Perfect 3-colorings of cubic graphs of order 8

M. Alaeiyan and A. Mehrabani

Abstract. Perfect coloring is a generalization of the notion of completely regular codes, given by Delsarte. A perfect $m$-coloring of a graph $G$ with $m$ colors is a partition of the vertex set of $G$ into $m$ parts $A_1, \ldots, A_m$ such that, for all $i, j \in \{1, \ldots, m\}$, every vertex of $A_i$ is adjacent to the same number of vertices, namely, $a_{ij}$ vertices, of $A_j$. The matrix $A = (a_{ij})_{i,j \in \{1, \ldots, m\}}$ is called the parameter matrix. We study the perfect 3-colorings (also known as the equitable partitions into three parts) of the cubic graphs of order 8. In particular, we classify all the realizable parameter matrices of perfect 3-colorings for the cubic graphs of order 8.

Key Words: perfect coloring, parameter matrices, Cubic graph, equitable partition

Mathematics Subject Classification 2010: 03E02, 05C15, 68R05

Introduction

The concept of a perfect $m$-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as “equitable partition” (see [11]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. We are looking for a positive answer to find the conjecture Delsarte for each cubic graphs of order 8. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including $J(6, 3)$, $J(7, 3)$, $J(8, 3)$, $J(8, 4)$, and $J(v, 3)$ ($v$ odd) (see [4, 5, 9]). Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of $n$-dimensional hypercube $Q_n$ for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the $n$-dimensional cube with a given parameter matrix (see [6, 7, 8]).
In this paper all graphs are finite, undirected, simple and connected. Let $G = (V, E)$ be an undirected graph. Two vertices $u, v \in V(G)$ are adjacent if there exists an edge $e = \{u, v\} \in E(G)$ to which they are both incident. The adjacent will be shown $u \leftrightarrow v$.

A cubic graph is a 3-regular graph. In [12], it is shown that the number of connected cubic graphs with 8 vertices is 5. Each graph is described by a drawing as shown in Figure 1.

![Connected cubic graphs of order 8](image)

**Figure 1:** Connected cubic graphs of order 8

**Definition 1** For a graph $G$ and an integer $m$, a mapping $T : V(G) \to \{1, \ldots, m\}$ is called a perfect $m$-coloring with matrix $A = (a_{ij})_{i,j \in \{1, \ldots, m\}}$ if it is surjective, and for all $i, j$, for every vertex of color $i$, the number of its neighbors of color $j$ is equal to $a_{ij}$. The matrix $A$ is called the parameter matrix of a perfect coloring. In the case $m = 3$, we use three colors: white, black and red. The sets of white, black and red vertices are denoted by $W, B$ and $R$, respectively. In this paper, we generally show a parameter matrix by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

**Remark 1** In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e. we identify the perfect 3-coloring with the matrices:

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ h & i & g \\ b & a & c \end{bmatrix}, \begin{bmatrix} e & f & d \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ i & g & h \\ c & a & b \end{bmatrix}, \begin{bmatrix} i & g & h \\ i & g & h \\ f & d & e \end{bmatrix},$$

obtained by switching the colors with the original coloring.
1 Preliminaries and Analysis

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of connected graph of order 8 with a given parameter matrix \( A \).

The simplest necessary condition for the existence of perfect 3-colorings of a cubic connected graph with the matrix

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\]

is:

\[a + b + c = d + e + f = g + h + i = 3.\]

Also, it is clear that we cannot have \( b = c = 0, d = f = 0, \) or \( g = h = 0, \) since the graph is connected. In addition, \( b = 0, c = 0, f = 0 \) if \( d = 0, g = 0, h = 0, \) respectively.

The number \( \theta \) is called an eigenvalue of a graph \( G \), if \( \theta \) is an eigenvalue of the adjacency matrix of this graph. The number \( \lambda \) is called an eigenvalue of a perfect coloring \( T \) into three colors with the matrix \( A \), if \( \lambda \) is an eigenvalue of \( A \). The following theorem demonstrates the connection between the introduced notions.

**Theorem 1** (\( [1] \)) If \( T \) is a perfect coloring of a graph \( G \) in \( m \) colors, then any eigenvalue of \( T \) is an eigenvalue of \( G \).

The next theorem can be useful to find the eigenvalues of a parameter matrix.

**Theorem 2** Let \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \) be a parameter matrix of a \( k \)-regular graph.

Then the eigenvalues of \( A \) are

\[\lambda_{1,2} = \frac{\text{tr}(A) - k}{2} \pm \sqrt{\left(\frac{\text{tr}(A) - k}{2}\right)^2 - \frac{\det(A)}{k}}, \quad \lambda_3 = k.\]

**Proof.** By using the condition \( a + b + c = d + e + f = g + h + i = k \), it is clear that one of the eigenvalues is \( k \). Therefore \( \det(A) = k\lambda_1\lambda_2 \). From \( \lambda_2 = \text{tr}(A) - \lambda_1 - k \), we get

\[\det(A) = k\lambda_1(\text{tr}(A) - \lambda_1 - k) = -k\lambda_1^2 + k(\text{tr}(A) - k)\lambda_1.\]

By solving the equation \( \lambda^2 + (k - \text{tr}(A))\lambda + \frac{\det(A)}{k} = 0 \), we obtain

\[\lambda_{1,2} = \frac{\text{tr}(A) - k}{2} \pm \sqrt{\left(\frac{\text{tr}(A) - k}{2}\right)^2 - \frac{\det(A)}{k}}.\]

\( \square \)
The eigenvalues of the all cubic graphs of order 8 are stated in the next theorem.

**Theorem 3 ([12])** The distinct eigenvalues of the graph $G_1$ are the numbers $3, \sqrt{5}, -1, -\sqrt{5}$. The distinct eigenvalues of the graph $G_2$ are the numbers $\sqrt{3}, 1, 1 - \sqrt{2}, -1, -\sqrt{3}, -3 + \sqrt{2}$. The distinct eigenvalues of the graph $G_3$ are the numbers $3, 1.5616, 0.618, 0, -1.618, -2.5616$. The distinct eigenvalues of the graph $G_4$ are the numbers $3, 1, -1, 3$. The distinct eigenvalues of the graph $G_5$ are the numbers $3, 1, 1 - \sqrt{2}, -1, -2, -3 + \sqrt{2}$.

The next proposition gives a formula for calculating the number of white, black and red vertices, in a perfect 3-coloring.

**Proposition 1 ([3])** Let $T$ be a perfect 3-coloring of a graph $G$ with the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

1. If $b, c, f \neq 0$, then
   \[
   |W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.
   \]

2. If $b = 0$, then
   \[
   |W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.
   \]

3. If $c = 0$, then
   \[
   |W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.
   \]

4. If $f = 0$, then
   \[
   |W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.
   \]

In this section, without loss of generality, we may assume $|W| \leq |B| \leq |R|$. 
Lemma 1 Let $G$ be a cubic connected graph of order 8. Then $G$ has no perfect 3-coloring $T$ with the matrix that $|W| = 1$.

Proof. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be a parameter matrix with $|W| = 1$. Consider the white vertex. It is clear that none of its adjacent vertices are white; i.e. $a = 0$. Therefore, we have two cases below.

(1) The adjacent vertices of the white vertex are the same color.

If they are black, then $b = 3$ and $c = 0$. From $c = 0$, we get $g = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$
\begin{bmatrix}
0 & 3 & 0 \\
1 & 1 & 1 \\
0 & 1 & 2
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 3 & 0 \\
1 & 0 & 2 \\
0 & 2 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 3 & 0 \\
1 & 1 & 1 \\
0 & 2 & 1
\end{bmatrix}
$$

If the adjacent vertices of the white vertex are red, then $c = 3, b = 0$. From $b = 0$, we get $d = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$
\begin{bmatrix}
0 & 0 & 3 \\
1 & 1 & 1 \\
1 & 2 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 3 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 3 \\
1 & 1 & 1 \\
0 & 2 & 1
\end{bmatrix}
$$

Finally, by using Remark [1] and the fact that $|W| \leq |B| \leq |R|$, it is obvious that there are only six matrices in (1), as shown $A_1, A_2, A_3, A_4, A_5, A_6$.

$$
A_1 = \begin{bmatrix}
0 & 3 & 0 \\
1 & 1 & 1 \\
0 & 1 & 2
\end{bmatrix}, \quad
A_2 = \begin{bmatrix}
0 & 3 & 0 \\
1 & 0 & 2 \\
0 & 1 & 2
\end{bmatrix}, \quad
A_3 = \begin{bmatrix}
0 & 3 & 0 \\
1 & 0 & 2 \\
0 & 2 & 1
\end{bmatrix}, \quad
A_4 = \begin{bmatrix}
0 & 0 & 3 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{bmatrix},
$$

$$
A_5 = \begin{bmatrix}
0 & 0 & 3 \\
0 & 0 & 3 \\
1 & 1 & 1
\end{bmatrix}, \quad
A_6 = \begin{bmatrix}
0 & 0 & 3 \\
0 & 0 & 3 \\
1 & 2 & 0
\end{bmatrix}
$$

(2) The adjacent vertices of the white vertex are different colors. It immediately gives that $b, c \neq 0$. Also, it can be seen that $d = g = 1$. An easy computation, as in (1), shows that there are only five matrices that can be a parameter matrix in this case, as shown $A_7, A_8, A_9, A_{10}$.
$A_{11}$.

$A_7 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $A_8 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $A_9 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, $A_{10} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $A_{11} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

By using the Proposition 1, it can be seen that no matrix can be a parameter.

□

We now present two lemmas which can be useful to reach our goal.

**Lemma 2** Let $G$ be a cubic connected graph of order 8. If $T$ is a perfect 3-coloring with the matrix $A$, and $|W| = |B| = 2$, $|R| = 4$, then $A$ should be one of the following matrices:

$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

**Proof.** First, suppose that $b, c \neq 0$. As $|W| = 2$, by Proposition 1, it follows that $\frac{b}{d} + \frac{c}{g} = 3$. From $b + c \leq 3$, we have $b = 2$, $c = g = d = 1$, or $c = 2$, $b = g = d = 1$. If $b = 2$, $c = g = d = 1$, we get a contradiction of $|B| = 2$. If $c = 2$, $b = d = g = 1$, then we conclude from $|B| = 2$ and $|R| = 4$ that $h = 1$, $f = 2$. Therefore $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Second, suppose that $b = 0$ and, in consequence, $d = 0$. As $|R| = 4$, by Proposition 1, it follows that $\frac{g}{c} + \frac{h}{f} = 1$. Therefore, $c = f = 2$, $g = h = 1$, or $c = f = 3$, $h = 2$, $g = 1$, or $c = f = 3$, $g = 2$, $h = 1$. If $c = f = 2$, $g = h = 1$, then $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$. In the other two cases, we get a contradiction of $|B| = 2$.

Third, suppose that $c = 0$ and, in consequence, $g = 0$. As $|B| = 2$, by Proposition 1, it follows that $\frac{d}{b} + \frac{f}{h} = 3$. Therefore $d = 2$, $b = f = h = 1$, or $f = 2$, $b = h = d = 1$. If $d = 2$, $b = f = h = 1$, then we get a contradiction of $|R| = 4$. If $f = 2$, $b = h = d = 1$, then $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.
Finally, note that the matrix \[
\begin{bmatrix}
0 & 1 & 2 \\
1 & 2 & 0 \\
1 & 0 & 2 \\
\end{bmatrix}
\] is the same as the matrix \[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 0 & 2 \\
0 & 1 & 2 \\
\end{bmatrix}
\] up to renaming the colors, by Remark 1. □

**Lemma 3** Let \( G \) be a cubic connected graph of order 8. If \( T \) is a perfect 3-coloring with the matrix \( A \), and \( |W| = 2, |B| = |R| = 3 \), then \( A \) should be the following matrix:

\[
\begin{bmatrix}
0 & 3 & 0 \\
2 & 0 & 1 \\
0 & 1 & 2 \\
\end{bmatrix}
\]

**Proof.** First, suppose that \( b, c \neq 0 \). As \( |W| = 2 \), by Proposition 1 it follows that \( \frac{b}{d} + \frac{c}{g} = 3 \). From \( b + c \leq 3 \), we get \( b = 2, c = g = d = 1 \), or \( c = 2, b = g = d = 1 \). If \( b = 2, c = g = d = 1 \), we get a contradiction of \( |B| = 3 \). If \( c = 2, b = d = g = 1 \), then from Proposition 1 we have \( f = 2, h = 3 \), which is a contradiction of \( g + h \leq 3 \).

Second, suppose that \( b = 0 \) and, in consequence, \( d = 0 \). As \( |R| = 3 \), by Proposition 1 it follows that \( \frac{g}{c} + \frac{h}{f} = \frac{5}{3} \). Therefore, \( c = 3, g = 2, h = f = 1 \), or \( f = 3, h = 2, c = g = 1 \). If \( c = 3, g = 2 \), then \( h = f = 1 \), and we get a contradiction of \( |W| = 2 \). If \( c = 0 \) and, in consequence, \( g = 0 \). As \( |B| = 3 \), by Proposition 1 it follows that \( \frac{d}{b} + \frac{f}{h} = \frac{5}{3} \). Therefore \( h = 3, f = 2, b = d = 1 \), or \( b = 3, d = 2, f = h = 1 \). If \( h = 3, f = 2 \), then \( A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \). In the other case, we get a contradiction of \( |W| = 2 \).

Finally, note that the matrix \[
\begin{bmatrix}
0 & 3 & 0 \\
2 & 0 & 1 \\
0 & 1 & 2 \\
\end{bmatrix}
\] is the same as the matrix \[
\begin{bmatrix}
0 & 3 & 0 \\
2 & 0 & 1 \\
0 & 1 & 2 \\
\end{bmatrix}
\] up to renaming the colors, by Remark 1. □

By using the Lemmas 1, 2 and 3 it can be seen that only the following matrices:

\[
A_1 = \begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 2 \\
1 & 1 & 1 \\
\end{bmatrix},
A_2 = \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 2 \\
1 & 1 & 1 \\
\end{bmatrix},
A_3 = \begin{bmatrix}
0 & 3 & 0 \\
2 & 0 & 1 \\
0 & 1 & 2 \\
\end{bmatrix},
A_4 = \begin{bmatrix}
2 & 1 & 0 \\
1 & 0 & 2 \\
0 & 1 & 2 \\
\end{bmatrix},
\] can be parameter ones.
2 Perfect 3-colorings of cubic graphs with 8 vertices

In this section we enumerate the parameter matrices of all perfect 3-colorings of cubic graphs with 8 vertices. As it has been shown in section 3, only matrices $A_1$, $A_2$, $A_3$ and $A_4$ can be parameter matrices. With consideration of cubic graphs eigenvalues and using Theorems 1, 2 and 3, it can be seen that the connected cubic graphs with 8 vertices can have a perfect 3-coloring with the matrices $A_1$, $A_2$ and $A_3$ which is represented by table 1.

<table>
<thead>
<tr>
<th>graphs</th>
<th>matrix $A_1$</th>
<th>matrix $A_2$</th>
<th>matrix $A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>$\sqrt{}$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\sqrt{}$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$G_3$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\sqrt{}$</td>
</tr>
<tr>
<td>$G_4$</td>
<td>$\sqrt{}$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$G_5$</td>
<td>$\sqrt{}$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

Table 1

**Theorem 4** There are no perfect 3-colorings with the matrix $A_1$ for the graph $G_5$.

**Proof.** Contrary to our claim, suppose that $T$ is a perfect 3-coloring with the matrix
\[
\begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 2 \\
1 & 1 & 1
\end{bmatrix}
\]
for the graph $G_5$. Without restriction of generality, suppose that $T(a_1) = 1$. Therefore, again without restriction of generality, suppose that $T(a_2) = T(a_8) = 3$ and $T(a_5) = 2$. From $T(a_5) = 2$, we can easily seen that $T(a_4) = T(a_6) = 3$. Therefore $T(a_4) = 2$, which is a contradiction with third row of the matrix $A_1$. $\Box$

**Theorem 5** The parameter matrices of cubic graphs of order 8 are listed in the following table.

<table>
<thead>
<tr>
<th>graphs</th>
<th>matrix $A_1$</th>
<th>matrix $A_2$</th>
<th>matrix $A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>$\sqrt{}$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\sqrt{}$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$G_3$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\sqrt{}$</td>
</tr>
<tr>
<td>$G_4$</td>
<td>$\sqrt{}$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$G_5$</td>
<td>$\times$</td>
<td>$\sqrt{}$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

Table 2
Proof. As it has been shown in the table 1, only the matrices $A_1$, $A_2$ and $A_3$ can be parameter matrices. Hence, from Theorem 4 it suffices to show that there are perfect 3-colorings with the matrices in the table 2. The graph $G_1$ has perfect 3-colorings with the matrices $A_1$ and $A_2$. Consider two mappings $T_1$ and $T_2$ as follows:

$$T_1(a_2) = T_1(a_6) = 1, T_1(a_4) = T_1(a_8) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_5) = T_1(a_7) = 3.$$

$$T_2(a_1) = T_2(a_2) = 1, T_2(a_5) = T_2(a_6) = 2,$$
$$T_2(a_3) = T_2(a_4) = T_2(a_7) = T_2(a_8) = 3.$$

It is clear that $T_1$ and $T_2$ are perfect 3-colorings with the matrices $A_1$ and $A_2$, respectively.

The graph $G_2$ has perfect 3-colorings with the matrices $A_1$ and $A_2$. Consider two mappings $T_1$ and $T_2$ as follows:

$$T_1(a_1) = T_1(a_3) = 1, T_1(a_5) = T_1(a_7) = 2,$$
$$T_1(a_2) = T_1(a_4) = T_1(a_6) = T_1(a_8) = 3,$$
$$T_2(a_1) = T_2(a_5) = 1, T_2(a_3) = T_2(a_7) = 2,$$
$$T_2(a_2) = T_2(a_4) = T_2(a_6) = T_2(a_8) = 3.$$

It is clear that $T_1$ and $T_2$ are perfect 3-coloring with the matrices $A_1$ and $A_2$, respectively.

The graph $G_3$ has perfect 3-colorings with the matrix $A_3$. Consider a mapping $T_1$ as follows:

$$T_1(a_2) = T_1(a_8) = 1, T_1(a_1) = T_1(a_3) = T_1(a_7) = 2,$$
$$T_1(a_4) = T_1(a_5) = T_1(a_6) = 3.$$

It is clear that $T_1$ is a perfect 3-colorings with the matrices $A_3$.

The graph $G_4$ has perfect 3-colorings with the matrices $A_1$ and $A_2$. Consider two mappings $T_1$ and $T_2$ as follows:

$$T_1(a_2) = T_1(a_7) = 1, T_1(a_5) = T_1(a_8) = 2,$$
$$T_1(a_1) = T_1(a_3) = T_1(a_4) = T_1(a_6) = 3,$$
$$T_2(a_2) = T_2(a_5) = 1, T_2(a_7) = T_2(a_8) = 2,$$
$$T_2(a_1) = T_2(a_3) = T_2(a_4) = T_2(a_6) = 3.$$
It is clear that $T_1$ and $T_2$ are perfect 3-colorings with the matrices $A_1$ and $A_2$, respectively.

The graph $G_5$ has perfect 3-colorings with the matrix $A_3$. Consider a mapping $T_1$ as follows:

$$
T_1(a_4) = T_1(a_8) = 1, T_1(a_2) = T_1(a_6) = 2,
T_1(a_1) = T_1(a_3) = T_1(a_5) = T_1(a_7) = 3.
$$

It is clear that $T_1$ is a perfect 3-colorings with the matrices $A_3$.

□

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