu [28], Tian and Jiang [27] and Tan and Li [25].

2 Preliminaries

Let K be a nonempty, closed, convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) := \sup\{||x-y|| : x, y \in K\}$. For each $x \in K$, let $r(x,K) := \sup\{||x-y|| : \ y \in K\}$ and let $r(K) := \inf\{r(x,K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced in 1980 by Bynum [7]; see also Lim [14] and references therein) is defined by

$$N(E) := \inf \frac{d(K)}{r(K)}$$

where infimum is taken over all closed convex and bounded subsets K of E with $d(K) > 0$. A space E such that $N(E) > 1$ is said to have uniform normal structure . It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [11, 15]). Let E be a normed space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0; \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1; ||y|| = \tau\right\}.$$

The space E is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \rho_E(t)/t = 0$. For some positive constant $q \in E$, E is called q- *uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q,\ t > 0$. It is well known that if E is smooth, then the duality mapping is singled-valued, and if E is uniformly smooth (see, e.g., [11, 15]), then the duality mapping is norm-to-norm uniformly continuous on bounded subset of E.

Lemma 1 *[11] Let E be a real normed space. Then*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle$$

for all $x, y \in E$ and for all $j(x+y) \in J(x+y)$.

Lemma 2 *(Xu [31]) Let E be a real q-uniformly smooth Banach space for some $q > 1$. Then there exists a positive constant d_q such that*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + d_q||y||^q$$

for any $x, y \in E$ and $j_q \in J_q(x)$.

Lemma 3 *(Xu [30]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0$$

where, (i) $\{a_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0, n \geq 0$, $\sum \gamma_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.

Lemma 4 *(Lim and Xu [12]) Suppose E is a Banach space with uniform normal structure, K is a nonempty bounded subset of E, and $T : K \to K$ is uniformly $\kappa-$Lipschitzian mapping with $\kappa < (E)^{1/2}$. Suppose also that there exists a nonempty bounded closed and convex subset C of K with the following property:*

$$x \in C \text{ implies } \omega_w(x) \subset C,$$

where $\omega_w(x)$ is the $\omega-$limit set of T at x, i.e.,

$$\omega_w(x) = \{y \in E : y = weak - \lim_j T^{n_j}x \text{ for some } n_j \to \infty\}.$$

Then T has a fixed point in C.

Lemma 5 *(Jung [16]) Let K be a nonempty, closed and convex subset of a reflexive Banach space E which satisfies Opial's condition, and suppose $T : K \to E$ is nonexpansive. Then $I - T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$, $x_n - Tx_n \to 0$ implies that $x = Tx$.*

The following lemma can easily be proven by simplifying the left side of (12), and, therefore, we omit the proof.

Lemma 6 *Let E be a real Banach space, $U : E \to E$ be a γ-Lipschitzian mapping with constant $\gamma > 0$ and $F : E \to E$ be κ-Lipschitzian and η-strongly accretive operator with $\kappa > 0$, $\eta \in (0,1)$. Then $(\mu F - \rho U)$ is strongly accretive with coefficient $(\mu\eta - \rho\gamma)$. That is, for $\rho \in (0, \mu\eta/\gamma)$,*

$$\langle(\mu F - \rho U)x - (\mu F - \rho U)y, j(x - y) \geq (\mu\eta - \rho\gamma)||x - y||^2, \ x, y \in E. \quad (12)$$

Let μ be a linear continuous functional on l^∞ and let $a = (a_1, a_2, \cdots) \in l^\infty$. We will sometimes write $\mu_n(a_n)$ in place of the value $\mu(a)$. A linear continuous functional μ such that $||\mu|| = 1 = \mu(1)$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2 \cdots) \in l^\infty$ is called a *Banach limit*. It is known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \cdots) \in l^\infty$ (see, for example, [11, 12]).

3 Main Results

We start with the following lemma.

Lemma 7 *Let $q > 0$ be a fixed number and E be real $q-$uniformly smooth Banach space with constant d_q. Let $T, S : E \to E$ be a nonexpansive mappings such that $F(T) \neq \emptyset$. Let $U : E \to E$ be a $\gamma-$Lipschitzian mapping with coefficient $\gamma \geq 0$ and $F : E \to E$ be an $\eta-$strongly accretive mapping which is also $\kappa-$Lipschizian. Assume $\mu \in \Big(0, \min\big\{1, (q\eta)/(d_q\kappa^q)^{1/(q-1)}\big\}\Big)$ and $\tau := \mu\Big(\eta - (\mu^{(q-1)}d_q\kappa^q)/q\Big)$ for each $t, \beta \in (0,1)$ with $\beta < t$ and $\rho \in (0, \tau/\gamma)$, and define a map $T_t : E \to E$ by*

$$T_t x = t\rho U(x) + (1 - t\mu F)T[\beta S(x) + (1-\beta)x], \ x \in E.$$

Then for any $x, y \in E$,

$$||T_t x - T_t y|| \leq [1 - t(\tau - \rho\gamma)]||x - y||,$$

which means that T_t is a contraction mapping.

Proof. Without loss of generality, we may assume that $\eta < 1/q$, since $\mu < (q\eta/(d_q\kappa^q))^{1/(q-1)}$. Then $0 < q\eta - \mu^{q-1}d_q\kappa^q < 1$. Also, since $\mu < 1$ and $t \in (0,1)$, we have $0 < t\mu(q\eta - \mu^{q-1}d_q\kappa^q) < 1$.

For each $t \in (0,1)$, define a map $K_t : E \to E$ by

$$K_t x := (1 - t\mu F)Tx, \ x \in E.$$

Then for any $x, y \in E$, we get

$$\begin{aligned}
||K_t x - K_t y||^q &= ||(1 - t\mu F)Tx - (1 - t\mu F)Ty||^q \\
&= ||(Tx - Ty) - t\mu(F(Tx) - F(Ty))||^q \\
&\leq ||Tx - Ty||^q - qt\mu\langle F(Tx) - F(Ty), j_q(Tx - Ty)\rangle \\
&\quad + t^q\mu^q d_q||Tx - Ty||^q \\
&\leq \big[1 - t\mu(q\eta - t^{q-1}\mu^{q-1}\kappa^q d_q)\big]||x - y||^q \\
&\leq \big[1 - qt\mu(q\eta - \frac{\mu^{q-1}\kappa^q d_q}{q})\big]||x - y||^q \\
&\leq \big[1 - t\mu(q\eta - \frac{\mu^{q-1}\kappa^q d_q}{q})\big]^q||x - y||^q \\
&= (1 - t\tau)^q||x - y||^q.
\end{aligned}$$

Letting $V_\beta := \beta S + (1-\beta)$, we obtain

$$\begin{aligned}
||K_t(V_\beta x) - K_t(V_\beta y)|| &\leq ||K_t[\beta Sx + (1-\beta)x] - K_t[\beta Sy + (1-\beta)y]|| \\
&\leq (1 - t\tau)\big[\beta||Sx - Sy|| + (1-\beta)||x - y||\big] \\
&\leq (1 - t\tau)||x - y||.
\end{aligned}$$

Thus,

$$\begin{aligned} ||T_t x - T_t y|| &= ||t\rho(Ux - Uy) + (K_t(V_\beta x) - K_t(V_\beta y))|| \\ &\leq t\rho||Ux - Uy|| + ||K_t(V_\beta x) - K_t(V_\beta y)|| \\ &\leq t\rho\gamma||x - y|| + (1 - t\tau)||x - y|| \\ &= \big[1 - t(\tau - \rho\gamma)\big]||x - y||, \end{aligned}$$

and it follows that T_t is a contraction mapping with coefficient $1 - t(\tau - \rho\gamma)$ in $(0, 1)$. Therefore, by Banach contraction mapping principle, there exists a unique fixed point x_t of T_t in E such that

$$x_t = t\rho U(x_t) + (1 - t\mu F)T[\beta S(x_t) + (1 - \beta)x_t]. \tag{13}$$

□

Theorem 1 *Let E, U, T, S and F be defined as in Lemma 7 and let $\{x_t\}_{t\in(0,1)}$ be defined by (13). Then $\{x_t\}$ converges strongly to $p \in F(T)$ which is a unique solution of the variational inequality*

$$\langle(\mu F - \rho U)p, j(p - x)\rangle \leq 0, \quad x \in F(T). \tag{14}$$

Proof. By Lemma 6, $(\mu F - \rho U)$ is strongly accretive, hence variational inequality (14) has a unique solution in $F(T)$. For $x^* \in F(T)$ and $\beta \leq t$, we have

$$\begin{aligned} ||x_t - x^*|| &= ||t\rho U(x_t) + (1 - t\mu F)T[\beta Sx_t + (1 - \beta)x_t] - x^*|| \\ &= ||t\rho[U(x_t) - U(x^*)] + t[\rho U(x^*) - \mu F(x^*)] \\ &\quad + (1 - t\mu F)T[\beta Sx_t + (1 - \beta)x_t] - (1 - t\mu F)x^*|| \\ &\leq t\rho\gamma||x_t - x^*|| + t||\rho U(x^*) - \mu F(x^*)|| \\ &\quad + (1 - \tau t)\big[\beta||Sx^* - x^*|| + ||x_t - x^*||\big] \\ &\leq [1 - t(\tau - \rho\gamma)]||x_t - x^*|| \\ &\quad + t\big[||\rho U(x^*) - \mu F(x^*)|| + ||Sx^* - x^*||\big]. \end{aligned}$$

Thus,

$$||x_t - x^*|| \leq \frac{1}{\tau - \rho\gamma}\big[||\rho U(x^*) - \mu F(x^*)|| + ||Sx^* - x^*||\big].$$

Therefore, $\{x_t\}$ is bounded, and this implies that $\{S(x_t)\}$, $\{T(x_t)\}$, $\{F(Tx_t)\}$ and $\{U(x_t)\}$ are also bounded. Furthermore, from (13) we get

$$||x_t - Tx_t|| \leq t||\rho U(x_t) - \mu F[\beta S(x_t) + (1 - \beta)x_t]|| + \beta||Sx_t - x_t|| \to 0 \tag{15}$$

as $t \to 0,\ \beta \to 0$.

Next, let $\{t_n\}$ be a sequence in $(0,1)$ such that $\{x_{t_n}\}$ satisfies (13). To simplify notations, we will write $\{x_n\}$ instead of $\{x_{t_n}\}$. Let $\phi : E \to \mathbb{R}$ be a map defined by

$$\phi(x) := \mu_n||x_n - x^*||^2, \ x \in E.$$

Then $\phi(x) \to \infty$ as $||x|| \to \infty$, ϕ is continuous and convex. Since E is reflexive, there exists $x^* \in E$ such that $\phi(x^*) = \min_{u\in E} \phi(u)$. Hence,

$$K^* := \{x \in E : \phi(x) = \min_{u\in E} \phi(u)\} \neq \emptyset.$$

Now let $x \in K^*$ and $v := \omega - lim_j T^{m_j} x$. Note that ϕ is lower semi-continuous and convex, and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ implies $\lim_{n\to\infty} ||x_n - T^m x_n|| = 0$ for any $m \geq 1$. Then by induction we have

$$\begin{aligned}
\phi(v) &= \liminf_{j\to\infty} \phi(T^{m_j} x_n) \leq \limsup_{m\to\infty} \phi(T^m x) \\
&= \limsup_{m\to\infty} (\mu_n||x_n - T^m x||^2) \\
&= \limsup_{m\to\infty} (\mu_n||x_n - T^m x + T^m x_n - T^m x||^2) \\
&\leq \limsup_{m\to\infty} (\mu_n||x_n - x||^2) = \phi(x) \\
&= \min_{u\in E} \phi(u).
\end{aligned}$$

Thus, $v \in K^*$. By Lemma 4, T has a fixed point in K^*, and, therefore, $K^* \cap F(T) \neq \emptyset$. Let $p \in K^* \cap F(T)$, then it follows that $\phi(p) \leq \phi(p - \epsilon(\rho U - \mu F)p)$, and using Lemma 1, we get

$$||x_n - p + \epsilon(\mu F - \rho U)p||^2 \leq ||x_n - p||^2 - 2\epsilon\langle(\mu F - \rho U)p, j(x_n - p + \epsilon(\mu F - \rho U)p)\rangle$$

which implies

$$\mu_n\langle(\rho U - \mu F)p, j(x_n - p) + \epsilon(\mu F - \rho U)p\rangle \leq 0.$$

Moreover,

$$\begin{aligned}
&\mu_n\langle(\rho U - \mu F)p, j(x_n - p)\rangle \\
&\quad = \mu_n\langle(\rho U - \mu F)p, j(x_n - p) - j(x_n - p + \epsilon(\mu F - \rho U)p)\rangle \\
&\quad\quad + \mu_n\langle(\rho U - \mu F)p, j(x_n - p + \epsilon(\mu F - \rho U)p)\rangle \\
&\quad \leq \mu_n\langle(\rho U - \mu F)p, j(x_n - p) - j(x_n - p + \epsilon(\mu F - \rho U)p)\rangle.
\end{aligned}$$

Since j is *norm-to-norm* uniformly continuous on bounded subsets of E, we obtain

$$\mu_n\langle(\rho U - \mu F)p, j(x_n - p)\rangle \leq 0. \tag{16}$$

Also, by (13), we have

$$
\begin{aligned}
&||x_n - p||^2 \\
&= \langle t_n \rho U(x_n) + (1 - t_n \mu F)T[\beta_n S(x_n) + (1-\beta_n)x_n] - p, j(x_n - p)\rangle \\
&= t_n \langle \rho U(p) - \mu F(p), j(x_n - p)\rangle + t_n \langle \rho U(x_n) - \rho U(p), j(x_n - p)\rangle \\
&\quad + \langle (1 - t_n \mu F)T[\beta_n S(x_n) + (1-\beta_n x_n)] - (1 - t_n\mu F)p, j(x_n - p)\rangle \\
&\leq t_n \langle (\rho U - \mu F)p, j(x_n - p)\rangle + t_n \rho\gamma ||x_n - p||^2 \\
&\quad + (1 - \tau t_n)[||x_n - p|| + \beta_n ||Sp - p||]||x_n - p|| \\
&\leq t_n \langle (\rho U - \mu F)p, j(x_n - p)\rangle + [1 - t_n(\tau - \rho\gamma)]||x_n - p||^2 \\
&\quad + \frac{\beta_n}{2}[||Sp - p||^2 + ||x_n - p||^2] \\
&\leq t_n \langle (\rho U - \mu F)p, j(x_n - p)\rangle \\
&\quad + \big(1 - \frac{t_n[2(\tau - \rho\gamma) - 1]}{2}\big)||x_n - p||^2 + \frac{\beta_n}{2}||Sp - p||||x_n - p||,
\end{aligned}
$$

and thus,

$$
\mu_n ||x_n - p||^2 \leq \mu_n \Big(\frac{2\langle (\rho U - \mu F)p, j(x_n - p)\rangle}{[2(\tau - \mu F) - 1]}\Big) + \mu_n \Big(\frac{\beta_n}{t_n} \cdot \frac{||Sp - p||||x_n - p||}{[2(\tau - \rho\gamma) - 1]}\Big).
$$

Combining (16) with condition that $\lim_{n\to\infty} \beta_n / t_n = 0$, we obtain $\mu_n ||x_n - p|| \to 0$ as $n \to \infty$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to $p \in F(T)$ as $k \to \infty$.

Further we show that $p \in F(T)$ is a unique solution to the variational inequality problem (14). Indeed, for any fixed $y \in F(T)$, since $\{x_n\}$ is bounded, there exists a positive constant Q such that $||x_n - y|| \leq Q$. Then

$$
\begin{aligned}
&||x_n - y||^2 \\
&= \langle t_n \rho U(x_n) + (1 - t_n \mu F)T[\beta_n S(x_n) + (1-\beta_n)x_n] - y, j(x_n - y)\rangle \\
&= t_n \langle \rho U(x_n) - \rho U(p) + \mu F(p) - \mu F(Tx_n), j(x_n) - p, j(x_n - p)\rangle \\
&\quad + t_n \langle \rho U(p) - \mu F(p), j(x_n - y)\rangle + t_n \mu \langle F(Tx_n) - F(Ty_n), j(x_n - y)\rangle \\
&\quad + \langle Ty_n - y, j(x_n - y)\rangle \\
&\leq t_n(\rho\gamma + \mu\kappa)||p - x_n||||x_n - y|| + t_n \langle (\rho U - \mu F)p, j(x_n - y)\rangle \\
&\quad + t_n \mu\kappa ||x_n - y_n||||x_n - y|| + ||Ty_n - y||||x_n - y|| \\
&\leq t_n(\rho\gamma + \mu\kappa)||p - x_n||||x_n - y|| + t_n \langle (\rho U - \mu F)p, j(x_n - y)\rangle \\
&\quad + t_n \mu\beta_n ||Sx_n - x_n||||x_n - y|| + \beta_n ||Sx_n - x_n||||x_n - y|| + ||x_n - y||^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\langle (\mu F - \rho U)p, j(x_n - y)\rangle &\leq (\rho\gamma + \mu\kappa)||x_n - p||Q \\
&\quad + \frac{\beta_n}{t_n}(\mu t_n + 1)||Sx_n - x_n||Q. \qquad (17)
\end{aligned}
$$

Since j is *norm-to-norm* uniformly continuous on bounded subsets of E, and $\{x_{n_k}\} \to p$ as $k \to \infty$, taking limit as $n_k \to \infty$ in (17), we obtain

$$\langle (\mu F - \rho U)p, j(p-y)\rangle \leq 0, \quad y \in F(T).$$

Hence, p is a unique solution of variational inequality (14). Now assume that there exists another subsequence of $\{x_n\}$, say $\{x_{n_k}\}$, such that $\lim\limits_{k\to\infty} x_{n_k} = p^*$. Then, using (15), we have $p^* \in F(T)$. Repeating the above arguments with p replaced with p^*, we can easily obtain that p^* also is a solution of the variational inequality problem (14). By uniqueness of the solution of variational inequality problem, we obtain that $p = p^*$, and this completes the proof. □

Next, we introduce an inertial-type viscosity approximation method for solving the hierarchical variational inequality problem. We begin with the following assumptions.

Assumptions 1 *Let E be a Banach space. Let sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\delta_n\} \subset [0,\infty)$, $\{\theta_n\} \subset [0,1)$ be such that*

(A1) $\sum_{n=1}^{\infty} \delta_n < \infty$ *with* $\delta_n = \circ(\alpha_n)$;

(A2) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A3) $\lim\limits_{n\to\infty} \beta_n/\alpha_n = 0$;

(A4) $\lim\limits_{n\to\infty} (1/\alpha_n)|\alpha_{n+1} - \alpha_n| = 0$ *and* $\lim\limits_{n\to\infty} (1/\alpha_n)|\beta_{n+1} - \beta_n| = 0$.

(A5) Let $x_0, x_1 \in E$ be arbitrary points. For the iterates x_{n-1} and x_n, $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$ where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \delta_n/\|x_n - x_{n-1}\|\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{18}$$

Remark 1 *From (18), for all $n \geq 1$ with $x_n \neq x_{n-1}$, we have*

$$\theta_n \leq \frac{\delta_n}{\|x_n - x_{n-1}\|}.$$

Hence,

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\delta_n}{\alpha_n} \to 0 \quad as \quad n \to \infty. \tag{19}$$

Theorem 2 *Let E be a real q-uniformly smooth Banach space with weakly sequentially continuous duality map. Let $T, S : E \to E$ be nonexpansive mappings with $F(T) \neq \emptyset$, F be an $\eta-$strongly accretive operator which is also $\kappa-$Lipschitzian, and $U : E \to E$ be $\gamma-$Lipschitzian mapping with $\gamma \geq 0$. Let μ, ρ and τ be as in Lemma 7. Suppose Assumptions 1 holds, and for any points $x_0, x_1 \in E$, define a sequence $\{x_n\}_{n=1}^{\infty}$ in E by*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = \beta_n S w_n + (1-\beta_n) w_n; \\ x_{n+1} = \alpha_n \rho U(x_n) + (I - \alpha_n \mu F) T y_n,\ n \geq 1. \end{cases} \tag{20}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which solves the variational inequality

$$(\langle (\rho U - \mu F)x^*, j(y - x^*)\rangle \leq 0,\ \ y \in F(T). \tag{21}$$

Proof. By Theorem 1, the variational inequality (21) have a unique solution x^* in $F(T)$. Assuming $\beta_n \leq \alpha_n$, from (20), we get

$$\begin{aligned} ||w_n - x^*|| &= ||x_n - x^* + \theta_n(x_n - x_{n-1})|| \\ &\leq ||x_n - x^*|| + \theta_n ||x_n - x_{n-1}|| \\ &= ||x_n - x^*|| + \alpha_n \cdot \frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}||, \end{aligned} \tag{22}$$

since the sequence $\Big\{\theta_n/\alpha_n ||x_n - x_{n-1}||\Big\}$ converges (see (19)). Then there exists a positive constant M such that for all $n \geq 1$, we have

$$\frac{\theta_n}{\alpha_n} ||x_n - x_{n-1}|| \leq M.$$

It follows from (22) that

$$||w_n - x^*|| \leq ||x_n - x^*|| + \alpha_n M$$

and

$$\begin{aligned} ||y_n - x^*|| &\leq \beta_n ||S w_n - x^*|| + (1-\beta_n) ||w_n - x^*|| \\ &\leq \beta_n ||S w_n - S x^*|| + \beta_n ||S x^* - x^*|| + (1-\beta_n) ||w_n - x^*|| \\ &\leq ||w_n - x^*|| + \beta_n ||S x^* - x^*|| && (23) \\ &\leq ||x_n - x^*|| + \alpha_n ||S x^* - x^*|| + \alpha_n M. && (24) \end{aligned}$$

Thus, from (20) and (24), we obtain

$$\begin{aligned}
||x_{n+1}-x^*|| &= ||\alpha_n\rho U(x_n)+(I-\alpha_n\mu F)Ty_n-x^*|| \\
&= ||\alpha_n(\rho U(x_n)-\mu Fx^*)+(I-\alpha_n\mu F)Ty_n-(I-\alpha_n\mu F)x^*|| \\
&\leq \alpha_n||\rho U(x^*)-\mu F(x^*)||+\alpha_n\rho\gamma||x_n-x^*|| \\
&\quad +(1-\tau\alpha_n)||y_n-x^*|| \\
&\leq \alpha_n||\rho U(x^*)-\mu F(x^*)||+\alpha_n\rho\gamma||x_n-x^*|| \\
&\quad +(1-\tau\alpha_n)[\alpha_n||Sx^*-x^*||+||x_n-x^*||+\alpha_n M] \\
&\leq [1-\alpha_n(\tau-\rho\gamma)]||x_n-x^*|| \\
&\quad +\alpha_n(\tau-\rho\gamma)\Big[\frac{||(\rho U-\mu F)x^*||+||Sx^*-x^*||+M}{\tau-\rho\gamma}\Big] \\
&\leq \max\Big\{||x_n-x^*||,\frac{||(\rho U-\mu F)x^*||+||Sx^*-x^*||+M}{\tau-\rho\gamma}\Big\}.
\end{aligned}$$

Hence, by induction for all $n\geq 1$,

$$||x_n-x^*||\leq \max\Big\{||x_1-x^*||,\frac{||(\rho U-\mu F)x^*||+||Sx^*-x^*||+M}{\tau-\rho\gamma}\Big\}.$$

Therefore, the sequence $\{x_n\}_{n=1}^\infty$ is bounded and so are $\{y_n\}$, $\{w_n\}$, $\{Tx_n\}$, $\{Ty_n\}$, $\{F(Tx_n)\}$, $\{F(Ty_n)\}$ and $\{U(x_n)\}$. Also, from (20) and (24), we have

$$\begin{aligned}
||w_n-w_{n-1}|| &= ||x_n-x_{n-1}+\theta_n(x_n-x_{n-1})+\theta_{n-1}(x_{n-1}-x_{n-2})|| \\
&\leq ||x_n-x_{n-1}||+\theta_n||x_n-x_{n-1}|| \\
&\quad +\theta_{n-1}||x_{n-1}-x_{n-2}||. \qquad (25)
\end{aligned}$$

Thus, by (20) and (25), we get

$$\begin{aligned}
||y_n-y_{n-1}|| &= ||\beta_n(Sw_n-Sw_{n-1})+(\beta_n-\beta_{n-1})Sw_{n-1} \\
&\quad +(\beta_{n-1}-\beta_n)w_{n-1}|| \\
&\leq ||w_n-w_{n-1}||+|\beta_n-\beta_{n-1}|(||Sw_{n-1}||+||w_{n-1}||) \\
&\leq ||x_n-x_{n-1}|||\beta_n-\beta_{n-1}|(||Sw_{n-1}||+||w_{n-1}||) \\
&\quad +\theta_n||x_n-x_{n-1}||+\theta_{n-1}||x_{n-1}-x_{n-2}||. \qquad (26)
\end{aligned}$$

Letting $\sup_{n\geq 1}\{M,||Sw_{n-1}||,||w_{n-1}||,||F(Ty_{n-1})||\}\leq M_1$ for some $M_1>0$, from (20) and (26), we obtain

$$\begin{aligned}||x_{n+1}-x_n|| = {}& ||\alpha_n\rho(U(x_n)-U(x_{n-1}))+\rho(\alpha_n-\alpha_{n-1})U(x_{n-1})\\ &+(I-\alpha_n\mu F)Ty_n-(I-\alpha_n\mu F)Ty_{n-1}+\mu(\alpha_n-\alpha_{n-1})F(Ty_{n-1})||\\ \leq {}& [1-\alpha_n(\tau-\rho\gamma)]||x_n-x_{n-1}||+[|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|]M_1\\ &+\theta_n||x_n-x_{n-1}||+\theta_{n-1}||x_{n-1}-x_{n-2}||\\ = {}& [1-\alpha_n(\tau-\rho\gamma)]||x_n-x_{n-1}||+\alpha_n(\tau-\rho\gamma)\\ &\times\Big[\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}+\frac{|\beta_n-\beta_{n-1}|}{\alpha_n}\Big]\frac{M_1}{(\tau-\rho\gamma)}\\ &+\alpha_n\Big(\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{\theta_{n-1}}{\alpha_n}||x_{n-1}-x_{n-2}||\Big). \end{aligned} \tag{27}$$

Applying Lemma 3 in (27) with (A2), (A4) and (19), we get

$$\lim_{n\to\infty}||x_{n+1}-x_n||=0.$$

Furthermore, since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\beta_n=0$, we obtain

$$\begin{aligned}||x_n-Tx_n|| &\leq ||x_n-x_{n+1}||+||x_{n+1}-Tx_n||\\ &= ||x_n-x_{n+1}||+||\alpha_n(\rho Ux_n-\mu FTy_n)+Ty_n-Tx_n||\\ &\leq ||x_n-x_{n+1}||+\alpha_n||\rho Ux_n-\mu FTy_n||+||Ty_n-Tx_n||\\ &\leq ||x_n-x_{n+1}||+\alpha_n||\rho Ux_n-\mu FTy_n||+||y_n-w_n||\\ &\leq ||x_n-x_{n+1}||+\alpha_n||\rho Ux_n-\alpha FTy_n||\\ &\quad+\beta_n||Sw_n-w_n||\to 0 \quad as\ n\to\infty. \end{aligned} \tag{28}$$

Since $\{x_n\}$ is bounded in E, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup z$ as $k\to\infty$ for some $z\in E$. Using (28) and the demiclosedness property of T at zero, we obtain $z\in F(T)$. Since j is weakly sequentially continuous, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle(\rho U-\mu F)x^*,j(x_n-x^*)\rangle &= \lim_{k\to\infty}\langle(\rho U-\mu F)x^*,j(x_{n_k}-x^*)\rangle\\ &= \langle(\rho U-\mu F)x^*,j(z-x^*)\rangle\leq 0. \end{aligned} \tag{29}$$

Combining (28) and (29), we obtain

$$\limsup_{n\to\infty}\langle(\rho U-\mu F)x^*,j(x_{n+1}-x^*)\rangle\leq 0. \tag{30}$$

Finally, let us show that $x_n \to x^*$. From (20), (22) and (23), we obtain

$$
\begin{aligned}
||x_{n+1}-x^*||^2 &= \langle \alpha_n\rho U(x_n) + (I-\alpha_n\mu F)Ty_n, j(x_{n+1}-x^*)\rangle \\
&= \langle (I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*, j(x_{n+1}-x^*)\rangle \\
&\quad +\alpha_n\rho\langle U(x_n)-U(x^*), j(x_{n+1}-x^*)\rangle \\
&\quad +\alpha_n\langle (\rho U-\mu F)x^*, j(x_{n+1}-x^*)\rangle \\
&\le ||(I-\alpha_n\mu F)Ty_n - (I-\alpha_n\mu F)x^*||||x_{n+1}-x^*|| \\
&\quad +\alpha_n\rho||U(x_n)-U(x^*)||||x_{n+1}-x^*|| \\
&\quad +\alpha_n\langle (\rho U-\mu F)x^*, j(x_{n+1}-x^*)\rangle \\
&\le (1-\alpha_n\tau)||y_n - x*||||x_{n+1}-x^*|| + \alpha_n\rho||x_n-x^*||||x_{n+1}-x^*|| \\
&\quad +\alpha_n\langle (\rho U-\mu F)x^*, j(x_{n+1}-x^*)\rangle \\
&\le [1-\alpha_n(\tau-\rho\gamma)]||x_n-x^*||||x_{n+1}-x^*|| \\
&\quad +\alpha_n(1-\alpha_n\tau)||x_{n+1}-x^*||\Big(\frac{\beta_n}{\alpha_n}||Sx^*-x^*|| + \frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\Big) \\
&\quad +\alpha_n\langle (\rho U-\mu F)x^*, j(x_{n+1}-x^*)\rangle \\
&\le \Big(1-\frac{2\alpha_n(\tau-\rho\gamma)}{1+\alpha_n(\tau-\rho\gamma)}\Big)||x_n-x^*||^2 \\
&\quad +\frac{2\alpha_n(\tau-\rho\gamma)}{1+\alpha_n(\tau-\rho\gamma)}||x_{n+1}-x^*||\Big(\frac{\beta_n}{\alpha_n}||Sx^*-x^*|| + \frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\Big) \\
&\quad +\frac{2\alpha_n}{1+\alpha_n(\tau-\rho\gamma)}\langle (\rho U-\mu F)x^*, j(x_{n+1}-x^*)\rangle.
\end{aligned}
$$

Therefore,

$$
||x_{n+1}-x^*||^2 \le (1-\vartheta_n)||x_n-x^*||^2 + \vartheta_n\Delta_n, \tag{31}
$$

where $\vartheta_n := [2\alpha_n(\tau-\rho\gamma)]/[1+\alpha_n(\tau-\rho\gamma)]$, and

$$
\begin{aligned}
\Delta_n &:= \Big(\frac{\langle (\rho U-\mu F)x^*, j(x_{n+1}-x^*)\rangle}{\tau-\rho\gamma} \\
&\qquad +M_2\Big[\frac{\beta_n}{\alpha_n}||Sx^*-x^*|| + \frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\Big]\Big),
\end{aligned}
$$

with $M_2 \ge \sup_{n\ge1}||x_{n+1}-x^*||$. Thus, applying Lemma 3 in (31) and using (A3), (19) and (30), we obtain $x_n \to x^*$ as $n\to\infty$, and this completes the proof. □

We have the following corollaries.

Corollary 1 *Let $E=\ell_p$ $(1<p<\infty)$, let $\{x_n\}$ be generated by $x_0, x_1 \in E$, and*

$$
\begin{cases}
w_n = x_n + \theta_n(x_n - x_{n-1});\\
y_n = \beta_n Sw_n + (1-\beta_n)w_n;\\
x_{n+1} = \alpha_n\rho U(x_n) + (I-\alpha_n\mu F)Ty_n,\ n\ge 1.
\end{cases}
$$

If Assumptions 1 hold, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $x^* \in F(T)$ *which solves the variational inequality*

$$(\langle(\rho U - \mu F)x^*, y - x^*\rangle \leq 0, \quad y \in F(T). \tag{32}$$

Corollary 2 (Tan and Li [25]) *Let* $E = H$ *be a real Hilbert space and* $\{x_n\}$ *be generated by* $x_0, x_1 \in E$ *and*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = \beta_n S w_n + (1 - \beta_n) w_n; \\ x_{n+1} = \alpha_n \rho U(x_n) + (I - \alpha_n \mu F) T y_n, \ n \geq 1. \end{cases}$$

If Assumption 1 holds, then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $x^* \in F(T)$ *which solves the variational inequality* (32)*.*

Corollary 3 (Wang and Xu [28]) *Let* $E = H$ *be a real Hilbert space, let* $\{x_n\}$ *be generated by* $x_1 \in E$ *and*

$$\begin{cases} y_n = \beta_n S x_n + (1 - \beta_n) x_n; \\ x_{n+1} = \alpha_n \rho U(x_n) + (I - \alpha_n \mu F) T y_n, \ n \geq 1. \end{cases}$$

If Assumptions 1 hold, then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $x^* \in F(T)$ *which solves the variational inequality* (32)*.*

Corollary 4 (Ceng et al. [8]) *Let* $E = H$ *be a real Hilbert space, let* $\{x_n\}$ *be generated by* $x_1 \in E$ *and*

$$x_{n+1} = \alpha_n \rho U(x_n) + (I - \alpha_n \mu F) T x_n, \ n \geq 1.$$

If Assumptions 1 hold, then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $x^* \in F(T)$ *which solves the variational inequality* (32)*.*

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