Armenian Journal of Mathematics
Volume 9, Number 2, 2017, 146–222

# Hyperidentities and Related Concepts, I

Yu. M. Movsisyan

**Abstract.** This survey article illustrates many important current trends and perspectives for the field including classification of hyperidentities, characterizations of algebras with hyperidentities, functional representations of free algebras, structure results for bigroups, categorical questions and applications. However, the paper contains new results and open problems, too.

*Key Words:* Hyperidentity, Hypervariety, Variety, Termal Hyperidentity, Essential hyperidentity, Grätzer Algebra, Bigroup, Bilattice, De Morgan algebra, Boole-De Morgan algebra, De Morgan function, quasi-De Morgan function, Free algebra.
*Mathematics Subject Classification* 2010: 03C05, 03C85, 06D05, 06D30, 06E05, 08A68,08A70, 08A62, 08C05, 20M07, 20N05

## 1 Introduction and Preliminaries

Model theory and algebra study the connections between formal languages and their interpretations in models and algebras. The simplest and most widespread formal language is the first order language (A. Church [45], A. I. Mal'tsev [113, 115, 116], G. Grätzer [82], C. Chang with H. Keisler[42], S. Burris with H. P. Sankappanavar [41], B. I. Plotkin [195]). The founders of the first order language (logic) are Löwenheim, Skolem, Gödel, Tarski, Mal'tsev and Birkhoff.

However, there exist very commonly encountered, classical algebraic structures that are not axiomatizable by the first order formulae (logic). For example, rings, associative rings, commutative rings, associative-commutative rings, fields, or fields of fixed characteristics are axiomatized by the first order formulae, but their multiplicative groupoids, semigroups and groups are not, because these classes of groupoids, semigroups and groups are not closed under elementary equivalency (A. I. Mal'tsev, S. Kogalovskii [99], G. Sabbagh [212]). The situation is analogous for near-fields (M. Hall [85]), Grätzer algebras ($G$-fields)[80, 69, 19], topological rings and topological fields (L. S. Pontryagin [198]). Characterizations of such semigroups and groups are the most important problems in modern algebra, logic and topology. L.Fuchs [68] called the characterization of multiplicative groups of fields a big problem.

This is why it is necessary to widen the formal language to allow to express phenomena that the first order logic can not capture.

An important extension of the first order logic (language) is the second order logic (language), described in detail in [42, 45, 115, 116] (also see [94]). The second order formulae consist of the same logical symbols of $\&, \vee, \neg, \rightarrow, \exists, \forall$ of individual and functional (predicate) variables, which are used in the first order formulae. The difference is that in the second order formulae, the quantifiers $\forall, \exists$ can be applied not only to individual variables, but also to functional (or to predicate) variables. Investigations of the second order formulae (logic) go back to L. Henkin, A. I. Mal'tsev, A. Church, S. Kleene, A. Tarski.

Starting with the 1960's the following second order formulae were studied in various domains of algebra and its applications (see [114, 115, 219, 220, 213, 214, 18, 63, 135, 136, 163, 21, 44, 94, 248, 24, 25, 182, 231, 137, 139, 144, 145, 237, 183, 232, 141, 160, 161, 162, 167, 175, 176, 102, 221, 118, 119, 129, 187, 49, 50, 51, 52, 53, 62, 76, 77, 178, 179, 168, 169, 170, 186, 187, 196, 253, 257, 258, 259, 260, 267, 275])

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{1}$$

$$\forall X_1, \ldots, X_k \exists X_{k+1}, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{2}$$

$$\exists x_1, \ldots, x_n \forall X_1, \ldots, X_m (w_1 = w_2), \tag{3}$$

$$\exists X_1, \ldots, X_k \forall X_{k+1}, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{4}$$

$$\forall X_1, \ldots, X_k \exists X_{k+1}, \ldots, X_t \forall X_{t+1}, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{5}$$

where $w_1, w_2$ are words (terms) in the functional variables $X_1, \ldots, X_m$ and the individual (object) variables $x_1, \ldots, x_n$. The first formula is called *hyperidentity* or $\forall(\forall)$-identity (see [137, 139, 221, 62, 100]) (also see [12]); the second (third, fourth, fifth) formula is called an $\forall\exists(\forall)$-identity ($(\exists)\forall$-identity, $\exists\forall(\forall)$-identity, $\forall\exists\forall(\forall)$- identity). Sometimes the $\forall\exists(\forall)$-identity is called a generalized identity [18], the $(\exists)\forall$- identity is called a *coidentity* [136, 137] (also see [12]) and $\exists\forall(\forall)$-identity is called a hybrid identity [21, 221, 164]. The satisfiability of these second order formulae in an algebra $\mathfrak{A} = (Q; \Sigma)$ is understood by functional quantifiers $(\forall X_i)$ and $(\exists X_j)$, meaning: "for every value $X_i = A \in \Sigma$ of the corresponding arity" and "there exists a value $X_j = A \in \Sigma$ of the corresponding arity". It is assumed that such a replacement is possible, that is

$$\{|X_1|, \ldots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{\mathfrak{A}},$$

where $|S|$ is the arity of $S$, and $T_{\mathfrak{A}}$ is called the arithmetic type of $\mathfrak{A}$ .

For the categorical definition of hyperidentities and $\forall\exists(\forall)$-identities see [135].

Second order formulae with analogous predicative quantifiers in models and algebraic systems are also often used in mathematical logic. For example, finiteness, the axiom of well-ordering, the continuum hypothesis, the property of being countable and others can be formulated within the second order logic.

A variety (or equational class) is a class of algebras (all of the same similarity type or signature) closed under the formation of products, subalgebras and homomorphic images. Equivalently, a variety is a class of algebras defined by a set of equations (identities). A hypervariety is a class of algebras (all of the same arithmetic type) defined by a set of hyperidentities. Since 1954 the following-type second order formulae were studied in algebras of term functions of various classes of varieties

$$\exists X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2), \tag{6}$$

which are called Mal'tsev (Mal'cev) conditions ( see [113], [81, 246, 234, 180, 247, 233], [91]), reducing to the hyperidentities of the class of term functions' algebras (termal or term algebras). (Note that the formula (6) is called functional equation in the Set theory [1, 2, 4].)

The formulae (1)–(6) are usually written without quantifiers, if the structures of the quantifiers are understood from the content. The formulae (2)–(6) are more general than hyperidentities. The numbers $m$ and $n$ in hyperidentity (1) are called the functional and object *rank*, respectively. A hyperidentity is said to be *non-trivial* if its functional *rank* is $> 1$, and it is called trivial otherwise (m=1). A hyperidentity is called $n$-ary, if its functional variables are $n$-ary. For $n = 1, 2, 3$ the $n$-ary hyperidentity is called unary,binary,ternary. A formula (hyperidentity, coidentity,...) is called a formula (hyperidentity, coidentity,...) of algebra $\mathfrak{A}$, if it is satisfied in algebra $\mathfrak{A}$. Hyperidentities (coidentities,...) are usually written without quantifiers: $w_1 = w_2$. Let $V$ be a variety or a class of algebras. A hyperidentity (coidentity,...) $w_1 = w_2$ is called a hyperidentity (coidentity,...) of $V$ if it is a hyperidentity (coidentity,...) for any algebra $\mathfrak{A} \in V$.

**Examples** 1. In any lattice the following hyperidentities are satisfied

$$X(x, x) = x,$$

$$X(x, y) = X(y, x),$$

$$X(x, X(y, z)) = X(X(x, y), z),$$

$$Y(y, x) = Y(y, X(x, Y(x, y))),$$

$$Y(X(x, z), y) = Y(X(x, z),\ X(y, Y(y, z))),$$

$$X(Y(X(x,y), z), Y(y,z)) = Y(X(x,y), z).$$

Hence, these hyperidentities are hyperidentities of the variety of lattices. The last non-trivial hyperidentity is called hyperidentity of interlacity (see [149]).

2. In any commutative and associative ring the following hyperidentities are satisfied

$$X(x,\, y) = X(y,\, x),$$

$$X(x,\, X(y,\, z)) = X(X(x,\, y), z),$$

$$X\left(X\left(Y(x,\, x),\, Y(x,\, x)\right),\, Y\left(X(x,\, x),\, X(x,\, x)\right)\right) =$$

$$= X\left(Y\left(X(x,\, x),\, X(x,\, x)\right),\, X\left(Y(x,\, x),\, Y(x,\, x)\right)\right).$$

3. In the termal algebra (i.e., the algebra of term functions) of any group (semigroup, Moufang loop) the following non-trivial hyperidentity is satisfied (see [24]):

$$X(Y(x,\, x),\, Y(x,\, x)) = Y(X(x,\, x),\, X(x,\, x)).$$

4. Let $B = \{0, 1\}$ and $P$ be the set of all binary Boolean functions. In algebra $(B; P)$ the following hyperidentities are satisfied

$$X(X(X(x, y), y), y) = X(x, y),$$

$$X(x, X(x, X(x, y))) = X(x, y).$$

In particular, these hyperidentities are satisfied in two-element Boolean algebra $(\{0, 1\}; \&, \vee, ', 0, 1)$. Hence, these hyperidentities are satisfied in any Boolean algebra too, by Birkhoff's subdirect representation theorem. See [140] for corresponding hyperidentities of $n$-ary Boolean functions. On the application of the results of [140] in modal logic see [88].

5. In any De Morgan algebra $Q(+, \cdot, ^{-}, 0, 1)$ the following non-trivial hyperidentity is satisfied

$$F(X(F(Y(x, y)), z)) = Y(F(X(F(x), z)), F(X(F(y), z))).$$

The concept of hyperidentity is present in many well known notions. For example, an algebra $\mathfrak{A} = (Q; \Sigma)$ is said to be *Abelian* (A.G.Kurosh [104]) or *entropic* (medial) if the following non-trivial hyperidentity

$$X\Bigg(Y(x_{11}, \ldots, x_{1n}), \ldots, Y(x_{m1}, \ldots, x_{mn})\Bigg) =$$

$$= Y\Bigg(X(x_{11}, \ldots, x_{m1}), \ldots, X(x_{1n}, \ldots, x_{mn})\Bigg)$$

is valid for all $m, n \in T_{\mathfrak{A}}$. An algebra $\mathfrak{A} = (Q; \Sigma)$ is said to be *idempotent* if the following hyperidentity of idempotency

$$X(\underbrace{x, \ldots, x}_{n}) = x$$

is valid for all $n \in T_{\mathfrak{A}}$.

A *mode* is an idempotent and entropic algebra (studied in monographs [208, 209]). A *distributive bisemilattice* (multisemilattice) [98] is a binary algebra with semilattice operations satisfying the following non-trivial hyperidentity of distributivity

$$X(x, Y(y, z)) = Y(X(x, y), X(x, z)).$$

A *doppelsemigroup* (see [7, 111, 190, 207, 276, 277]) is an algebra with two binary operations satisfying the following hyperidentity of associativity

$$X(x, Y(y, z)) = Y(X(x, y), z).$$

Binary algebras with the hyperidentity of associativity

$$X(x, Y(y, z)) = Y(X(x, y), z)$$

under the name of Γ-semigroups (or gamma-semigroups) also were considered by various authors [15, 112, 181, 224, 226, 227] (see earlier paper [214, 44], too).

The investigation of hyperidentities is a relatively new, actively developing field of pure and applied algebra. The concept of hyperidentity offers a high-level approach to algebraic questions, leading to new results, applications and problems. In particular, the investigation of hyperidentities is useful from the point of view of new technologies too, via optimization problems of block diagrams [145]. For applications of hyperidentities in discrete mathematics and topology see [60, 61, 71, 130, 155, 156, 157, 158, 159, 178, 275]. For characterization of Sheffer functions and primal algebras by hyperidentities see (K. Denecke, R. Pöschel [60, 61]).

Any algebra $\mathfrak{A} = (Q; \Sigma)$ may be interpreted as a many-sorted algebra $(Q; \Sigma_i, \ldots, \Sigma_n, \ldots)$ (where $\Sigma_n$ is a set of all n-ary operations of the given algebra ) with the following operations $(f, x_1, \ldots, x_n) \to f(x_1, \ldots, x_n)$ where $f \in \Sigma_n$, $x_1, \ldots, x_n \in Q$, $n \in T_{\mathfrak{A}}$. Moreover the hyperidentities of the given algebra become the identities of the corresponding many-sorted algebra and vice versa. It is possible the another approach too (see section 11). In this way the theory of hyperidentities as a second order theory of algebras is converted into a first order theory of many-sorted algebras. Simultaneously there is a bijection between hyperidentities of termal (term) algebra $\mathcal{F}(\mathfrak{A})$ and identities of the clone $Cl(\mathfrak{A})$ of an algebra $\mathfrak{A}$ (a *clone* is also a many-sorted algebra defined in section 10). One of the specifics of a hyperidentity (coidentity) is that if a hyperidentity (coidentity) is valid in algebra $\mathfrak{A}$ then it is also valid in every reduct $\mathfrak{B}$ of $\mathfrak{A}$ with the condition $T_{\mathfrak{B}} = T_{\mathfrak{A}}$.

Hyperidentities are also "identities" of algebras in the category of bihomomorphisms $(\varphi, \tilde{\psi})$, where

$$\varphi A(x_1, \ldots, x_n) = (\tilde{\psi} A)(\varphi x_1, \ldots, \varphi x_n),$$

which were studied in the monograph [137]. More about the application of such morphisms in the cryptography can be found in [10].

Hyperidentities in binary algebras with quasigroup operations were first considered by V. D. Belousov [18] (as a special case of $\forall\exists(\forall)$-identities which earlier is considered by R. Schauffler ([219, 220]) in coding theory) and then J. Aczel [3], about the classification of associative and distributive hyperidentities in binary algebras with quasigroup operations. Currently, more general results about these and other classifications of hyperidentities can be found in [137, 139, 144] and [145]. Observe that in algebras with quasigroup operations many $\forall\exists\forall(\forall)$- identities are equivalent to hyperidentities (see[139]).

The multiplicative groups of fields have been characterized in [138] and [144] by hyperidentities. The hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean algebras, De Morgan algebras and weakly idempotent lattices have been characterized in the works [140, 143, 144, 142, 160, 152, 153, 154, 166], [167, 171].

A hyperidentity $\omega_1 = \omega_2$ is called *termal* or *polynomial* hyperidentity of the algebra $\mathfrak{A}$ if it is valid in the term algebra $\mathcal{F}(\mathfrak{A})$. Let $V$ be a variety. A hyperidentity $\omega_1 = \omega_2$ is called a termal hyperidentity of $V$ if it is a termal hyperidentity for any algebra $\mathfrak{A} \in V$. Termal hyperidentities for varieties were first considered by W. Taylor ([248]) (as a special case of Mal'tsev conditions for varieties) for characterization of classes of varieties which are closed under formation of equivalent varieties, products of varieties, reducts of varieties and subvarieties. Since the operations of an algebra are included in the set of term operations (clone) of the algebra, the concept of termal hyperidentity of a variety is stronger than the concept of hyperidentity. In particular, the variety of rings (even commutative rings) does not have termal hyperidentities except $w = w$, but has hyperidentities.

Termal hyperidentities of varieties of groups and semigroups have been characterized by G. Bergman [24] (also see [25]). Termal hyperidentities of the variety of lattices and of the variety of semilattices were studied by R. Padmanabhan and P. Penner ([182, 189, 183]).

Hyperidentities in algebras as an individual direction of investigations, were first presented in the monographs [137, 139]. The problem of characterization of termal hyperidentities of important classes of groups, semigroups, loops, quasigroups has been posed in the book [137] (p.129, problem 26). The hyperidentities of algebras and varieties, termal and essential hyperidentities, pre-hyperidentities of various varieties of groups, semigroups, quasigroups, loops and related algebras were also studied by many authors (see references of this paper).

We briefly describe the structure of the paper. This paper is a survey of the results and problems on hyperidentities and related formulae (equations) and on related concepts. In the section 2 we review some standard concepts and results. In the section 3 we introduce the concept of De Morgan algebra and prove the Stone type representation theorem for De Morgan algebras. In the section 4 we introduce the concept of De Morgan function and characterize the finitely generated free de Morgan algebras by De Morgan functions. In the section 5 we introduce the concepts of Boole-De Morgan algebra, consider the natural examples and prove the Stone type representation theorem for Boole-De Morgan algebras too. In the section 6 we introduce the concept of quasi-De Morgan function and describe the finitely generated free Boole-De Morgan algebras by quasi-De Morgan functions. In the section 7, followed by the concept of bilattices (see [67, 70, 72, 73, 74, 75, 131, 149, 150, 165, 167, 175, 176, 201, 203, 210, 11, 13, 16, 17, 33, 26, 90]), we introduce the concept of bigroup and characterize bigroup of binary operations trough Grätzer algebras [80]. From this point of view we also consider the Steiner, Stein and Belousov quasigroups in the next section. In the sections 9,10,11 the categorical concepts of hyperidentities and related equations and their general properties are considered.

To limit the size of the paper the proofs of results are mostly omitted.

## 2 Free distributive lattices and free Boolean algebras (preliminary concepts and results)

Let us start from the definition of a Boolean algebra.

**Definition 1** *An algebra* $(Q; \{+, \cdot, \ ', 0, 1\})$ *with two binary, one unary and two nullary operations is called a Boolean algebra if* $(Q; \{+, \cdot, 0, 1\})$ *is a bounded distributive lattice with the least element* 0 *and the greatest element* 1*, and the algebra* $(Q; \{+, \cdot, \ ', 0, 1\})$ *satisfies the following identities:*

$$x + x' = 1,$$

$$x \cdot x' = 0.$$

For a lattice $(L; \{+, \cdot\})$ a partial order $\leq$ is defined in the following way:

$$x \leq y \Leftrightarrow x + y = y, \ x, y \in L.$$

For the definition and existence of free algebras $F_V(X)$ of the given variety $V$ with the set of free generators $X$ see [25, 29, 41, 46, 82, 83, 103, 127, 192, 195, 238]. If $V$ is the variety of Boolean algebras then the free algebra $F_V(X)$ is called free Boolean algebra with the set of free generators $X$. For the variety $V$ of distributive lattices (bounded distributive lattices) the free algebra $F_V(X)$ is called free distributive lattice (free bounded distributive lattice) with the set of free generators $X$. The problem of determining the cardinality of free distributive lattices goes back to R. Dedekind [55]( for a modern survey of the field, see [101]).

Let $B = \{0, 1\}$. Define the operations $+, \cdot, \, '$, on $B$ in the following way: $0+0=0,\ 0+1=1+0=1+1=1,\ 0\cdot 1=0\cdot 0=1\cdot 0=0,\ 1\cdot 1=1,\ 0'=1,\ 1'=0$. We get the two-element Boolean algebra $\mathbf{B} = (B; \{+, \cdot, \, ', 0, 1\})$.

The two-element Boolean algebra is the only subdirectly irreducible Boolean algebra up to isomorphism, the two-element lattice is the only subdirectly irreducible distributive lattice up to isomorphism, the two-element semilattice is the only subdirectly irreducible semilattice up to isomorphism [83, 103, 127].

For a set $X$ denote the set of all its subsets by $2^X$ or $\mathcal{P}(X)$. If we consider subsets of a given set $X$ then for a subset $s \subseteq X$ we denote $s' = X \setminus s$.

A function $f : B^n \to B$ is called a *Boolean function* of $n$ variables, where $B^n$ is the set of all $n$-element sequences of $B$. The following result is well known.

**Theorem 1** ([48]) *For every Boolean function $f : B^n \to B$ there exists a unique set $S \subseteq 2^{\{1,\dots,n\}}$ such that*

$$f(x_1, \dots, x_n) = \sum_{s\in S} \left( \prod_{i\in s} x_i \cdot \prod_{i\in \overline{s}} x_i' \right),$$

*where the operations on the right hand side are the operations of Boolean algebra* $\mathbf{B}$.

Note that those terms are called disjunctive normal forms for Boolean functions.

It is commonly known that the free Boolean algebra on $n$ free generators is isomorphic to the Boolean algebra of Boolean functions of $n$ variables ([30, 55, 83, 195]). The free bounded distributive lattice on $n$ free generators is isomorphic to the bounded lattice of monotone Boolean functions of $n$ variables ([30, 55, 83, 195]). Let us present the last result in detailed.

For $u = (u_1, \dots, u_n),\ v = (v_1, \dots, v_n) \in B^n$ we define: $u \le v$ iff $u_i \le v_i$ for all $i = \overline{1, n}$. Here and afterwards $n \ge 1$ is a positive integer.

**Definition 2** *A Boolean function $f : B^n \to B$ is called monotone if*

$$x \le y \Rightarrow f(x) \le f(y),$$

*where $x, y \in B^n$.*

If $u = (u_1, \dots, u_n),\ v = (v_1, \dots, v_n) \in B^n$ then we will say $u \preceq v$ if there exists $k$ $(1 \le k \le n)$ such that $u_i = v_i$ for all $i \ne k$ and $u_k = 0,\ v_k = 1$. A Boolean function $f : B^n \to B$ is monotone iff

$$x \preceq y \Rightarrow f(x) \le f(y),$$

where $x, y \in B^n$.

Denote the set of all monotone Boolean functions of $n$ variables by $\mathcal{M}_n$. We can define $f + g$ and $f \cdot g$ for any two Boolean functions of $n$ variables by the standard way. It is obvious that if $f$ and $g$ are monotone Boolean functions, then $f + g$ and $f \cdot g$ are monotone, too. Thus, we get the algebra $\mathfrak{L}_n = \mathcal{M}_n(+, \cdot)$ which obviously is a bounded distributive lattice. Also let $m_n = |\mathcal{M}_n|$ be the number of monotone boolean functions of $n$ variables.( Note that the numbers $m_n$ are called Dedekind's numbers.) For instance, $m_1 = 3, m_2 = 6, m_3 = 20, m_4 = 168, m_5 = 7581, m_6 = 7828354$ ([101]).

Now let $S \subseteq 2^{\{1,\dots,n\}}$ be an antichain (or Sperner set [55, 239]) with respect to the order $\subseteq$. It means that $S$ consists of subsets of $\{1, \dots, n\}$, none of which is contained in any other subset from $S$. Note that the empty set is also considered as an antichain. For an antichain $S \subseteq 2^{\{1,\dots,n\}}$ define the following monotone Boolean function:

$$f_S(x_1, \dots, x_n) = \sum_{s \in S} \prod_{i \in s} x_i. \tag{7}$$

For $S = \emptyset$ we set $f_\emptyset = 0$, and for $S = \{\emptyset\}$ we set $f_{\{\emptyset\}} = 1$. Notice that $f_S$ does not depend on the order of the elements in the set $S$. It is easy to see that if $S_1 \neq S_2$ are two antichains, then $f_{S_1} \neq f_{S_2}$. To see this without loss of generality suppose that there exists $s \in S_1$ such that $s \notin S_2$. We can also suppose that there does not exist $s' \in S_2$ with $s' \subseteq s$. Otherwise, we would take $s'$ instead of $s$ (in that case $s' \notin S_1$, because $S_1$ is an antichain). Take the following values of the variables

$$x_i = \begin{cases} 1, & \text{if } i \in s, \\ 0, & \text{if } i \notin s. \end{cases}$$

For that values of variables we have: $f_{S_1} = 1$ and $f_{S_2} = 0$.

The form (7) is uniquely determined by the antichain $S \subseteq 2^{\{1,\dots,n\}}$. And conversely, every monotone Boolean function can be obtained in that way.

**Proposition 1** *For every monotone Boolean function of $n$ variables, there exists a unique antichain $S \subseteq 2^{\{1,\dots,n\}}$ such that $f = f_S$.*

**Proof**: For $a = (a_1, \dots, a_n) \in B^n$ let $s_a = \{i : a_i = 1\}$. Consider the set $A = \{s_a : a \in B^n,\ f(a) = 1\}$. Let $S$ be the subset of $A$, consisting exactly of all minimal sets in $A$. Then $S \subseteq 2^{\{1,\dots,n\}}$ is an antichain. Notice that $f(a_1, \dots, a_n) = 1$ iff for some $s \in S$ we have $a_i = 1$ for all $i \in s$. The same is valid for $f_S$. Therefore, $f(a_1, \dots, a_n) = f_S(a_1, \dots, a_n)$ for all $(a_1, \dots, a_n) \in B^n$, and so $f = f_S$. The uniqueness follows from the argument stated above. $\square$

Define the Boolean functions:

$$\delta_n^i = x_i,\ i = \overline{1,n}.$$

**Theorem 2** *The algebra $\mathfrak{L}_n$ is a bounded free distributive lattice with the system of free generators: $\Delta = \{\delta_n^1, \ldots, \delta_n^n\}$.*

**Proof**: Let $\mathfrak{L}$ be any distributive lattice and $\varphi : \Delta \to \mathfrak{L}$ be a mapping. We show that there exists a unique homomorphism $\psi : \mathfrak{L}_n \to \mathfrak{L}$ such that $\psi|_\Delta = \varphi$. For any $f \in \mathfrak{L}_n$ there exists a unique antichain $S$ such that

$$f(x_1, \ldots, x_n) = f_S(x_1, \ldots, x_n) = \sum_{s\in S} \prod_{i\in s} x_i.$$

Take

$$\psi(f) = \sum_{s\in S} \prod_{i\in s} \varphi(\delta_i).$$

Obviously $\varphi(f) = \psi(f)$ for $f \in \Delta$ and $\psi$ is a homomorphism. The uniqueness of $\psi$ is also obvious. $\square$

A problem posed by B. I. Plotkin in 1970s has required finding the varieties of algebras with similar functional representations of finitely generated free algebras. In paper [156] we have introduced the concept of De Morgan function and proved that the free De Morgan algebra on $n$ free generators is isomorphic to the De Morgan algebra of De Morgan functions of $n$ variables (see definitions below). This is a solution of the problem posed by B. I. Plotkin.

# 3 De Morgan algebras

**Definition 3** *An algebra $(Q; \{+, \cdot, ^-, 0, 1\})$ with two binary, one unary and two nullary operations is called a De Morgan algebra if $(Q; \{+, \cdot, 0, 1\})$ is a bounded distributive lattice with the least element $0$ and the greatest element $1$, and the algebra $(Q; \{+, \cdot, ^-\})$ satisfies the following identities*

$$\overline{x+y} = \overline{x} \cdot \overline{y},$$

$$\overline{\overline{x}} = x,$$

*where $\overline{\overline{x}} = \overline{(\overline{x})}$ ([14, 28, 30, 38, 39, 40, 65, 93, 123, 132, 205, 215]).*

For example, the standard fuzzy algebra $F = ([0, 1]; \max(x, y), \min(x, y), 1 - x, 0, 1)$ is a De Morgan algebra.

Note that in any De Morgan algebra $(Q;\{+,\cdot,^-,0,1\})$ we have $\overline{0}=1$, $\overline{1}=0$ and

$$x \leq y \Leftrightarrow \overline{y} \leq \overline{x},\ x,y \in Q.$$

If $V$ is the variety of De Morgan algebras then the free algebra $F_V(X)$ is called free De Morgan algebra with the set of free generators $X$. Thus, De Morgan algebra $\mathcal{F} = F(+,\cdot,^-,0,1\})$ is called a free De Morgan algebra with the system of free generators $X \subseteq F$ if the algebra $\mathcal{F}$ is generated by the subset $X \subseteq F$ and for every De Morgan algebra $\mathfrak{S} = S(+,\cdot,^-,0,1\})$ and for every mapping $\mu : X \to S$ there exists a unique homomorphism $\nu : \mathcal{F} \to \mathfrak{S}$ with $\nu|_X = \mu$.

A characterization of De Morgan algebras can be found in [28, 39].

Let $\mathfrak{B} = (Q;\{+,\cdot,0,1\})$ be a bounded distributive lattice. Denote its dual bounded distributive lattice by $\mathfrak{B}^{\mathfrak{op}}$, i.e., $\mathfrak{B}^{\mathfrak{op}} = (Q;\{\cdot,+,1,0\})$. Consider the direct product $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}} = (Q \times Q;\{\vee,\wedge,(0,1),(1,0)\})$ where $(x_1,y_1) \vee (x_2,y_2) = (x_1+x_2, y_1 \cdot y_2)$, $(x_1,y_1) \wedge (x_2,y_2) = (x_1 \cdot x_2, y_1+y_2)$, for any $x_1,x_2,y_1,y_2 \in Q$. Defining one more operation $^-$ on the set $Q \times Q$ by $\overline{(x,y)} = (y,x)$, we convert the bounded distributive lattice $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}}$ into the De Morgan algebra $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}}$.

Now we prove a Stone-type representation theorem for De Morgan algebras.

**Theorem 3** *Every De Morgan algebra is isomorphic to a subalgebra of the De Morgan algebra $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}}$ for some bounded distributive lattice $\mathfrak{B}$.*

**Proof:** Suppose $\mathfrak{A} = (Q;\{+,\cdot,^-,0,1\})$ is a De Morgan algebra. From Birkhoff's representation theorem for distributive lattices ([83]) it follows that there exists a set $I$ such that the distributive lattice $(Q;\{+,\cdot,\ 0,1\})$ is isomorphic to a subalgebra of the distributive lattice $(2^I;\{\cup,\cap,\ \emptyset,I\}) = \mathfrak{B}$. Let $\sigma : Q \to 2^I$ be an embedding of the mentioned distributive lattice in $\mathfrak{B}$. We define an embedding of the De Morgan algebra $\mathfrak{A}$ into the De Morgan algebra $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}}$ by the following rule:

$$\varphi(s) = (\sigma(s), \sigma(\overline{s})),\ s \in Q.$$

Indeed, for all $s,t \in Q$ we have:

$$\varphi(s+t) = (\sigma(s+t), \sigma(\overline{s+t})) = (\sigma(s+t), \sigma(\overline{s}\cdot\overline{t})) =$$
$$= (\sigma(s)\cup\sigma(t), \sigma(\overline{s})\cap\sigma(\overline{t})) = (\sigma(s),\sigma(\overline{s})) \vee (\sigma(t),\sigma(\overline{t})) = \varphi(s)\vee\varphi(t),$$

$$\varphi(s\cdot t) = (\sigma(s\cdot t), \sigma(\overline{s\cdot t})) = (\sigma(s\cdot t), \sigma(\overline{s}+\overline{t})) =$$
$$= (\sigma(s)\cap\sigma(t), \sigma(\overline{s})\cup\sigma(\overline{t})) = (\sigma(s),\sigma(\overline{s})) \wedge (\sigma(t),\sigma(\overline{t})) = \varphi(s)\wedge\varphi(t),$$

$$\varphi(\overline{s}) = (\sigma(\overline{s}), \sigma(\overline{\overline{s}})) = (\sigma(\overline{s}), \sigma(s)) = \overline{\varphi(s)},$$

.

These equalities show that $\varphi$ is a homomorphism. Obviously, $\varphi$ is injective, hence it is an embedding. $\square$

Let us consider the following De Morgan algebras:
$\mathbf{2} = (\{0,1\}; \{+, \cdot, ^-, 0, 1\})$,
$\mathbf{3} = (\{0,a,1\}; \{+, \cdot, ^-, 0, 1\})$, where $\overline{a} = a$, and
$\mathbf{4} = (\{0,a,b,1\}; \{+, \cdot, ^-, 0, 1\})$, where $\overline{a} = a,\ \overline{b} = b,\ a + b = 1,\ a \cdot b = 0$.
Let us remind the following result.

**Theorem 4** ([93]) *Every non-trivial subdirectly irreducible De Morgan algebra is isomorphic to one of the following algebras:* $\mathbf{2}, \mathbf{3}, \mathbf{4}$*, where* $\mathbf{2}$ *is the unique non-trivial subdirectly irreducible Boolean algebra.* $\square$

# 4 Free De Morgan algebras and De Morgan functions

Denote $D = B \times B = \{(0,0), (1,0), (0,1), (1,1)\} = \{0, a, b, 1\}$, where $0 = (0,0), a = (1,0), b = (0,1), 1 = (1,1)$. Defining $0 + x = x,\ 1 \cdot x = x,\ x \in D,\ a + b = 1,\ ab = 0,\ \overline{0} = 1,\ \overline{1} = 0,\ \overline{a} = a,\ \overline{b} = b$ we get the De Morgan algebra $\mathbf{4} = D(+, \cdot, ^-, 0, 1)$. Notice that $\overline{(u,v)} = (\overline{v}, \overline{u})$, $(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$ and $(u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2)$ (here the operations on the right hand side are the operations in the Boolean algebra $\mathbf{2}$). For $x \in D$ let

$$x^* = \begin{cases} x, & \text{if } x = 0, 1, \\ a, & \text{if } x = b, \\ b, & \text{if } x = a. \end{cases}$$

Also for $c = (c_1, \ldots, c_n), d = (d_1, \ldots, d_n) \in D^n$ we say that $d$ is a permitted modification of $c$ if for some $k$ $(1 \leq k \leq n)$ we have $d_i = c_i$ for all $1 \leq i \leq n,\ i \neq k$ and

$$d_k = \begin{cases} a, & \text{if } c_k = 0, \\ 1, & \text{if } c_k = b. \end{cases}$$

**Definition 4** *A function* $f : D^n \to D$ *is called a De Morgan function of* $n$ *variables if the following conditions hold:*

*(1) if* $x_i \in \{0,1\},\ i = \overline{1,n}$*, then* $f(x_1, \ldots, x_n) \in \{0,1\}$*,*

*(2) if* $x_i \in D,\ i = \overline{1,n}$ *then* $f(x_1^*, \ldots, x_n^*) = (f(x_1, \ldots, x_n))^*$*,*

*(3) if* $x \in D^n$ *with* $f(x) \neq b$ *and* $y$ *is a permitted modification of* $x$ *then* $f(y) \in \{f(x), a\}$*.*

Note that it follows from the first condition that every De Morgan function is an extension of some Boolean function. And notice that the constant functions $f = 1$ and $f = 0$ are De Morgan functions, but the constant functions $f = a$ and $f = b$ are not. This means that 0 and 1 are only constant De Morgan functions. Other examples of De Morgan functions are $f(x) = x,\ g(x) = \overline{x},\ h(x, y) = x \cdot y,\ q(x, y) = x + y$, where the operations on the right hand side are the operations in the algebra **4**.

As Boolean functions, De Morgan functions can be given by tables. Also note that there is an algorithm which for a given table of a function $f : D^n \to D$ determines whether $f$ is a De Morgan function.

Below, for $x_i \in D$ by $(y_i, z_i)$ we mean the couple from $B \times B$ which is equal to $x_i$. But often we will consider $B = \{0, 1\}$ as a subset of $D$ (when it can cause no confusion).

**Definition 5** *The function $f : D^n \to D$ is called a quasi-De Morgan function of $n$ variables if there exists a Boolean function $\varphi : B^{2n} \to B$ such that*

$$f(x_1, \ldots, x_n) = (\varphi(y_1, \ldots, y_n, \overline{z}_1, \ldots, \overline{z}_n), \varphi(z_1, \ldots, z_n, \overline{y}_1, \ldots, \overline{y}_n)), \qquad (8)$$

*for all $x_1, \ldots, x_n \in D$, $x_i = (y_i, z_i)$, $i = 1, ..., n$.*

**Proposition 2** *The function $f : D^n \to D$ is quasi-De Morgan function iff it satisfies the conditions* (1) *and* (2)*.*

**Proof**: Let $f$ be a quasi-De Morgan function. If $x_i \in \{0, 1\}$, then $y_i = z_i$ and $\varphi(y_1, \ldots, y_n, \overline{z}_1, \ldots, \overline{z}_n) = \varphi(z_1, \ldots, z_n, \overline{y}_1, \ldots, \overline{y}_n)$. Hence, $f(x_1, \ldots, x_n) \in B$. Thus, condition (1) holds for $f$. Now let us check condition (2). To do this notice that $(u, v)^* = (v, u)$. Hence,

$$f(x_1^*, \ldots, x_n^*) = (\varphi(z_1, \ldots, z_n, \overline{y}_1, \ldots, \overline{y}_n), \varphi(y_1, \ldots, y_n, \overline{z}_1, \ldots, \overline{z}_n)) =$$
$$(\varphi(y_1, \ldots, y_n, \overline{z}_1, \ldots, \overline{z}_n), \varphi(z_1, \ldots, z_n, \overline{y}_1, \ldots, \overline{y}_n))^* = (f(x_1, \ldots, x_n))^*.$$

Now suppose conditions (1) and (2) hold for $f$, and let us prove that there exists a Boolean function $\varphi$ with condition (8). First we prove that there are at most $2^{4^n}$ functions for which the conditions (1) and (2) hold. To see this notice that there are $2^n$ $n$-tuples $(u_1, \ldots, u_n) \in B^n$. For such $n$-tuples $f$ can take only two values (by condition (1)). Further, if the $n$-tuple $(v_1, \ldots, v_n) \in D^n$ contains $a$ or $b$, then $(v_1^*, \ldots, v_n^*) \neq (v_1, \ldots, v_n)$ and $f(v_1^*, \ldots, v_n^*)$ is uniquely determined by $f(v_1, \ldots, v_n)$ (by condition (2)). There are $4^n - 2^n$ such $n$-tuples. Thus, the number of such functions does not exceed $2^{2^n} \cdot 4^{\frac{4^n - 2^n}{2}} = 2^{2^n} \cdot 2^{4^n - 2^n} = 2^{4^n}$. It is clear that for a quasi-De Morgan function $f$ there exists exactly one Boolean function $\varphi$ with condition (8). Therefore, there are $2^{4^n} = 2^{2^{2n}}$ quasi-De Morgan functions of $n$ variables. And all quasi-De Morgan functions satisfy conditions (1) and (2). Hence, all functions $f : D^n \to D$ satisfying (1) and (2) are quasi-De Morgan functions. $\square$

**Corollary 1** *There are exactly $2^{4^n}$ quasi-De Morgan functions of $n$ variables.*

As we mentioned in the proof, for a quasi-De Morgan function $f : D^n \to D$ there exists a unique Boolean function $\varphi : B^{2n} \to B$ which satisfies (8). To emphasize that $\varphi$ is the function corresponding to $f$, we denote it by $\varphi_f$.

**Theorem 5** *The function $f : D^n \to D$ is a De Morgan function iff it is a quasi-De Morgan function and $\varphi_f$ is a monotone Boolean function.*

**Proof**: If $f$ is a De Morgan function, then by Proposition 2 it is a quasi-De Morgan function. Let us prove that $\varphi_f$ is monotone. Let $u = (u_1, \ldots, u_{2n})$, $v = (v_1, \ldots, v_{2n}) \in B^{2n}$ and for some $k$ $(1 \le k \le 2n)$ $u_i = v_i$, if $i \ne k$, $u_k = 0$, $v_k = 1$. We show that $\varphi_f(u) \le \varphi_f(v)$. Suppose it is not true, i.e. $\varphi_f(u) = 1,\ \varphi_f(v) = 0$. For $1 \le i \le n$ denote

$$c_i = \begin{cases} (u_i, \overline{u}_{n+i}), \text{ if } 1 \le k \le n, \\ (\overline{u}_{n+i}, u_i), \text{ if } n+1 \le k \le 2n, \end{cases}$$

and

$$d_i = \begin{cases} (v_i, \overline{v}_{n+i}), \text{ if } 1 \le k \le n, \\ (\overline{v}_{n+i}, v_i), \text{ if } n+1 \le k \le 2n. \end{cases}$$

Suppose $1 \le k \le n$. Then $d = (d_1, \ldots, d_n)$ is a permitted modification of $c = (c_1, \ldots, c_n)$.

$$f(c) = (\varphi_f(u_1, \ldots, u_n, u_{n+1}, \ldots, u_{2n}), \varphi_f(\overline{u}_{n+1}, \ldots, \overline{u}_{2n}, \overline{u}_1, \ldots, \overline{u}_n)) =$$

$$(1, \varphi_f(\overline{u}_{n+1}, \ldots, \overline{u}_{2n}, \overline{u}_1, \ldots, \overline{u}_n)) \ne b.$$

Similarly

$$f(d) = (0, \varphi_f(\overline{v}_{n+1}, \ldots, \overline{v}_{2n}, \overline{v}_1, \ldots, \overline{v}_n)).$$

By condition (3) we have $f(d) = f(c)$ or $f(d) = a$. A contradiction!

Now suppose $n+1 \le k \le 2n$. Then $c$ is a permitted modification of $d$. We have

$$f(d) = (\varphi_f(\overline{v}_{n+1}, \ldots, \overline{v}_{2n}, \overline{v}_1, \ldots, \overline{v}_n), \varphi_f(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n})) =$$

$$(\varphi_f(\overline{v}_{n+1}, \ldots, \overline{v}_{2n}, \overline{v}_1, \ldots, \overline{v}_n), 0) \ne b.$$

And also

$$f(c) = (\varphi_f(\overline{u}_{n+1}, \dots, \overline{u}_{2n}, \overline{u}_1, \dots, \overline{u}_n), 1).$$

Again, by condition (3), we have $f(c) = f(d)$ or $f(c) = a$. A contradiction!

In both cases, we have arrived at a contradiction. Consequently, $\varphi_f$ is a monotone Boolean function.

The first part of the theorem is proved. Now let us prove the second part.

Suppose that $f$ is a quasi-De Morgan function and $\varphi_f$ is a monotone Boolean function. We verify that condition (3) holds for $f$. To see this let $d = (d_1, \dots, d_n) \in D^n$ be a permitted modification of $c = (c_1, \dots, c_n) \in D^n$. It means that for some $k$ $(1 \le k \le n)$ we have $c_i = d_i$ if $i \ne k$ and

$$d_k = \begin{cases} a, & \text{if } c_k = 0, \\ 1, & \text{if } c_k = b. \end{cases}$$

Let $c_i = (u_i, v_i)$, $d_i = (p_i, q_i)$. Then $u_i \le p_i$ and $v_i = q_i$ for all $i = \overline{1, n}$. Therefore, $(u_1, \dots, u_n, \overline{v}_1, \dots, \overline{v}_n) \le (p_1, \dots, p_n, \overline{q}_1, \dots, \overline{q}_n)$ and $(v_1, \dots, v_n, \overline{u}_1, \dots, \overline{u}_n) \ge (q_1, \dots, q_n, \overline{p}_1, \dots, \overline{p}_n)$. Hence,

$$\varphi_f(u_1, \dots, u_n, \overline{v}_1, \dots, \overline{v}_n) \le \varphi_f(p_1, \dots, p_n, \overline{q}_1, \dots, \overline{q}_n)$$

and

$$\varphi_f(v_1, \dots, v_n, \overline{u}_1, \dots, \overline{u}_n) \ge \varphi_f(q_1, \dots, q_n, \overline{p}_1, \dots, \overline{p}_n).$$

Thus, the first coordinate of $f(c)$ is less than (or equal to) the first coordinate of $f(d)$ and the second coordinate of $f(c)$ is greater than (or equal to) the second coordinate of $f(d)$. Thus, if $f(c) = 0$, then $f(d) \in \{0, a\}$; if $f(c) = a$, then $f(d) = a$; and if $f(c) = 1$, then $f(d) \in \{1, a\}$. $\square$

**Corollary 2** *There are $m_{2n}$ De Morgan functions of $n$ variables.* $\square$

Denote the set of all of $n$-variable De Morgan functions by $\mathcal{D}_n$. For the functions $f, g : D^n \to D$ define $f + g$, $f \cdot g$ and $\overline{f}$ in the standard way, i.e. $(f + g)(x) = f(x) + g(x)$, $(f \cdot g)(x) = f(x) \cdot g(x)$, $\overline{f}(x) = \overline{f(x)}$, $x \in D^n$, where the operations on the right hand side are the operations of De Morgan algebra **4**. Notice that $\mathcal{D}_n$ is closed under those operations, i.e. if $f, g \in \mathcal{D}_n$, then $f + g, f \cdot g, \overline{f} \in \mathcal{D}_n$. We can verify it straightforwardly, using the definition of De Morgan function. But it is easier to prove that using Theorem 5. If $f, g \in \mathcal{D}_n$, then

$$f(x_1, \dots, x_n) = (\varphi_f(y_1, \dots, y_n, \overline{z}_1, \dots, \overline{z}_n), \varphi_f(z_1, \dots, z_n, \overline{y}_1, \dots, \overline{y}_n)),$$

and

$$g(x_1, \dots, x_n) = (\varphi_g(y_1, \dots, y_n, \overline{z}_1, \dots, \overline{z}_n), \varphi_g(z_1, \dots, z_n, \overline{y}_1, \dots, \overline{y}_n)).$$

Hence

$$(f+g)(x_1,\dots,x_n)= \\ =((\varphi_f+\varphi_g)(y_1,\dots,y_n,\overline{z}_1,\dots,\overline{z}_n),(\varphi_f+\varphi_g)(z_1,\dots,z_n,\overline{y}_1,\dots,\overline{y}_n))$$

and

$$(f\cdot g)(x_1,\dots,x_n)= \\ =((\varphi_f\cdot\varphi_g)(y_1,\dots,y_n,\overline{z}_1,\dots,\overline{z}_n),(\varphi_f\cdot\varphi_g)(z_1,\dots,z_n,\overline{y}_1,\dots,\overline{y}_n)).$$

As $\varphi_f,\varphi_g$ are monotone boolean functions, $\varphi_f\cdot\varphi_g$ and $\varphi_f+\varphi_g$ are monotone, as well. So by Theorem 5 $f+g$ and $f\cdot g$ are De Morgan functions.
Further, we get

$$\begin{aligned}&\overline{f}(x_1,\dots,x_n)=\\ &\overline{(\varphi_f(y_1,\dots,y_n,\overline{z}_1,\dots,\overline{z}_n),\varphi_f(z_1,\dots,z_n,\overline{y}_1,\dots,\overline{y}_n))}=\\ &(\overline{\varphi_f(z_1,\dots,z_n,\overline{y}_1,\dots,\overline{y}_n)},\overline{\varphi_f(y_1,\dots,y_n,\overline{z}_1,\dots,\overline{z}_n)})=\\ &(\varphi'_f(y_1,\dots,y_n,\overline{z}_1,\dots,\overline{z}_n),\varphi'_f(z_1,\dots,z_n,\overline{y}_1,\dots,\overline{y}_n)).\end{aligned}$$

And $\varphi_f$ is monotone; therefore, $\varphi'_f$ is also monotone. Hence, $\overline{f}$ is a De Morgan function.

Thus, we get the algebra $\mathfrak{D}_n=\mathcal{D}_n(+,\cdot,^{-},0,1)$, which obviously is a De Morgan algebra. Also, for $f,g\in\mathcal{D}_n$ we have: $\varphi_{f+g}=\varphi_f+\varphi_g$, $\varphi_{f\cdot g}=\varphi_f\cdot\varphi_g$, $\varphi_{\overline{f}}=\varphi'_f$.

Let $a=(a_1,a_2)$, $b=(b_1,b_2)\in 2^{\{1,\dots,n\}}\times 2^{\{1,\dots,n\}}$. We say that $a\subseteq b$, if $a_1\subseteq b_1$ and $a_2\subseteq b_2$. In this way, we get a partially ordered set $2^{\{1,\dots,n\}}\times 2^{\{1,\dots,n\}}(\subseteq)$. For the antichain, $S\subseteq 2^{\{1,\dots,n\}}\times 2^{\{1,\dots,n\}}$, define the function, $f_S:D^n\to D$, by the following way:

$$f_S(x_1,\dots,x_n)=\sum_{s=(s_1,s_2)\in S}\left(\prod_{i\in s_1}x_i\cdot\prod_{i\in s_2}\overline{x}_i\right). \tag{9}$$

Notice that $f_S$ does not depend on the order of the elements in the set $S$ ([140]).

Note that we set $f_\varnothing=0$ and $f_{\{(\varnothing,\varnothing)\}}=1$.

Let us consider the functions

$$\delta_n^i=x_i,\ i=\overline{1,n},$$

as functions $D^n\to D$. Obviously, $\delta_n^i$ is a De Morgan function. And according to (9), for any antichain $S\subseteq 2^{\{1,\dots,n\}}\times 2^{\{1,\dots,n\}}$ we have

$$f_S=\sum_{s=(s_1,s_2)\in S}\left(\prod_{i\in s_1}\delta_n^i\cdot\prod_{i\in s_2}\overline{\delta_n^i}\right).$$

Hence, $f_S \in \mathcal{D}_n$, i.e. $f_S$ is a De Morgan function for any antichain $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$.

For $s = (s_1, s_2) \in 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $s' = s_1 \cup \{n+i : i \in s_2\} \in 2^{\{1,\ldots,2n\}}$, and for $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $S' = \{s' : s \in S\} \subseteq 2^{\{1,\ldots,2n\}}$. In this way we give a bijective mapping from the set of all antichains of $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$ to the set of all antichains of $2^{\{1,\ldots,2n\}}(\subseteq)$. And so the number of all antichains of the partially ordered set $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$ is $m_{2n}$.

Now, for any De Morgan function $f \in \mathcal{D}_n$ from Proposition 1 and Theorem 5 we conclude that there exists an antichain $S' \subseteq 2^{\{1,\ldots,2n\}}$ such that

$$f(x_1,\ldots,x_n) = (\varphi_f(y_1,\ldots,y_n,\overline{z}_1,\ldots,\overline{z}_n), \varphi_f(z_1,\ldots,z_n,\overline{y}_1,\ldots,\overline{y}_n)) =$$

$$\left(\sum_{s'\in S'}\left(\prod_{\substack{i\in s'\\1\le i\le n}} y_i \cdot \prod_{\substack{i\in s'\\n+1\le i\le 2n}} \overline{z}_{i-n}\right), \sum_{s'\in S'}\left(\prod_{\substack{i\in s'\\1\le i\le n}} z_i \cdot \prod_{\substack{i\in s'\\n+1\le i\le 2n}} \overline{y}_{i-n}\right)\right) =$$

$$\sum_{s'\in S'}\left(\prod_{i\in s'\ 1\le i\le n} (y_i, z_i) \cdot \prod_{i\in s'\ n+1\le i\le 2n} (\overline{z}_{i-n}, \overline{y}_{i-n})\right) =$$

$$\sum_{s=(s_1,s_2)\in S}\left(\prod_{i\in s_1} x_i \cdot \prod_{i\in s_2} \overline{x}_i\right) = f_S(x_1,\ldots,x_n),$$

where $S$ is the antichain of $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$, corresponding to $S'$. Moreover, the number of all De Morgan functions of $n$ variables is the same as the number of all antichains of $2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}(\subseteq)$. Hence, we get the following result.

**Theorem 6** *For any De Morgan function $f$ of $n$ variables there exists a unique antichain $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ such that $f = f_S$.* □

In particular, $f_{S_1} \neq f_{S_2}$ if $S_1 \neq S_2$.

Thus every nonconstant De Morgan function can be uniquely presented in the form (9). This form is called the canonical form (or disjunctive normal form (or briefly - DNF)) of De Morgan function $f$. Notice that from Theorem 5 and from the proofs of Theorem 6 and Proposition 1 we get an algorithm which, for a De Morgan function, gives its disjunctive normal form. From this we conclude that every nonconstant De Morgan function can be uniquely presented in conjunctive normal form (CNF), i.e. in the following form:

$$\prod_{(s_1,s_2)\in S}\left(\sum_{i\in s_1} x_i + \sum_{i\in s_2} \overline{x}_i\right).$$

For a De Morgan function CNF is unique (we can prove this analogously to DNF).

The next result is related to the Plotkin's problem.

**Theorem 7** (Functional representation theorem) ([156]) *The algebra $\mathfrak{D}_n$ is a free De Morgan algebra with the system of free generators: $\Delta = \{\delta_n^1, \ldots, \delta_n^n\}$. Hence, every free $n$-generated De Morgan algebra is isomorphic to the De Morgan algebra $\mathfrak{D}_n$.*

**Problem 1** *To develop the De Morgan analogue of the theory of Boolean functions.*

# 5 Boole-De Morgan algebras

**Definition 6** An algebra $(Q; \{+, \cdot, ^-, \,', 0, 1\})$ with two binary, two unary and two nullary operations is called a *Boole-De Morgan* algebra if $(Q; \{+, \cdot, ^-, 0, 1\})$ is a De Morgan algebra and $(Q; \{+, \cdot, \,', 0, 1\})$ is a Boolean algebra and the two unary operations commute, i.e., $(\overline{x})' = \overline{(x')}$.

This concept is introduced in [146, 147] under the name of Boolean bisemigroup (also see [157]).

If $V$ is the variety of Boole-De Morgan algebras then the free algebra $F_V(X)$ is called free Boole-De Morgan algebra with the set of free generators $X$.

Let us consider some natural examples of Boole-De Morgan algebras. First note that every Boolean algebra can be considered as a Boole-De Morgan algebra with two equal unary operations. In particular if $\mathbf{B} = (B; \{+, \cdot, \,', 0, 1\})$ is the two-element Boolean algebra and $\overline{x} = x'$ (i.e., the unary operations $^-$ and $'$ are equal) then the algebra $(B; \{+, \cdot, ^-, \,', 0, 1\})$ is a Boole-De Morgan algebra and we will denote it by $BM_2$.

Now we define a Boole-De Morgan algebra on the four-element set $D = \{0, a, b, 1\}$ which will be used in the proof of the main theorem of Section 4. Defining $0 + x = x + 0 = x,\ 0 \cdot x = x \cdot 0 = 0$ and $1 \cdot x = x \cdot 1 = x,\ 1 + x = x + 1 = 1$ and $x + x = x,\ x \cdot x = x$ for all $x \in D$, and $a + b = b + a = 1,\ a \cdot b = b \cdot a = 0,\ \overline{0} = 1,\ \overline{1} = 0,\ \overline{a} = a,\ \overline{b} = b,\ 1' = 0,\ 0' = 1,\ a' = b,\ b' = a$ we get the Boole-De Morgan algebra $BM_4 = (D; \{+, \cdot, ^-, \,', 0, 1\})$.

For a Boolean algebra $\mathfrak{B} = (Q; \{+, \cdot, \,', 0, 1\})$ consider the direct product $\mathfrak{B} \times \mathfrak{B}$. Defining one more unary operation $^-$ on the set $Q \times Q$ by $\overline{(x, y)} = (y', x')$ we get the Boole-De Morgan algebra $\mathfrak{B} \times \mathfrak{B}$.

Let us recall the definition of $n$-ary term operations. Let $\mathfrak{A} = (Q; \Sigma)$ be an arbitrary algebra. The $n$-ary term operations (or term functions) of algebra $\mathfrak{A}$ are defined by induction:

1) all $n$-ary identical operations (or projections) of set $Q$

$$\delta_n^i(x_1, \ldots, x_n) = x_i,\ i = 1, \ldots, n,$$

are $n$-ary term operations of $\mathfrak{A}$;

2) if $f_1, \ldots, f_m$ are $n$-ary term operations of $\mathfrak{A}$, then the superposition

$$f(x_1, \ldots, x_n) = f_0(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

is again an $n$-ary term operation of $\mathfrak{A}$, for every $m$-ary operation $f_0 \in \Sigma$;

3) there are no other $n$-ary term operations of $\mathfrak{A}$.

An operation $h$ on the set $Q$ is called a term operation (function) of algebra $\mathfrak{A}$, if $h$ is an $n$-ary term operation of $\mathfrak{A}$ for some $n$.
In particular, for $n = 2$ we get the definition of binary term operation of the given algebra $\mathfrak{A}$. For example, for a nontrivial lattice $\mathfrak{A}$ the set $\mathcal{T}(\mathfrak{A})$ of binary term operations of $\mathfrak{A}$ is equal to $\{x + y, x \cdot y, x, y\}$.

The set $\mathcal{T}(\mathfrak{A})$ of binary term operations of the nontrivial lattice $\mathfrak{A}$ is a Boole-De Morgan algebra (of order 4) where the operations are defined below. For any two binary terms $f(x, y)$ and $g(x, y)$ the binary operations are defined as the following binary superpositions:

$$(f + g)(x, y) = f(x, g(x, y)),\ (f \cdot g)(x, y) = f(g(x, y), y).$$

The nullary operations are the terms $y$ and $x$. The unary operations are the commutation and dualization. The commutation is defined by $\overline{f}(x, y) = f(y, x)$ and for a binary term $f(x, y)$ to get its dual term $f'(x, y)$ we shall change all variables $x$ by $y$ and vice versa, and also change all operations $+$ by $\cdot$ and vice versa. So we obtain the Boole-De Morgan algebra $\mathcal{T}(\mathfrak{A})$: $\overline{x + y} = x + y$, $\overline{x \cdot y} = x \cdot y$, $\overline{x} = y$, $\overline{y} = x$, $(x + y)' = x \cdot y$, $(x \cdot y)' = x + y$, $x' = y$, $y' = x$. Note, that every lattice identity of the Boole-De Morgan algebra $\mathcal{T}(\mathfrak{A})$ is equivalent to the certain hyperidentity.

Let $\mathfrak{B} = (Q; \{+, \cdot, \ ', 0, 1\})$ be a Boolean algebra. Denote its dual Boolean algebra by $\mathfrak{B}^{\mathrm{op}}$, i.e., $\mathfrak{B}^{\mathrm{op}} = (Q; \{\cdot, +, \ ', 1, 0\})$. Consider the direct product $\mathfrak{B} \times \mathfrak{B}^{\mathrm{op}} = (Q \times Q; \{\vee, \wedge, \ ', (0, 1), (1, 0)\})$ where $(x_1, y_1) \vee (x_2, y_2) = (x_1 + x_2, y_1 \cdot y_2)$, $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2)$, $(x_1, y_1)' = (x_1', y_1')$ for any $x_1, x_2, y_1, y_2 \in Q$. Defining one more unary operation $^{-}$ on the set $Q \times Q$ by $\overline{(x, y)} = (y, x)$, we convert the Boolean algebra $\mathfrak{B} \times \mathfrak{B}^{\mathrm{op}}$ into the Boole-De Morgan algebra $\mathfrak{B} \times \mathfrak{B}^{\mathrm{op}}$.

Now we prove a Stone-type representation theorem for Boole-De Morgan algebras.

**Theorem 8** ([146]) *Every Boole-De Morgan algebra is isomorphic to a subalgebra of the Boole-De Morgan algebra* $\mathfrak{B} \times \mathfrak{B}^{\mathrm{op}}$ *for some Boolean algebra* $\mathfrak{B}$.

**Proof:** Suppose $\mathfrak{A} = (Q; \{+, \cdot, ^{-}, \ ', 0, 1\})$ is a Boole-De Morgan algebra. From Stone's representation theorem for Boolean algebras ([83]) it follows that there exists a set $I$ such that the Boolean algebra $(Q; \{+, \cdot, \ ', 0, 1\})$ is isomorphic to a subalgebra of the Boolean algebra $(2^I; \{\cup, \cap, \ ', \emptyset, I\}) = \mathfrak{B}$ where for a set $X \subseteq I$ we define $X' = I \backslash X$. Let $\sigma : Q \to 2^I$ be an embedding of the mentioned Boolean algebra in $\mathfrak{B}$. We define an embedding of the Boole-De Morgan algebra $\mathfrak{A}$ in the Boole-De Morgan algebra $\mathfrak{B} \times \mathfrak{B}^{op}$ by the following rule:

$$\varphi(s) = (\sigma(s), \sigma(\overline{s})),\ s \in Q.$$

Indeed, for all $s, t \in Q$ we have

$$\varphi(s+t) = (\sigma(s+t), \sigma(\overline{s+t})) = (\sigma(s+t), \sigma(\overline{s} \cdot \overline{t})) =$$
$$= (\sigma(s) \cup \sigma(t), \sigma(\overline{s}) \cap \sigma(\overline{t})) = (\sigma(s), \sigma(\overline{s})) \vee (\sigma(t), \sigma(\overline{t})) = \varphi(s) \vee \varphi(t),$$

$$\varphi(s \cdot t) = (\sigma(s \cdot t), \sigma(\overline{s \cdot t})) = (\sigma(s \cdot t), \sigma(\overline{s} + \overline{t})) =$$
$$= (\sigma(s) \cap \sigma(t), \sigma(\overline{s}) \cup \sigma(\overline{t})) = (\sigma(s), \sigma(\overline{s})) \wedge (\sigma(t), \sigma(\overline{t})) = \varphi(s) \wedge \varphi(t),$$

$$\varphi(\overline{s}) = (\sigma(\overline{s}), \sigma(\overline{\overline{s}})) = (\sigma(\overline{s}), \sigma(s)) = \overline{\varphi(s)},$$

$$\varphi(s') = (\sigma(s'), \sigma(\overline{s'})) = (\sigma(s'), \sigma((\overline{s})')) = ((\sigma(s))', (\sigma(\overline{s}))') =$$
$$= (\sigma(s), \sigma(\overline{s}))' = (\varphi(s))'.$$

These equalities show that $\varphi$ is a homomorphism. Obviously, $\varphi$ is injective, hence it is an embedding. $\square$

**Theorem 9** ([157]) *The Boole-De Morgan algebras $BM_2$ and $BM_4$ are subdirectly irreducible, and those algebras are the only nontrivial subdirectly irreducible Boole-De Morgan algebras up to isomorphism.*

For a Boole-De Morgan algebra $(Q; \{+, \cdot, ^-, \ ', 0, 1\})$ we define one more unary operation $^*$ by the following way: $x^* = (\overline{x})' = \overline{(x')}$. It is easy to see that $(x+y)^* = x^* + y^*$, $(x \cdot y)^* = x^* \cdot y^*$, $\overline{x^*} = (\overline{x})^*$, $(x^*)' = (x')^*$. Thus the mapping $x \to x^*$ is an automorphism of the Boole-De Morgan algebra $(Q; \{+, \cdot, ^-, \ ', 0, 1\})$. Also it is easy to see that $x^* = x$ if and only if $x' = \overline{x}$.

# 6 Free Boole-De Morgan algebras and quasi-De Morgan functions

Recall that $D = \{0, a, b, 1\}$ and $B = \{0, 1\}$. Let us construct a one-to-one correspondence between the sets $D$ and $B \times B$ as follows:

$$0 \leftrightarrow (0,0),\ a \leftrightarrow (1,0),\ b \leftrightarrow (0,1),\ 1 \leftrightarrow (1,1).$$

We define the operations $+, \cdot, ^-, \ '$ on the set $B \times B$ as follows:

$$\overline{(u,v)} = (v', u'),\ (u,v)' = (u', v'),$$

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2),\ (u_1, v_1) \cdot (u_2, v_2) = (u_1 \cdot u_2, v_1 \cdot v_2)$$

(here the operations on the right hand side are the operations of the Boole-De Morgan algebra $BM_2$). We get the Boole-De Morgan algebra $(B \times B; \{+, \cdot, ^{-}, {}', 0, 1\}) = \mathbf{B}\times\mathbf{B}$ (see Section 5), which is isomorphic to the algebra $BM_4$ (the one-to-one correspondence described above is an isomorphism). However, if the ordered pair $(y, z) \in B \times B$ corresponds to $x \in D$ then we will write $x = (y, z)$ (this causes no confusion).

For $x \in D$ let

$$x^* = \begin{cases} x, & \text{if } x = 0, 1, \\ a, & \text{if } x = b, \\ b, & \text{if } x = a. \end{cases}$$

The unary operation $^*$ can also be defined on $B \times B$ taking into account the isomorphism described above. As a result we get $(u, v)^* = (v, u) = (v^*, u^*),\ u, v \in B$. It is clear that $x^* = (\overline{x})' = \overline{x'}$ (which agrees with the notation from the previous section).

The following two concepts of quasi-De Morgan function and De Morgan function are introduced in section 4.

**Definition 7** A function $f : D^n \to D$ is called a *quasi-De Morgan function* if the following conditions hold:

(1) if $x_i \in \{0, 1\},\ i = 1, \ldots, n$, then $f(x_1, \ldots, x_n) \in \{0, 1\}$,

(2) if $x_i \in D,\ i = 1, \ldots, n$, then $f(x_1^*, \ldots, x_n^*) = (f(x_1, \ldots, x_n))^*$.

In terms of clone theory, Condition (1) means that the function $f$ preserves the unary relation $\{0, 1\} \subseteq D$, and Condition (2) means that $f$ preserves the binary relation $\{(0, 0), (a, b), (b, a), (1, 1)\} \subseteq D^2$, which is the graph of the automorphism $x \mapsto x^*$.

**Definition 8** A quasi-De Morgan function $f : D^n \to D$ is called a *De Morgan function* of $n$ variables if it satisfies the following condition:

(3) if $x, y \in D^n$ with $f(x) \neq b$ and $y$ is a permitted modification of $x$ then $f(y) \in \{f(x), a\}$.

In terms of clone theory the Condition (3) means that $f$ preserves the order relation $\rho = \{(b, b), (b, 0), (b, 1), (b, a), (0, 0), (0, a), (1, 1), (1, a), (a, a)\} \subseteq D^2$.

Notice that Condition (1) is a consequence of Condition (2), but however it is more convenient to write it as a separate condition.

Note that it follows from Condition (1) that every quasi-De Morgan function is an extension of some Boolean function. Notice that the constant functions $f = 1$ and $f = 0$ are quasi-De Morgan functions, but the constant functions $f = a$ and $f = b$ are not. This means that 0 and 1 are the only constant quasi-De Morgan functions. Further examples of quasi-De Morgan functions are $f(x) = x,\ g(x) = \overline{x},\ h(x, y) = x \cdot y,\ q(x, y) = x + y,\ p(x) = x'$, where the operations on the right hand side are the operations of the Boole-De Morgan algebra $BM_4$. Also note that the function $p$ is an example of quasi-De Morgan function which is not a De Morgan function.

Below, for $x_i \in D$ we denote by $(y_i, z_i)$ the pair from $B \times B$ which corresponds to $x_i$, i.e., $x_i = (y_i, z_i)$.

A function $f : D^n \to D$ is a quasi-De Morgan function if and only if there exists a Boolean function $\varphi : B^{2n} \to B$ such that

$$f(x_1, \ldots, x_n) = (\varphi(y_1, \ldots, y_n, z_1', \ldots, z_n'), \varphi(z_1, \ldots, z_n, y_1', \ldots, y_n')), \tag{10}$$

for all $x_1, \ldots, x_n \in D$.

Denote the set of all quasi-De Morgan functions of $n$ variables by $\mathcal{BM}_n$. For the functions $f, g : D^n \to D$ define $f + g,\ f \cdot g,\ \overline{f}$ and $f'$ by the standard way, i.e., $(f + g)(x) = f(x) + g(x),\ (f \cdot g)(x) = f(x) \cdot g(x),\ \overline{f}(x) = \overline{f(x)},\ f'(x) = (f(x))',\ x \in D^n$, where the operations on the right hand side are the operations of the Boole-De Morgan algebra $BM_4$.

**Theorem 10** *The set $\mathcal{BM}_n$ is closed under the operations $+, \cdot, ^-, '$, i.e., if $f, g \in \mathcal{BM}_n$, then $f + g, f \cdot g, \overline{f}, f' \in \mathcal{BM}_n$.*

**Proof**: We will use the facts that $(x + y)^* = x^* + y^*,\ (x \cdot y)^* = x^* \cdot y^*,\ \overline{x^*} = (\overline{x})^*,\ (x^*)' = (x')^*$. We have

$$(f + g)(x_1^*, \ldots, x_n^*) = f(x_1^*, \ldots, x_n^*) + g(x_1^*, \ldots, x_n^*) =$$
$$(f(x_1, \ldots, x_n))^* + (g(x_1, \ldots, x_n))^* = ((f + g)(x_1, \ldots, x_n))^*,$$
$$(f \cdot g)(x_1^*, \ldots, x_n^*) = f(x_1^*, \ldots, x_n^*) \cdot g(x_1^*, \ldots, x_n^*) =$$
$$(f(x_1, \ldots, x_n))^* \cdot (g(x_1, \ldots, x_n))^* = ((f \cdot g)(x_1, \ldots, x_n))^*,$$
$$\overline{f}(x_1^*, \ldots, x_n^*) = \overline{f(x_1^*, \ldots, x_n^*)} = \overline{(f(x_1, \ldots, x_n))^*} =$$
$$(\overline{f(x_1, \ldots, x_n)})^* = (\overline{f}(x_1, \ldots, x_n))^*,$$
$$f'(x_1^*, \ldots, x_n^*) = (f(x_1^*, \ldots, x_n^*))' = ((f(x_1, \ldots, x_n))^*)' =$$
$$((f(x_1, \ldots, x_n))')^* = (f'(x_1, \ldots, x_n))^*.$$

These equalities prove the statement of the theorem. $\square$

Thus, we get an algebra: $\mathfrak{BM}_n = (\mathcal{BM}_n, \{+, \cdot, ^-, ', 0, 1\})$ (here 0 and 1 are the constant quasi-De Morgan functions), which obviously is a Boole-De Morgan algebra.

For a set $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ define the function $f_S : D^n \to D$ by the following way:

$$f_S(x_1, \ldots, x_n) = \sum_{(s_1,s_2)\in S} \left( \prod_{i\in s_1} x_i \cdot \prod_{i\in \overline{s}_1} x_i' \cdot \prod_{i\in s_2} \overline{x}_i \cdot \prod_{i\in \overline{s}_2} x_i^* \right), \qquad (11)$$

where the operations on the right hand side are the operations of $BM_4$ (cf. [140, 143, 152]).
Notice that $f_S$ does not depend on the order of the elements in the set $S$.

Also we set $f_\varnothing = 0$ and $f_{\{(\varnothing,\varnothing)\}} = 1$.

Let us consider the projection functions

$$\delta_n^i(x_1, \ldots, x_n) = x_i, \ i = 1, \ldots, n,$$

as functions $D^n \to D$. Obviously, $\delta_n^i$ is a quasi-De Morgan function for each $i$. According to (11), for any set $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ we have

$$f_S = \sum_{(s_1,s_2)\in S} \left( \prod_{i\in s_1} \delta_n^i \cdot \prod_{i\in \overline{s}_1} (\delta_n^i)' \cdot \prod_{i\in s_2} \overline{\delta_n^i} \cdot \prod_{i\in \overline{s}_2} (\delta_n^i)^* \right).$$

Hence, $f_S \in \mathcal{BM}_n$, i.e., $f_S$ is a quasi-De Morgan function for any set $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$.

For $s = (s_1, s_2) \in 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $s' = s_1 \cup \{n+i : i \in s_2\} \in 2^{\{1,\ldots,2n\}}$, and for $S \subseteq 2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}$ let $S' = \{s' : s \in S\} \subseteq 2^{\{1,\ldots,2n\}}$. In this way we give a one-to-one correspondence between the sets $\mathcal{P}\left(2^{\{1,\ldots,n\}} \times 2^{\{1,\ldots,n\}}\right)$ and $\mathcal{P}\left(2^{\{1,\ldots,2n\}}\right)$.

Now, for any quasi-De Morgan function $f \in \mathcal{BM}_n$, we conclude that there exists a set $S' \subseteq 2^{\{1,\ldots,2n\}}$ such that:

$$f(x_1,\ldots,x_n)=(\varphi_f(y_1,\ldots,y_n,z'_1,\ldots,z'_n),\varphi_f(z_1,\ldots,z_n,y'_1,\ldots,y'_n))=$$

$$\left(\sum_{s'\in S'}\left(\prod_{\substack{i\in s'\\1\le i\le n}} y_i\cdot\prod_{\substack{i\in\overline{s'}\\1\le i\le n}} y'_i\cdot\prod_{\substack{i\in s'\\n+1\le i\le 2n}} z'_{i-n}\cdot\prod_{\substack{i\in\overline{s'}\\n+1\le i\le 2n}} z_{i-n}\right),\right.$$

$$\left.\sum_{s'\in S'}\left(\prod_{\substack{i\in s'\\1\le i\le n}} z_i\cdot\prod_{\substack{i\in\overline{s'}\\1\le i\le n}} z'_i\cdot\prod_{\substack{i\in s'\\n+1\le i\le 2n}} y'_{i-n}\cdot\prod_{\substack{i\in\overline{s'}\\n+1\le i\le 2n}} y_{i-n}\right)\right)=$$

$$\sum_{s'\in S'}\left(\prod_{\substack{i\in s'\\1\le i\le n}}(y_i,z_i)\cdot\prod_{\substack{i\in\overline{s'}\\1\le i\le n}}(y_i,z_i)'\cdot\prod_{\substack{i\in s'\\n+1\le i\le 2n}}\overline{(y_{i-n},z_{i-n})}\cdot\prod_{\substack{i\in\overline{s'}\\n+1\le i\le 2n}}(y_{i-n},z_{i-n})^*\right)=$$

$$\sum_{(s_1,s_2)\in S}\left(\prod_{i\in s_1}x_i\cdot\prod_{i\in\overline{s}_1}x'_i\cdot\prod_{i\in s_2}\overline{x}_i\cdot\prod_{i\in\overline{s}_2}x^*_i\right)=f_S(x_1,\ldots,x_n),$$

where $S$ is the subset of $2^{\{1,\ldots,n\}}\times 2^{\{1,\ldots,n\}}$ corresponding to $S'$.

Moreover, the number of all quasi-De Morgan functions of $n$ variables is the same as the number of all subsets of $2^{\{1,\ldots,n\}}\times 2^{\{1,\ldots,n\}}$. Hence, we get the following result.

**Theorem 11** *For any quasi-De Morgan function $f$ of $n$ variables there exists a unique set $S\subseteq 2^{\{1,\ldots,n\}}\times 2^{\{1,\ldots,n\}}$ such that $f=f_S$.*

In particular, $f_{S_1}\neq f_{S_2}$ if $S_1\neq S_2$.

Thus, every quasi-De Morgan function can be uniquely presented in the form (11). This form is called the disjunctive normal form (or briefly - DNF) of quasi-De Morgan function $f$. Notice that from the definition 5 and Theorem 11 we get an algorithm which, given a quasi-De Morgan function, gives its disjunctive normal form. We can also prove that every quasi-De Morgan function can be uniquely presented in conjunctive normal form (CNF), i.e., in the following form:

$$\prod_{(s_1,s_2)\in S}\left(\sum_{i\in s_1}x_i+\sum_{i\in\overline{s}_1}x'_i+\sum_{i\in s_2}\overline{x}_i+\sum_{i\in\overline{s}_2}x^*_i\right).$$

Below we will use the concept of essential variable (and essential dependence) of quasi-De Morgan functions. The definitions are the same as in case of Boolean functions and so we do not give them here.

Using arguments similar to those given above we can prove the following theorem.

**Theorem 12** *For a quasi-De Morgan function $f$ the corresponding Boolean function $\varphi_f$ does not essentially depend on the last $n$ variables if and only if $f$ can be represented as a term function with functional symbols $+, \cdot, '$, i.e., $f$ is a term function of the Boolean algebra $(D; \{+, \cdot, ', 0, 1\})$.*

Now we can formulate the following functional representation theorem for finitely-generated free Boole-De Morgan algebras, which relates to the Plotkin's problem.

**Theorem 13** (Functional representation theorem)([157]) *The algebra $\mathfrak{BM}_n$ is a free Boole-De Morgan algebra with the system of free generators $\Delta = \{\delta_n^1, \ldots, \delta_n^n\}$. Hence, every free $n$-generated Boole-De Morgan algebra is isomorphic to the Boole-De Morgan algebra $\mathfrak{BM}_n$.*

**Problem 2** *To develop the quasi-De Morgan analogue of the theory of Boolean functions.*

# 7 Bigroups. Bigroups of binary operations

By analogy of bilattices [67, 72, 73, 74, 75, 210, 176] we introduce the concepts of a bisemigroup, a bimonoid, a De Morgan bisemigroup and a bigroup.

Let $Q$ be an arbitrary non-empty set; let $O_p^{(n)}Q$ be a set of all $n$-ary operations on $Q$, and

$$O_pQ = \bigcup_n O_p^{(n)}Q;$$

For every non-empty subset $\Sigma \subseteq O_pQ$, the pair $(Q; \Sigma)$ is called an algebra.

A bisemigroup is an algebra $Q(\cdot, \circ)$ equipped with two binary associative operations $\cdot$ and $\circ$. If both of these operations have an identity element, then the bisemigroup is called a bimonoid. A commutative bisemigroup is a bisemigroup in which both operations are commutative. A bisemilattice is a commutative bisemigroup in which both operations are idempotent. In any bisemilattice $Q(\cdot, \circ)$, the binary operations determine two partial orders $\leqslant_1$ and $\leqslant_2$. A bisemilattice is called a bilattice, if the partial orders $\leqslant_1$ and $\leqslant_2$ are lattice orders. Since every lattice order is characterized by two binary operations, every bilattice is a binary algebra with four operations and corresponding identities. A De Morgan bisemigroup ([38, 179]) is an algebra $Q(\cdot, \circ, ^-, 0, 1)$ such that $Q(\cdot, \circ)$ is a bimonoid with identity elements 0 (for operation $\cdot$), 1 (for operation $\circ$) and such that the identities

$$\overline{\overline{x}} = x,$$
$$\overline{x \cdot y} = \overline{x} \circ \overline{y},$$
$$\overline{x \circ y} = \overline{x} \cdot \overline{y},$$
$$0 \circ x = 0,$$

$1 \cdot x = 1$

hold. A De Morgan bisemigroup $Q(\cdot, \circ, ^-, 0, 1)$ is a De Morgan algebra if $Q(\cdot, \circ)$ is a distributive lattice.

A bimonoid $Q(\cdot, \circ)$ with identity elements 0 (for the operation $\cdot$) and 1 (for the operation $\circ$) is called a bigroup, if for every $x \in Q$

$0 \circ x = 0,$

$1 \cdot x = 1,$

and the following conditions are valid:

a) $Q \setminus \{1\}$ is a group with an identity element 0 under the multiplication $\cdot$ ;

b) $Q \setminus \{0\}$ is a group with an identity element 1 under the multiplication $\circ$ ;

A bigroup of order $> 3$ is called non-trivial.

A bimonoid $Q(\cdot, \circ)$ with identity elements 0 (for operation $\cdot$) and 1 (for operation $\circ$) is called a half-bigroup, if for every $x \in Q$:

$0 \circ x = 0,$

$1 \cdot x = 1,$

and at least one of the two conditions a), b) is valid.

An element $a \in Q$ is called invertible in a bimonoid $Q(\cdot, \circ)$, if $a$ is invertible in both semigroups $Q(\cdot)$ and $Q(\circ)$. We denote the inverse element of $a$ in the semigroup $Q(\cdot)$ by $a^{-1}$, and that in the semigroup $Q(\circ)$ we denote by ${}^{-1}a$. An invertible element of bimonoid $Q(\cdot, \circ)$ is also called an invertible element of De Morgan bisemigroup $Q(\cdot, \circ, ^-, 0, 1)$.

The set $O_p^{(2)}Q$ of all binary operations on $Q$ is a bimonoid under the following operations:

$$f \cdot g(x, y) = f(x, g(x, y)), \tag{12}$$

$$f \circ g(x, y) = f(g(x, y), y), \tag{13}$$

in which the identity elements are the identical operations $\delta_2^2$ and $\delta_2^1$, where $\delta_2^1(x, y) = x$, and $\delta_2^2(x, y) = y$ for all $x, y \in Q$. Sometimes we denote: $\delta_2^2 = E$ and $\delta_2^1 = F$.

The Mal'tsev-type characterization of bisemigroup $O_p^{(2)}Q$ is considered in [266]. For applications of the mentioned binary superpositions see [18, 22, 23, 32, 71, 121, 138, 144, 148, 172, 173, 174, 184, 188, 202, 240, 256].

Any subset $S \subseteq O_p^{(2)}Q$ which is closed under these two operations is called a bisemigroup of the operations (on the set $Q$). The bisemigroup of operations (on the set $Q$) is called a bimonoid of operations (on the set $Q$) if it contains the identical operations $\delta_2^1$ and $\delta_2^2$. Besides, the dual operation $f^*(x, y) = f(y, x)$ defines the unary operation $^- : f \to f^*$, which is an antiautomorphism of the bimonoid $O_p^{(2)}Q$, since

$$(f \cdot g)^* = f^* \circ g^*,$$

$$(f \circ g)^* = f^* \cdot g^*.$$

So the set $O_p^{(2)}Q$ of all binary operations on the $Q$ is a De Morgan bisemigroup. Every subalgebra of a De Morgan bisemigroup $O_p^{(2)}Q$ is called a De Morgan bisemigroup of operations (on the set $Q$).

The bimonoid $S$ of operations (on the set $Q$) is a bigroup, if both of the following conditions are valid (cf. [18]):

c) $S\backslash\{\delta_2^1\}$ is a group with an identity element $\delta_2^2$ under the multiplication (12) ;

d) $S\backslash\{\delta_2^2\}$ is a group with an identity element $\delta_2^1$ under the multiplication (13) ;

This bigroup is called a bigroup of operations (on the set $Q$).

**Example.** Let $Q(+,\cdot)$ be a field and for every $a \in Q$

$$A_a(x,y) = (1-a)x + ay.$$

If $\Sigma = \{A_a | \, a \in Q\}$ then the algebra $(Q;\Sigma)$ is a bigroup of operation (on the set $Q$).

The bimonoid $S$ of operations (on the set $Q$) is a half-bigroup, if at least one of the two conditions c), d) is valid. This half-bigroup is called a half-bigroup of operations (on the set $Q$).

The binary operation $A$ on $Q$ is called a left-quasigroup operation, if the equation $A(x,\, a) = b$ has a unique solution $x \in Q$ for any $a, b \in Q$. If $A$ is a left-quasigroup operation on $Q$, then $Q(A)$ is called a left-quasigroup. The definitions of a right-quasigroup operation and a right-quasigroup are analogous. A left-quasigroup with a left identity element is called a left-loop. A right-quasigroup with a right identity element is called a right-loop. The binary operation $A$ on $Q$ is called a quasigroup operation, if it is a left and a right-quasigroup operation simultaneously. If $A$ is a quasigroup operation on $Q$, then $Q(A)$ is called a quasigroup ([8, 9, 20, 120, 137, 222, 236]). A quasigroup with an identity element is called a loop.

On applications of right (left) quasigroups in geometry and topology (knot theory) see [185, 124].

The operation $A \in O_p^{(2)}Q$ is called right invertible, if it is invertible in the semigroup $O_p^{(2)}Q(\cdot)$; this inverse is denoted by $A^{-1}$, which is called the right inverse of $A$. The operation $A \in O_p^{(2)}Q$ is called left invertible, if it is invertible in the semigroup $O_p^{(2)}Q(\circ)$; this inverse is denoted by ${}^{-1}A$, which is called the left inverse of $A$.

The following propositions are consequences of the above definitions.

**Proposition 3 ([148])** *The binary operation $A$ on $Q$ is a right-quasigroup operation if and only if $A$ is right invertible. The binary operation $A$ on $Q$ is a left-quasigroup operation if and only if $A$ is left invertible. A groupoid $Q(A)$ is quasigroup if and only if the operation $A \in O_p^{(2)}Q$ is right and left invertible simultaneously. Hence, in a bigroup $S$ of operations, any operation $A \in S, A \neq \delta_2^1, \delta_2^2$ is a quasigroup operation.*

**Proposition 4** *1) If the operation $A \in O_p^{(2)}Q$ is right invertible, then*

$$A(x, A^{-1}(x, y)) = y, \quad A^{-1}(x, A(x, y)) = y$$

*for every $x, y \in Q$.*

*2) If the operation $A \in O_p^{(2)}Q$ is left invertible, then*

$$A({}^{-1}A(y, x), x) = y, \quad {}^{-1}A(A(y, x), x) = y$$

*for every $x, y \in Q$.*

**Proposition 5** *1) For every right invertible operation $A \in O_p^{(2)}Q$ and for every operations $B, C \in O_p^{(2)}Q$*

$$C(A, B) = (C^* \circ (B \cdot A^{-1})) \cdot A,$$

*where $C(A, B)(x, y) = C(A(x, y), B(x, y))$ for every $x, y \in Q$;*

*2) For every left invertible operation $A \in O_p^{(2)}Q$ and for every operations $B, C \in O_p^{(2)}Q$*

$$C(A, B) = (C \cdot (B \circ^{-1} A)) \circ A;$$

*3) For every right invertible operation $B \in O_p^{(2)}Q$ and for every operations $A, C \in O_p^{(2)}Q$*

$$C(A, B) = (C \circ (A \cdot B^{-1})) \cdot B;$$

*4) For every left invertible operation $B \in O_p^{(2)}Q$ and for every operations $A, C \in O_p^{(2)}Q$*

$$C(A, B) = (C^* \cdot (A \circ^{-1} B)) \circ B.$$

***Proof.*** 1) We have

$$(C^* \circ (B \cdot A^{-1})) \cdot A(x, y) = C^* \circ (B \cdot A^{-1})(x, A(x, y)) =$$

$$= C^*(B \cdot A^{-1}(x, A(x, y)), A(x, y)) = C^*(B(x, A^{-1}(x, A(x, y))), A(x, y)) =$$
$$= C^*(B(x, y), A(x, y)) = C(A(x, y), B(x, y)) = C(A, B)(x, y).$$

2) We have

$$(C \cdot (B \circ^{-1} A)) \circ A(x, y) = C \cdot (B \circ^{-1} A)(A(x, y), y) =$$

$$= C(A(x, y), B \circ^{-1} A(A(x, y), y)) = C(A(x, y), B({}^{-1}A(A(x, y), y), y)) =$$
$$= C(A(x, y), B(x, y)) = C(A, B)(x, y).$$

The proof of 3) and 4) are analogous. □

**Lemma 1** *If $A$ and $B$ are left-quasigroup operations on $Q$, then $A \circ B$ is a left-quasigroup operation on $Q$. If $A$ and $B$ are right-quasigroup operations on $Q$, then $A \cdot B$ is a right-quasigroup operation on $Q$. Hence, the set of all left-quasigroup operations on $Q$ is a subgroup of the semigroup $O_p^{(2)}Q(\circ)$, and the set of all right-quasigroup operations on $Q$ is a subgroup of the semigroup $O_p^{(2)}Q(\cdot)$.*

**Example.** If $Q = \{1, 2\}$, there are four right-quasigroup operations on $Q$ formed a Klein four-group under multiplication (12).

**Lemma 2** *Every bigroup of operations is a De Morgan bisemigroup of operations.*

Two binary operations $A, B$ on $Q$ are called orthogonal ([66, 41]), if the system of equations

$$A(x,\, y) = a,$$

$$B(x,\, y) = b$$

has a unique solution $(x,\, y) \in Q^2$ for any $a, b \in Q$. In this case we also say that the groupoids $Q(A)$ and $Q(B)$ are orthogonal.

It is obvious that two commutative operations defined on the same non-trivial set can not be orthogonal. In particular, two non-trivial commutative quasigroups can not be orthogonal. Two non-trivial groupoids with identity elements can not be orthogonal. In particular, two non-trivial loops defined on the same set cannot be orthogonal.

The following lemma is a generalization of the corresponding result in [121] and [240].

**Lemma 3** *If $A$ is a left-quasigroup operation on $Q$ and $B$ is a right-quasigroup operation on $Q$, then operations $B$ and $A \cdot B$ are orthogonal. In particular the quasigroup operations $A$ and $A \cdot A$ are orthogonal. Hence, if $Q(A)$ is a loop, then $Q(A \cdot A)$ can not be a loop. If $A$ is a right-quasigroup operation on $Q$ and $B$ is a left-quasigroup operation on $Q$, then the operations $B$ and $A \circ B$ are orthogonal. In particular, the quasigroup operations $A$ and $A \circ A$ are orthogonal. Hence, if $Q(A)$ is a loop, then $Q(A \circ A)$ can not be a loop.*

**Lemma 4** *In a half-bigroup $S$ of operations (on a set $Q$) every operation $A \in S$ is orthogonal to any quasigroup operation $B \in S$. In particular, any two quasigroup operations in a half-bigroup of operations (on the set $Q$) are orthogonal. Hence, any two operations in a bigroup of operations (on a set $Q$) are orthogonal.*

**Theorem 14** *A half-bigroup of operations $S(\cdot, \circ)$ (on a set $Q$) has a commutative quasigroup operation if and only if it has a quasigroup operation $H \in S$ with condition $H \cdot H = E$ or $H \circ H = F$. In particular, a finite half-bigroup of operations (on a set $Q$) with a commutative quasigroup operation has an odd order.*

*Proof.* Let $S(\cdot, \circ)$ be a half-bigroup of operations, and let $S \setminus \{\delta_2^1\}$ be a group with an identity element $\delta_2^2$ under the multiplication (12). If $A$ is a commutative quasigroup operation, $A \in S$ and $H =^{-1} A$, then $H \cdot H = E$:

$$A^{-1} = (A^*)^{-1} = (^{-1}A)^* = ((^{-1}A \cdot^{-1} A) \circ A) \cdot A^{-1},$$

$$(^{-1}A \cdot^{-1} A) \circ A = E,$$

$$^{-1}A \cdot^{-1} A = E,$$

where $B^*(x, y) = B(y, x)$ for all $x, y \in Q$.

If $S \setminus \{\delta_2^2\}$ is a group with an identity element $\delta_2^1$ under the multiplication (13), $A \in S$ is a commutative quasigroup operation, and $H = A^{-1}$, then $H \circ H = F$. The proof of this fact is similar. The remaining part of the proof is clear.

A bimonoid $S$ of operations (on a set $Q$) is called a (non-trivial) local bigroup if for every quasigroup operation $A \in S$ there exists some (non-trivial) bigroup of operations (on set $Q$), which includes $A$.

**Proposition 6** *If $A$ and $B$ are idempotent binary operations on $Q$, then $A^*$, $A \cdot B$ and $A \circ B$ are also idempotent operations. So the set of all idempotent binary operations on $Q$ is a De Morgan bisemigroup of operations on $Q$. For every quasigroup operation $A$ the following operations are idempotent*

$$(A \cdot A) \circ^{-1} A,$$

$$^{-1}A \cdot A^{-1},$$

$$(A \circ A) \cdot A^{-1},$$

$$A \cdot (A^{-1} \circ^{-1} A)^* \cdot A,$$

$$A^{-1} \circ^{-1} A,$$

$$A \circ (^{-1}A \cdot A^{-1})^* \circ A.$$

**Example.** If $Q = \{1, 2\}$, then the De Morgan bisemigroup of idempotent operations (on the set $Q$) is a Boole-De Morgan algebra of order 4.

If $S$ is a bigroup of binary operations (on the set $Q$) and $A \in S$, then $A^* \in S$, where $A^*(x,y) = A(y,x)$ for every $x, y \in Q$. Indeed, if $A$ is a quasigroup operation, then

$$({}^{-1}(A^{-1}))^{-1} = {}^{-1}(({}^{-1}A)^{-1}) = A^*.$$

This fact follows from the following result too.

**Lemma 5** *For any operation $B \in O_p^{(2)}Q$ and for any quasigroup operation $C \in O_p^{(2)}Q$ we have*

$$B^* = ((B \circ C^{-1}) \cdot C) \circ^{-1} C,$$

$$B^* = ((B \cdot^{-1} C) \circ C) \cdot C^{-1}.$$

***Proof.*** Indeed,

$$(((B \circ C^{-1}) \cdot C) \circ^{-1} C)(x,y) = (B \circ C^{-1}) \cdot C({}^{-1}C(x,y), y) =$$

$$= B \circ C^{-1}({}^{-1}C(x,y), C({}^{-1}C(x,y), y))) =$$

$$= B \circ C^{-1}({}^{-1}C(x,y), x) = B(C^{-1}({}^{-1}C(x,y), x), x) = B(y,x) = B^*(x,y),$$

since

$$C^{-1}({}^{-1}C(x,y), x) = y,$$

$$C({}^{-1}C(x,y), y) = x.$$

The second equality is proved similarly. □

**Theorem 15** *Every quasigroup operation $A$ of a non-trivial bigroup $S$ of binary operations (on the set $Q$) is idempotent ($|S| > 3$).*

***Proof.*** Let $S_0$ be the subset of binary idempotent operation of the set $S$. We have: $E, F \in S_0$. Besides, it is evident, that $S_0 \setminus \{F\}$ is a subgroup of the group $S \setminus \{F\}$ under the right multiplication of binary operations, and $S_0 \setminus \{E\}$ is a subgroup of the group $S \setminus \{E\}$ under the left multiplication of binary operations. According to Proposition 6, the operations

$$S_A = (A \circ A) \cdot A^{-1}, \quad T_A =^{-1} A \cdot A^{-1}$$

are idempotent. Besides, $T_A \neq F$, since from the condition ${}^{-1}A \cdot A^{-1} = F$ the equalities: ${}^{-1}A = F$ and $A = F$ follow, which is a contradiction, since $A$ is a quasigroup operation. According to Proposition 6, $U_A = A \circ T_A^* \circ A \in S_0$. Let us consider the operation $V_A = A \circ T_A^*$. If $V_A = F$, then $A =^{-1} (T_A^*) \in S_0$. If $V_A \neq F$, then

$$V_A \circ V_A = A \circ T_A^* \circ A \circ T_A^* = U_A \circ T_A^* \in S_0.$$

On the other hand:

$$S_{V_A} \cdot V_A = (V_A \circ V_A) \cdot V_A^{-1} \cdot V_A = V_A \circ V_A.$$

Let us consider the following two cases:

1) The group $S \setminus \{E\}$ has not a second order element. In this case we have: $S_{V_A} \neq F$, since if $S_{V_A} = F$, then

$$F \cdot V_A = V_A \circ V_A,$$

$$F = V_A \circ V_A,$$

and $V_A$ has the second order. Hence, there exists $S_{V_A}^{-1}$ and since $V_A \neq F$, there exists $V_A^{-1}$ too. We have

$$S_{V_A} = (V_A \circ V_A) \cdot V_A^{-1} \in S_0.$$

Indeed,

$$S_{V_A}(x,x) = (V_A \circ V_A) \cdot V_A^{-1}(x,x) = V_A \circ V_A(x, V_A^{-1}(x,x)) =$$

$$= V_A(V_A(x, V_A^{-1}(x,x)), V_A^{-1}(x,x)) = V_A(x, V_A^{-1}(x,x)) = x.$$

Now,

$$V_A = S_{V_A}^{-1} \cdot (V_A \circ V_A) \in S_0,$$

and

$$A \circ T_A^* \in S_0.$$

Further, $T_A \neq F$ and $T_A^* \neq E$. Hence, there exists ${}^{-1}(T_A^*)$, therefore

$$(A \circ T_A^*) \circ^{-1} (T_A^*) \in S_0,$$

and $A \in S_0$;

2) In the group $S \setminus \{E\}$ there exists a second order element (quasigroup operation), moreover this operation is unique. Indeed, if $C \circ C = F$ and $D \circ D = F$, then the operations $C^{-1}$ $D^{-1}$ commute, which contradicts the orthogonality of operations $C^{-1}$ $D^{-1}$. In fact, if $H = C^{-1}$, then: $H^{-1} \circ H^{-1} = (C^{-1})^{-1} \circ (C^{-1})^{-1} = C \circ C = F$ and $(H^{-1} \circ H^{-1}) \cdot H = F \cdot H = F$. From here:

$${}^{-1}H = ((H^{-1} \circ H^{-1}) \cdot H) \circ^{-1} H$$

and according to Lemma 5, ${}^{-1}H = (H^{-1})^*$ or ${}^{-1}H = {}^{-1}(H^*)$, and $H = H^*$. Hence, the operation $H = C^{-1}$ is a commutative quasigroup.

Let $C$ be a second order quasigroup operation in the group $S \setminus \{E\}$. The operation $C$ is contained in the center of the group $S \setminus \{E\}$, since the element $X \circ C \circ X^{-1}$ also has the second order and therefore $X \circ C \circ X^{-1} = C$. Hence, $X \circ C = C \circ X$ for every $X \in S \setminus \{E\}$.

If $S_{V_A} \neq F$, then, as above, we obtain $A \in S_0$.

Let $A$ be an operation such that $S_{V_A} = F$. Then $V_A \circ V_A = F$, and $V_A = C$, according to uniqueness of the operation $C$. On the other hand we have

$$F = V_A \circ V_A = V_A \circ A \circ T_A^* = A \circ V_A \circ T_A^* = A \circ A \circ T_A^* \circ T_A^*.$$

Hence,

$$A \circ A =^{-1} (T_A^*) \circ^{-1} (T_A^*) \in S_0.$$

If $A \neq C$, then $A \circ A \neq F$ and

$$A^{-1} = (A \circ A)^{-1} \cdot S_A \in S_0.$$

Thus: $A^{-1} \in S_0$ and $A \in S_0$.

Now we only need to prove that the operation $C$ is idempotent, too.

We can assume that $|S| \geq 5$, i.e. the bigroup $S$ contains at least three quasigroup operations. Hence, there exists a quasigroup operation $C_1 \in S$, $C_1 \neq C$, such that the operation $C_2 = C_1 \circ C$ differs from $C$ and $C_1$. Since $C_1$ and $C_2$ differ from $C$, then, as it was proved above, they are idempotent. Therefore, $C =^{-1} C_1 \circ C_2 \in S_0$, i.e. the operation $C$ is idempotent, too. □

For finite $Q$ the result follows from [18].

Now we characterize a bigroup of operations (on the set $Q$) through special algebras introduced by G.Grätzer [80] for characterization of minimal doubly transitive groups of permutations. Such algebras are called Grätzer algebras. The group $G$ of permutations of the set $Q$ is called minimally double transitive, if for every $(a, b)$ and $(c, d)$, where $a \neq b$ and $c \neq d$, there exists a unique permutation $\alpha \in G$ such that $\alpha(a) = c$ and $\alpha(b) = d$, where $a, b, c, d \in Q$ [125, 126, 86, 92, 95, 97, 110, 250, 251, 252, 261, 269, 274]. We use here the concept of Grätzer algebra ($G$-field) for characterization of $A(x, y)$, in which $x \neq y$ and $A$ is a quasigroup operation in the bigroup of operations.

The algebra $Q(-, \cdot)$ with two binary operations is called a (left) Grätzer algebra or (left) $G$-field ([80], [19, 69]), if the following conditions are valid:

G1) $Q(\cdot)$ is a semigroup with the zero element 0, i.e. $x \cdot 0 = 0 \cdot x = 0$ for any $x \in Q$;

G2) $|Q| \geq 2$ and $Q'(\cdot)$ is a group, where $Q' = Q \backslash \{0\}$ (the identity element of this group is denoted by 1);

G3) $x - 0 = x$ for every element $x \in Q$;

G4) $x(y - z) = xy - xz$ for every elements $x, y, z \in Q$;

G5) $x - (x - y) = y, \quad x - (y - z) = (x - y) - (x - y)(y - x)^{-1} z, \; x \neq y,$ for every $x, y, z \in Q$.

Here $a^{-1}$ is the inverse element of $a \neq 0$ in the group $Q'(\cdot)$. Note that $x - y = 0$ if and only if $x = y$. Indeed: $x - x = x - (x - 0) = 0$. And conversely, from the equality $x - y = 0$ it follows

$$y = x - (x - y) = x - 0 = x.$$

Besides, $0 \neq 1$ (since if $0 = 1$ then $x = x \cdot 1 = x \cdot 0 = 0$, which contradicts to the condition: $|Q| \geq 2$).

Let us denote $-x = 0 - x$. We obtain: $-(-x) = x$ and $x(-y) = -(xy)$ for every $x, y \in Q$.

**Proposition 7** *In the Grätzer algebra $Q(-, \cdot)$ the groupoid $Q(-)$ is a right quasigroup, i,e. the equation $a - x = b$ has a unique solution $x \in Q$ for every $a, b \in Q$.*

***Proof.*** It is evident that $x = a - b$ is a solution of the equation $a - x = b$. Indeed, as according to the definition of Grätzer algebra we have: $a-(a-b) = b$. If $a - x_1 = b$ and $a - x_2 = b$, then $a - x_1 = a - x_2$ and

$$x_1 = a - (a - x_1) = a - (a - x_2) = x_2.$$

□

If $Q(-)$ is a quasigroup, then the Grätzer algebra $Q(-, \cdot)$ is called a Grätzer $q$-algebra, and $Q(-)$ is called a Grätzer quasigroup.

The algebra $Q(+, \cdot)$ with two binary operations is called (left) near-field ([85]), if the following conditions are valid:

NF1) $Q(+)$ is a non-trivial abelian group with an identity element 0;

NF2) $Q(\cdot)$ is a semigroup and $Q' = Q \backslash \{0\}$ is a group under multiplication $\cdot$ with the identity element $1 \in Q$;

NF3) In $Q(+, \cdot)$ the identity of left distributivity is valid.

Near-fields were first considered in 1905 by Dickson ([64]). A complete classification of finite near-fields was obtained in 1936 by Zassenhaus ([268]). In particular, they have an order $p^n$, where $p$ is a prime.

**Examples.** 1) If we define the following multiplication on the Abelian group $Z_2(+)$:

$$0 \cdot 0 = 1 \cdot 0 = 0, 0 \cdot 1 = 1 \cdot 1 = 1,$$

we obtain a near-field, which is not a field. If a near-field contains at least 3 elements, then $x \cdot 0 = 0 \cdot x = 0$ for any $x$. Finite near-fields of order 3, 4, 5 are fields.

2) If the near-field $Q(+, \cdot)$ contains at least 3 elements, then $Q(-, \cdot)$ is a Grätzer q-algebra, where $x - y = x + (-y)$. The obtained Grätzer $q$-algebra $Q(-, \cdot)$ is called a derivative of the near-field $Q(+, \cdot)$. There exists a Grätzer algebra of order 3, which is not a Grätzer $q$-algebra. For example,

if $Q = \{0, 1, 2\}$, $Q(\cdot)$ is a semigroup with the zero element 0, $Q' = Q - \{0\}$ is a two-element group under multiplication $\cdot$, with identity element 1, and $1 - 2 = 2, 2 - 1 = 1, x - 0 = 0 - x = x, x - x = 0$, then $Q(-)$ is a right but not a left quasigroup and $Q(-, \cdot)$ is a Grätzer algebra. Consequently, there exists a finite Grätzer algebra which is not a derivative of a finite near-field. However, the multiplicative groups of finite Grätzer algebras and finite near-fields are the same (see [151]).

3) If we define the following new product on the finite field $GF(3^2)$:

$$x \circ y = xy^3,$$

if x is not a square, then

$$x \circ y = xy.$$

Otherwise, we obtain a near-field whose multiplicative group is the quaternion group.

**Lemma 6** *Every bigroup $S$ of binary operations (on the set $Q$) forms a Grätzer algebra under the following operations:*

$$A \odot B = B \cdot A,$$

$$A - B = \begin{cases} (B \cdot A^{-1})^* \cdot A, & \text{if } A \neq F, \\ B, & \text{if } A = F, \end{cases}$$

*where $A, B \in S$.*

***Proof.*** The axioms of $G1$ and $G2$ are evident. Let us prove the condition $G3$: if $A = F$, then $A - F = F$. If $A \neq F$, then

$$A - F = (F \cdot A^{-1})^* \cdot A = F^* \cdot A = E \cdot A = A.$$

Let us check $G4$:

$$A \odot (B - C) = (A \odot B) - (A \odot C).$$

If $A = F$, then the equation is valid. Let $A \neq F$. Then,

$$A \odot (B - C) = (B - C) \cdot A = \begin{cases} C \cdot A, & \text{if } B = F, \\ (C \cdot B^{-1})^* BA, & \text{if } B \neq F, \end{cases}$$

$$(A \odot B) - (A \odot C) = BA - CA =$$

$$= \begin{cases} F - CA = CA, & \text{if } B = F, \\ (CA(BA)^{-1})^* BA = (CB^{-1})^* BA, & \text{if } B \neq F. \end{cases}$$

Now we can check the first identity of $G5$:

$$F - (F - C) = F - C = C.$$

If $A \neq F$, then

$$A - (A - C) = A - (CA^{-1})^* A = ((CA^{-1})^* A \cdot A^{-1})^* A =$$

$$= ((CA^{-1})^*)^* A = CA^{-1} \cdot A = C \cdot A^{-1} A = C \cdot E = C.$$

Now about the validity of the second identity of $G5$. Denote:

$$L_1 = A - (B - C),$$

$$L_2 = (A - B) - (A - B) \odot (B - A)^{-1} \odot C,$$

and consider the following three cases:
1) $A = F$ ( and $B \neq F$),
2) $B = F$ ( and $A \neq F$),
3) $A \neq F$, $B \neq F$ (and $A \neq B$).

In the first case we have:

$$L_1 = L_2 = B - C;$$

In the second case:

$$L_1 = L_2 = A - C;$$

For the third case we find:

$$L_1 = A - (CB^{-1})^* B = ((CB^{-1})^* BA^{-1})^* A,$$

$$L_2 = (A - B) - C(B - A)^{-1}(A - B).$$

Denote:

$$T = (B - A)^{-1}(A - B)$$

we obtain:

$$T = ((AB^{-1})^* B)^{-1}(BA^{-1})^* A = B^{-1}((AB^{-1})^*)^{-1}(BA^{-1})^* A =$$

$$= B^{-1} U (BA^{-1})^* A,$$

where

$$U = ((AB^{-1})^*)^{-1}.$$

Using the equality $(X \circ Y)^* = X^* \cdot Y^*$, we find:

$$L_2 = (A - B) - CT = (BA^{-1})^* A - CT = (CT((BA^{-1})^* A)^{-1})^* (BA^{-1})^* A =$$

$$= (CB^{-1} U (BA^{-1})^* A((BA^{-1})^* \cdot A)^{-1})^* (BA^{-1})^* A =$$

$$= (CB^{-1}U)^*(BA^{-1})^*A = ((CB^{-1}U) \circ (BA^{-1}))^*A.$$

On the other hand, if $D$ is a quasigroup operation, then

$$D^* = ({}^{-1}(D^{-1}))^{-1};$$

Further:

$$AB^{-1} = (U^{-1})^*, \quad BA^{-1} = ((U^{-1})^*)^{-1} = ^{-1} U.$$

Therefore:

$$L_2 = ((CB^{-1} \cdot U) \circ^{-1} U)^*A.$$

However, according to Lemma 5:

$$W({}^{-1}V) \circ V = W^*V,$$

and by $W = CB^{-1}$, $V = ^{-1} U$, we find:

$$(CB^{-1} \cdot U) \circ^{-1} U = (CB^{-1})^*({}^{-1}U).$$

Hence,

$$L_2 = ((CB^{-1})^*({}^{-1}U))^*A = ((CB^{-1})^*BA^{-1})^*A = L_1.$$

□

We call the following result a local characterization of the bigroups of binary operations (cf.[146]).

**Theorem 16** *Let $S$ be a non-trivial bigroup of operations (on set $Q$). For every two different elements $a, b \in Q$ there exists a Grätzer algebra $H_{a,b}(-, \cdot)$, $H_{a,b} \subseteq Q$, such that $a, b \in H_{a,b}$ and for every $x, y \in H_{a,b}$, for every operation $C \in S$:*

$$C(x, y) = x - (x - y)c,$$

*where $c \in H_{a,b}$. Moreover the Grätzer algebras $H_{a,b}(-, \cdot)$ and $H_{c,d}(-, \cdot)$ are isomorphic for every $a, b, c, d \in Q$, where $a \neq b$ $c \neq d$.*

***Proof.*** Let us define:

$$H_{a,b} = \{A(a, b) | A \in S\} \subseteq Q.$$

From Proposition 5 it follows that $C(A, B) \in S$ for every $A, B, C \in S$, where

$$C(A, B)(u, v) = C(A(u, v), B(u, v)).$$

Therefore, the subset $H_{a,b}$ is a subalgebra of the algebra $(Q; S)$. It is evident, $F(a, b) = a \in H_{a,b}$ and $E(a, b) = b \in H_{a,b}$.

For every element $d \in H_{a,b}$ there exists a unique operation $A \in S$ with the equality: $d = A(a, b)$. Indeed, let $d \neq a$ and $d = B(a, b)$, where $B \in S$

and $B \neq A$. Since $B \neq F$, there exists $B^{-1}$. Consider the operation: $U = B^{-1} \cdot A \in S$. Then:

$$U(a,b) = B^{-1} \cdot A(a,b) = B^{-1}(a, A(a,b)) = B^{-1}(a,d) = b.$$

According to the above theorem, the operation $C \in S$ is an idempotent. We have:

$$\begin{cases} U(a,b) = b, \\ E(a,b) = b, \end{cases} \qquad \begin{cases} U(b,b) = b, \\ E(b,b) = b, \end{cases}$$

that is the system of equations

$$\begin{cases} U(x,y) = b, \\ E(x,y) = b, \end{cases}$$

has two solutions: $(a,b)$ and $(b,b)$, which contradicts the orthogonality of operations $U$ and $E$. Now we have to consider the following case only: $d = a$. In this case $F(a,b) = a$ and if $A(a,b) = a$, then we obtain a contradiction, according to the idempotency of the operation $A \in S$.

Thus, there exists a biective mapping $\varphi$ between the sets $S$ and $H_{a,b}$ for every $a, b \in Q$, where $a \neq b$. Namely: $\varphi(A) = x$, where $x = A(a,b)$.

Let us define the following two operations on the set $H_{a,b}$:

$$x \cdot y = B \cdot A(a,b),$$

$$x - y = \begin{cases} (B \cdot A^{-1})^* A(a,b), & \text{if } x \neq a, \\ y, & \text{if } x = a, \end{cases}$$

where $x = A(a,b)$, $y = B(a,b)$, i.e. $x = \varphi A$, $y = \varphi B$. According to Lemma 6, $S(-, \odot)$ is a bigroup. Moreover:

$$\varphi(A \odot B) = \varphi(B \cdot A) = x \cdot y = \varphi(A) \cdot \varphi(B),$$

$$\varphi(A - B) = x - y = \varphi(A) - \varphi(B),$$

that is the mapping $\varphi : S \to H_{a,b}$ is an isomorphism from Grätzer algebra $S(-, \odot)$ to the algebra $H_{a,b}(-, \cdot)$. Hence, the algebra $H_{a,b}(-, \cdot)$ is also a Grätzer algebra.

Evidently, $a = \varphi(F)$, $b = \varphi(E)$; If $d = \varphi(A)$, then $d^{-1} = \varphi(A^{-1})$ $b - d = \varphi(A^*)$:

$$b - d = (A \cdot E^{-1})^* \cdot E(a,b) = A^*(a,b).$$

Now we have only to prove that:

$$C(x,y) = x - (x - y)c,$$

where $x = A(a,b)$, $y = B(a,b)$, $c = C(a,b)$, $C, B, A \in S$. We have:

$$C(x,y) = C(A(a,b), B(a,b)) = C(A,B)(a,b).$$

If $x = a$, then:

$$C(a,y) = C(a, B(a,b)) = C \cdot B(a,b) = y \cdot c,$$

i.e.

$$C(a,y) = y \cdot c = a - (a-y)c.$$

If $x \neq a$, then:

$$x - y = (B \cdot A^{-1})^* A(a,b),$$
$$(x-y)c = C(B \cdot A^{-1})^* A(a,b),$$

i.e.

$$x - (x-y)c = (C(B \cdot A^{-1})^* A \cdot A^{-1})^* A(a,b) =$$
$$= (C(BA^{-1})^*)^* A(a,b) = (C^* \circ (BA^{-1}))A(a,b) = C(A,B)(a,b),$$

according to Proposition 5. □

**Problem 3** *For what bigroups of operations does Theorem 16 hold for Grätzer q-algebras?*

**Problem 4** *For what bigroups of operations does Theorem 16 hold for near-fields?*

**Problem 5** *For what bigroups of operations does Theorem 16 hold for fields?*

**Problem 6** *Characterize the De Morgan bisemigroup of idempotent operations on the arbitrary set $Q$.*

**Problem 7** *To which loops are isotopic Grätzer quasigroups?*

**Problem 8** *To develop the bigroup analogue of the group theory.*

# 8 On Steiner, Stein, and Belousov quasigroups

We now apply the results of the previous section to quasigroups.

The quasigroup $Q(\circ)$ is called $TS$-quasigroup([20, 36, 191]), if the identities

$$x \circ (x \circ y) = y,$$

$$x \circ y = y \circ x$$

are valid. In general, a $TS$-quasigroup is not idempotent.

An idempotent $TS$-quasigroup is called a Steiner quasigroup ([20, 36, 43, 96, 191], [204], [241]). In a non trivial Steiner quasigroup $Q(\circ)$, for any $a \neq b$ in $Q$ the set $\{a, b, a \circ b\}$ is a three-element subquasigroup, which is isomorphic to a three element quasigroup with the following multiplication table:

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 2 | 1 |
| 1 | 2 | 1 | 0 |
| 2 | 1 | 0 | 2 |

The variety of all Steiner quasigroups has permutable, uniform and regular congruences. Any finite simple Steiner quasigroup with at least four elements is functionally complete ([204]).

**Lemma 7** *If $Q(A)$ is a $TS$-quasigroup, then $A \cdot A = \delta_2^2$ and $A \circ A = \delta_2^1$ . So if $Q(A)$ is a non trivial $TS$-quasigroup, then the set $\{\delta_2^1, \delta_2^2, A\}$ is a bigroup of operations.*

**Corollary 3** *If $Q(\circ)$ is a non trivial Steiner quasigroup, then for every $u, v \in Q, u \neq v$ there exists a three-element field $H_{u,v}(+, \cdot), H_{u,v} \subseteq Q$ such that $u, v \in H_{u,v}$ and for every $x, y \in H_{u,v}$:*

$$x \circ y = (y - x)a + x, a \in H_{u,v}.$$

**Example.** Let $Q = \{0, 1, 2\}$. For the abelian group $Q(+)$ of order 3 and for the idempotent $TS$-quasigroup operation $A(x, y) = x + x + y + y$ on $Q$ we have $A(x, y) = (y - x)2 + x$, where $Q(+, \cdot)$ is a field of order 3. Here $H = Q$.

The quasigroup $Q(\circ)$ is called a Stein quasigroup([18, 43, 240]), if the identities

$$x \circ (x \circ y) = y \circ x,$$

$$x \circ (y \circ x) = y$$

are valid. Every Stein quasigroup is idempotent. A non-trivial Stein quasigroup is not commutative. In a non-trivial Stein quasigroup $Q(\circ)$, for any $a \neq b$ in $Q$, the set $\{a, b, a \circ b, b \circ a\}$ is a four-element subquasigroup, which is isomorphic to the four-element quasigroup with the following multiplication table:

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 2 | 3 | 1 |
| 1 | 3 | 1 | 0 | 2 |
| 2 | 1 | 3 | 2 | 0 |
| 3 | 2 | 0 | 1 | 3 |

**Lemma 8** *If $Q(A)$ is a Stein quasigroup, then $A \cdot A = A \circ A = A^* = A^{-1} = {}^{-1}A$ and $A^* \cdot A^* = A = A^* \circ A^*$. So if $Q(A)$ is a non-trivial Stein quasigroup, then the set $\{\delta_2^1, \delta_2^2, A, A^*\}$ is a bigroup of operations.*

**Corollary 4** *If $Q(\circ)$ is a non-trivial Stein quasigroup, then for every $u, v \in Q, u \neq v$ there exists a four-element field $H_{u,v}(+, \cdot), H_{u,v} \subseteq Q$ such that $u, v \in H_{u,v}$ and for every $x, y \in H_{u,v}$:*

$$x \circ y = (y - x)a + x, a \in H_{u,v};$$

In order to move to the next step we present the concept of the following variety of quasigroups.

The quasigroup $Q(\circ)$ is called a Belousov quasigroup, if the identities

$$x \circ (x \circ y) = y \circ x,$$

$$(x \circ y) \circ y = x,$$

$$x \circ (y \circ x) = (y \circ x) \circ y$$

are valid. A non-trivial Belousov quasigroup is not a Stein quasigroup and is not commutative.

**Lemma 9** *If $Q(A)$ is a non-trivial Belousov quasigroup, then it is idempotent, and $A \cdot A = A^*$, $A \cdot A^* = A \circ A^*$, $A \circ A = \delta_2^1$, $A^* \cdot A^* = \delta_2^2, A^* \circ A^* = A$ . So if $Q(A)$ is a non-trivial Belousov quasigroup, then the set $\{\delta_2^1, \delta_2^2, A, A^*, A \cdot A^* = A \circ A^*\}$ is a bigroup of operations.*

**Lemma 10** *In every Belousov quasigroup $Q(\circ)$ the identities $(x \circ y) \circ (y \circ x) = y, (x \circ y) \circ (x \circ (y \circ x)) = y \circ x, (y \circ x) \circ (x \circ (y \circ x)) = x \circ y$ are valid. In a non-trivial Belousov quasigroup $Q(\circ)$, for any $a \neq b$ in $Q$ the set $\{a, b, a \circ b, b \circ a, a \circ (b \circ a)\}$ is a five-element subquasigroup, which is isomorphic to the five-element quasigroup with the following multiplication table:*

| | *0* | *1* | *2* | *3* | *4* |
|---|---|---|---|---|---|
| *0* | *0* | *2* | *4* | *1* | *3* |
| *1* | *4* | *1* | *3* | *0* | *2* |
| *2* | *3* | *0* | *2* | *4* | *1* |
| *3* | *2* | *4* | *1* | *3* | *0* |
| *4* | *1* | *3* | *0* | *2* | *4* |

If we take such subquasigroups as blocks, we obtain a block design on $Q$.

**Corollary 5** *If $Q(\circ)$ is a non trivial Belousov quasigroup, then for every $u, v \in Q, u \neq v$ there exists a five-element field $H_{u,v}(+, \cdot), H_{u,v} \subseteq Q$ such that $u, v \in H_{u,v}$ and for every $x, y \in H_{u,v}$:*

$$x \circ y = (y - x)a + x, a \in H_{u,v};$$

It follows from the last corollary 5 (or Lemma 10) that a non-trivial Belousov quasigroup has at least five elements. The variety of Belousov quasigroups is called a Belousov variety, which is a subvariety of the Mikado variety ([69]). Hence, the Belousov variety has a solvable word problem and is congruence-permutable. Every Belousov quasigroup of prime order is a simple algebra.

For applications of similar quasigroups in cellular automata see in [133].

**Problem 9** *To which loops are isotopic Belousov quasigroups?*

# 9 Bimonoid of term operations (functions)

Let $\mathfrak{A} = (Q; \Sigma)$ be an arbitrary algebra. Let us recall that $n$-ary term operations of algebra $\mathfrak{A}$ are defined by the following induction:

1) all $n$-ary identical operations (or projections) of the set $Q$

$$\delta_n^i(x_1, \ldots, x_n) = x_i, \qquad i = 1, \ldots, n,$$

are $n$-ary term operations of $\mathfrak{A}$;

2) if $f_1, \ldots, f_m$ are $n$-ary term operations of $\mathfrak{A}$, then the superposition

$$\mu_m^n(f, f_1, \ldots, f_m)(x_1, \ldots, x_n) = f(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

is again an $n$-ary term operation of $\mathfrak{A}$, for every $m$-ary $f \in \Sigma$.

3) there are no other $n$-ary term operations of $\mathfrak{A}$.

The operation $h \in O_pQ$ is called a term operation of $\mathfrak{A} = (Q; \Sigma)$, if $h$ is an $n$-ary term operation of $\mathfrak{A}$ for some $n$. For $n = 1, 2, 3$ the $n$-ary term operation is called unary, binary, ternary.

If we denote the set of all $n$-ary term operations of algebra $\mathfrak{A} = (Q; \Sigma)$ by $\mathcal{F}^n(\Sigma)$ and

$$\mathcal{F}(\Sigma) = \mathcal{F}^1(\Sigma) \bigcup \mathcal{F}^2(\Sigma) \bigcup \ldots$$

is a set of all its term operations, then the algebra $\mathcal{F}(\mathfrak{A}) = (Q, \mathcal{F}(\Sigma))$ is called an algebra of term operations (functions) of $\mathfrak{A}$ (or a termal (term) algebra of $\mathfrak{A}$).

It is obvious that the multiplications (12) and (13) of binary operations considered in the previous sections are obtained from the superposition $\mu_2^2$:

$$f \cdot f_1 = \mu_2^2(f, \delta_2^1, f_1),$$

$$f \circ f_1 = \mu_2^2(f, f_1, \delta_2^2).$$

If $f$ and $g$ are binary term operations of any algebra, then $f(x, g(x, y))$ and $f(g(x, y), y)$ are also binary term operations, hence for every binary term operations $f$ and $g$ there exists binary term operations $h$ and $h'$ with identities:

$$f(x, g(x, y)) = h(x, y), \tag{14}$$

$$f(g(x, y), y) = h'(x, y). \tag{15}$$

So the set $\mathcal{F}^2(\Sigma)$ of all binary term operations of any algebra $\mathfrak{A} = (Q;\, \Sigma)$ is a bimonoid of operations on $Q$.

The equations (14) and (15) have the meaning of $\forall\exists(\forall)$-identities in the termal algebra.

The bimonoid $\mathcal{F}^2(\Sigma)$ is called a bimonoid of binary term operations of the algebra $\mathfrak{A} = (Q;\, \Sigma)$ (or bimonoid of the algebra $\mathfrak{A}$ in short).

Besides, the dual operation to every binary term operation is also a binary term operation, i.e. for every binary term operation $f$ there exists a binary term operation $f^*$ with identity: $f(x, y) = f^*(y, x)$, and the mapping $^- : f \to f^*$ is an antiautomorphism of the bimonoid $\mathcal{F}^2(\Sigma)$.

So the set $\mathcal{F}^2(\Sigma)$ of all binary term operations of any algebra $\mathfrak{A} = (Q;\, \Sigma)$ is a De Morgan bisemigroup of operations on $Q$ with an involution $^- : f \to f^*$, and every commutative binary term operation is a fix point of this involution.

**Problem 10** *For which algebras $\mathfrak{A} = (Q;\, \Sigma)$ is the bimonoid $\mathcal{F}^2(\Sigma)$ a bigroup (local bigroup) of operations?*

**Problem 11** *For which algebras $\mathfrak{A} = (Q;\, \Sigma)$ is the bimonoid $\mathcal{F}^2(\Sigma)$ a lattice (free lattice)?*

**Problem 12** *For which algebras $\mathfrak{A} = (Q;\, \Sigma)$ is the bimonoid $\mathcal{F}^2(\Sigma)$ a modular lattice?*

**Problem 13** *For which algebras $\mathfrak{A} = (Q;\, \Sigma)$ is the bimonoid $\mathcal{F}^2(\Sigma)$ a distributive lattice, i.e. a De Morgan algebra ?*

**Problem 14** *For which algebras $\mathfrak{A} = (Q;\, \Sigma)$ is the De Morgan bisemigroup $\mathcal{F}^2(\Sigma)$ a Boole-De Morgan algebra ?*

1) If $\mathfrak{A} = (Q;\, \Sigma)$ is an abelian group, then

$$\mathcal{F}(\Sigma) = \left\{ \sum_{i=1}^{n} c_i x_i \mid n \geqslant 1,\ c_1,\, \ldots,\, c_n \in Z \right\};$$

Similarly $\mathcal{F}(\Sigma)$ is defined for the unitary module over a ring with identity. For every binary term operations $f, g$ of Abelian group the following identity is valid:

$$f(g(x,\, y), g(u,\, v)) = g(f(x,\, u), f(y,\, v)).$$

The same identity is also valid for a commutative semigroup.

**Proposition 8** *The bimonoid of binary term operations of Abelian group (commutative Moufang loop) with the identity $x^2 = e$ is a local bigroup of order 4.*

2) If $\mathfrak{A} = (Q;\, \Sigma)$ is a two-element Boolean algebra, then

$$\mathcal{F}(\Sigma) = O_p Q;$$

Similarly $\mathcal{F}(\Sigma)$ is defined for a finite field and a finite Post algebra. For every binary term operation $f$ of a two-element Boolean algebra the following identities are valid:

$$f(x,\, f(x,\, f(x,\, y)) = f(x,\, y),$$

$$f(f(f(x,\, y),\, y),\, y) = f(x,\, y).$$

There exists a binary term operation $f \neq \delta_2^1,\, \delta_2^2$ with equations $f(x,\, f(x,\, y)) = f(x,\, y)$, $f(f(x,\, y),\, y) = f(x,\, y)$. Hence the bimonoid of binary term operations of a two-element Boolean algebra is not a half-bigroup. Besides there exists a binary term operation $f$ with condition $f(x,\, f(x,\, y)) \neq f(x,\, y)$. Hence the bimonoid of binary term operations of two-element Boolean algebra is not a lattice.

**Proposition 9** *The bimonoid of binary term operations of a two-element Boolean algebra is a local bigroup of order 16.*

3) If $\mathfrak{A} = (Q;\, \Sigma)$ is a non-trivial lattice or a semilattice, then

$$\mathcal{F}^2(\Sigma) = \Sigma \bigcup \left\{\delta_2^1,\, \delta_2^2\right\}.$$

For every binary term operations $f, g$ of any lattice (or any semilattice) the following identities are valid:

$$f(x,\, x) = x,$$

$$f(x,\, f(y,\, z)) = f(f(x,\, y), z),$$

$$f(f(x,\, y), f(u,\, v)) = f(f(x,\, u), f(y,\, v)),$$

$$f(g(f(x,\, y),\, z),\, g(y,\, z)) = g(f(x,\, y),\, z),$$

$$f(x, f(x, y)) = f(x, y),$$

$$f(f(x, y), y) = f(x, y),$$

$$f(x, g(x, y)) = g(x, f(x, y)),$$

$$f(g(x, y), y) = g(f(x, y), y),$$

$$f(x, f(g(x, y), y)) = f(x, y),$$

$$f(f(x, g(x, y)), y) = f(x, y).$$

For every binary term operations $f, g$ of any distributive lattice the following identities are valid:

$$f(x,\ g(y,\ z)) = g(f(x,\ y), f(x,\ z)),$$
$$f(g(y,\ z), x) = g(f(y,\ x), f(z,\ x)).$$

For every binary term operations $f, g$ of any modular lattice the following identity is valid:

$$f(g(x,\ f(y,\ z)),\ g(y,\ z)) = g(f(x,\ g(y,\ z),\ f(y,\ z)).$$

**Proposition 10** *The bimonoid of binary term operations of any semilattice is a lattice of order 3, hence this lattice is distributive and consequently is a De Morgan algebra.*

**Proposition 11** *The bimonoid of binary term operations of any non-trivial lattice is a Boolean bisemigroup of order 4.*

4) If $\mathfrak{A} = Q(\circ)$ is a non-commutative and idempotent semigroup, then

$$\mathcal{F}^2(\{\circ\}) = \left\{\circ,\ \delta_2^1,\ \delta_2^2,\ f_1,\ f_2,\ f_3\right\},$$

where

$f_1(x,\ y) = y \circ x,$
$f_2(x,\ y) = x \circ y \circ x,$
$f_3(x,\ y) = y \circ x \circ y.$

For every binary term operations $f, g$ of any idempotent semigroup the following identities are valid:

$$f(x,\ x) = x,$$
$$f(x,\ f(x,\ y)) = f(x,\ y),$$
$$f(f(x,\ y),\ y) = f(x,\ y),$$
$$f(x, g(x, y)) = g(x, f(x, y)),$$
$$f(g(x, y), y) = g(f(x, y), y),$$
$$f(x, f(g(x, y), y)) = f(x, y),$$
$$f(f(x, g(x, y)), y) = f(x, y).$$

**Theorem 17** *The bimonoid of binary term operations of any non-commutative and idempotent semigroup is a distributive lattice of order 6, i.e. a De Morgan algebra of order 6.*

**Corollary 6** *For every binary term operations $f, g, h$ of any idempotent semigroup the following identity is valid:*

$$f(x, g(h(x, y), y)) = f(f(x, h(x, y)), g(f(x, h(x, y)), y)).$$

# 10 Bimonoid of variety. Clone of variety, variety of clones

If $\omega$ is every $n$-ary and $\omega_1, \ldots, \omega_n$ are every $m$-ary operations on $Q$, then a new $m$-ary operation (superposition) $\mu_n^m(\omega, \omega_1, \ldots, \omega_n)$ on $Q$ can be composed. It is easy to prove the following identities:

$$\mu_n^m(\mu_p^n(w, v_1, \ldots, v_p), u_1, \ldots, u_n) =$$
$$= \mu_p^m(w, \mu_n^m(v_1, u_1, \ldots, u_n), \ldots, \mu_n^m(v_p, u_1, \ldots, u_n), \tag{16}$$
$$\mu_n^m(\delta_n^i, u_1, \ldots, u_n) = u_i, \tag{17}$$
$$\mu_n^n(v, \delta_n^1, \ldots, \delta_n^n) = v. \tag{18}$$

A clone of operations on the set $Q$ is any set of operations on $Q$ which is closed under superpositions and contains all the identical operations (projections) $\delta_n^i$ for all $n$ and $1 \leqslant i \leqslant n$ (see [46, 108], [117, 127, 139, 195], [211, 242, 238]).

For example, the set of all term operations of an algebra $\mathfrak{A} = (Q; \Sigma)$, the set $O_pQ$ and the set $J(Q)$ of all identical operations (projections) are the clones of operations on $Q$. The intersection of clones of operations on $Q$ is also the clone of operations on $Q$. Hence, the set of all clones of operations on $Q$, ordered by set inclusion, forms a complete lattice, with $O_pQ$ as the greatest element and the $J(Q)$ as the least. Moreover, it is the algebraic lattice, denoted by $Lat(Q)$. The lattice $Lat(Q)$ is described by E.L. Post [199, 200](for case $|Q| = 2$) ( see [27] for short proof), Yu.I. Yanov and A.A. Muchnik [264](for case $Q$ is finite, $|Q| > 2$), and I.G. Rosenberg [211](for case $Q$ is infinite).

For an arbitrary set $\Sigma \subseteq O_pQ$ there exists the least clone of operations on $Q$ containing $\Sigma$, which is called the clone of operations generated by $\Sigma$. The clone of all term operations of the algebra $\mathfrak{A} = (Q; \Sigma)$, which is the clone generated by $\Sigma$, is called the clone of the algebra $\mathfrak{A}$ and is denoted by $Cl(\mathfrak{A})$. For clones $Cl(\mathfrak{A})$ see ([127, 195, 242]).

Two algebras $\mathfrak{A} = (Q; \Sigma)$ and $\mathfrak{B} = (Q; \Sigma')$ with the same elements are called rationally equivalent or clone equivalent if $Cl(\mathfrak{A})$=$Cl(\mathfrak{B})$.

For the investigation of algebraic properties an appropriate "environment" is needed, for which appropriate categories of algebras are considered, i.e. corresponding homomorphisms (morphisms) between algebras are needed.

If the first order properties of algebras are studied, i.e. the properties expressed by first order formulae, then algebras are considered in the category of $\Omega$-algebras and their $\Omega$-homomorphisms.

Let $\Omega$ be some set of symbols of operations, such that a nonnegative integer $n$ is assigned to each member $\omega$ of $\Omega$. This integer is called the arity of $\omega$ and $\omega$ is said to be an $n$-ary operation symbol, and denoted $n = |\omega|$. An algebra $\mathfrak{A} = (Q, \Sigma)$ is called an $\Omega$-algebra, if there exists a surjective mapping $f : \Omega \to \Sigma$, which preserves the arity, i.e. $|f(\omega)| = |\omega|$ for any $\omega \in \Omega$. Hence the $\Omega$-algebra can be treated as a triple $(Q; \Sigma, f)$. The set $\Omega$ is called a signature, a language or a type for the $\Omega$-algebra. A morphism between two $\Omega$-algebras $\mathfrak{A}_1 = (Q_1, \Sigma_1)$ and $\mathfrak{A}_2 = (Q_2, \Sigma_2)$ with corresponding surjective mappings $f_1 : \Omega \to \Sigma_1$ and $f_2 : \Omega \to \Sigma_2$ is defined as a mapping $\varphi : Q_1 \to Q_2$ with condition

$$\varphi\left(f_1(\omega)\left(x_1, \ldots, x_n\right)\right) = f_2(\omega)\left(\varphi x_1, \ldots, \varphi x_n\right)$$

for any $n$-ary $\omega \in \Omega$ and $x_1, \ldots, x_n \in Q_1$. Usually the operation $f(\omega) \in \Sigma$ is also denoted by $\omega$ and the definition of a morphism between two $\Omega$-algebras takes the usual, simple form:

$$\varphi\,\omega\left(x_1, \ldots, x_n\right) = \omega\left(\varphi x_1, \ldots, \varphi x_n\right).$$

Such morphisms are called $\Omega$-homomorphisms or homomorphisms between two $\Omega$-algebras. The arising category is called a category of $\Omega$-algebras or a category of algebras with the type $\Omega$. The notions of subalgebra (subobject) of $\Omega$-algebra, of direct product of $\Omega$-algebras, filtered and ultraproducts of $\Omega$-algebras, varieties of $\Omega$-algebras and others are understood in this category.

The $\Omega$-models and $\Omega$-algebraic systems and their categories are defines similarly.

The category of $\Omega$-algebras ($\Omega$-models, $\Omega$-algebraic systems) is a largely investigated category of algebras (see [29, 46, 116, 82, 42, 41, 127, 12, 195, 233, 91, 238]).

If $\mathfrak{A} = (Q, \Sigma)$ is an algebra and $\Sigma' \subseteq \Sigma$, then the algebra $(Q, \Sigma')$ is called the reduct of $\mathfrak{A}$.

If $V$ is a variety of $\Omega$-algebras and $F_V(X)$ is a free algebra (of this variety) of countable rang, then its clone is called the clone of variety $V$ and is denoted by $Cl(V)$ ([246, 195]).

The bimonoid (De Morgan bisemigroup) of binary term operations of free algebra $F_V(X)$ is called a bimonoid (De Morgan bisemigroup) of variety $V$. The following result follows from the theorem 15.

**Theorem 18** *If the bimonoid of variety $V$ is a non-trivial bigroup, then the variety $V$ satisfies the hyperidentity of idempotency: $X(x, x) = x$.*

**Problem 15** *Characterize the bimonoids of classical varieties $V$ (of groups, semigroups, quasigroups, loops, rings). Is every De Morgan bisemigroup included in the De Morgan bisemigroup of binary term operations of $F_V(X)$ for some variety $V$ of groups ?*

**Problem 16** *For which varieties $V$ are their bimonoids bigroups (local bigroups)?*

**Problem 17** *For which varieties $V$ are their bimonoids lattices (free lattices)?*

**Problem 18** *For which varieties $V$ are their bimonoids modular lattices?*

**Problem 19** *For which varieties $V$ are their bimonoids distributive lattices, i.e. a De Morgan algebras ?*

**Problem 20** *For which varieties $V$ are their De Morgan bisemigroups Boole-De Morgan algebras?*

The notion of abstract clone or briefly clone, generalizes the situation of clone of operations. Let $Q$ be an arbitrary set of operation symbols with a condition:

$$\mathcal{N} = \{|\omega| \mid \omega \in Q\},$$

where $\mathcal{N}$ is the set of all natural numbers. We will denote the subset of all $n$-ary elements of the set $Q$ by $Q_n$. Such a set $Q$ is called a clone if:

a) for any natural $n \geqslant 1$ and $m \geqslant 1$ the following operations are defined

$$\mu_n^m : Q_n \times \underbrace{Q_m \times \ldots \times Q_m}_{n} \to Q_m$$

and nullary operations are defined as the elements $\delta_n^i \in Q_n$ for any natural $n \geqslant 1$ and for any natural $i$ with the property $1 \leqslant i \leqslant n$;

b) these operations satisfy the three identities mentioned above: (16), (17), (18).

Thus, all clones form varieties of graduated or heterogeneous algebras, the general theory of which is developed in [114], [87, 31, 195]. The notion of a clone is equivalent to the notion of an algebraic theory in the sense of Lawvere ([109, 195, 249, 255]).

Any subvariety of this variety is called a variety of clones. If $Q$ and $Q'$ are arbitrary clones, then the homomorphism of clones $Q \to Q'$ will be the mapping that preserves the arity of every element and is compatible with the operations of clones. In accordance with clone homomorphisms we understand clone congruence, fully invariant clone congruence, free clones, etc.

The representation of a clone $\Gamma$ in a clone $\Gamma'$ is a homomorphism of the clone $\Gamma$ into the clone $\Gamma'$:

$$\varphi : \Gamma \to \Gamma';$$

The representation $\varphi$ is faithful if $\varphi$ is injective.

**Proposition 12** *(Cayley's theorem for clones). Every clone has faithful representation in the clone of all operations of some set. Every clone $\Gamma$ has a faithful representation in the clone of some $\Gamma$-algebra.*

# 11 $T$-algebras and bihomomorphisms, $T$-hyperidentities. Hypervarieties of $T$-algebras and $\Omega$-algebras

The property of an algebra is called a second order property, if it is expressed (described) by second order formulae. In the study of second order properties of algebras, naturally aries the following category of algebras, which is called the category of $T$-algebras (where $T \subseteq \mathcal{N}$) and their bihomomorphisms $(\varphi, \tilde{\psi})$ ([137, 139]) or the category of algebras with the same arithmetic type $T$.

If $\mathfrak{A} = (Q; \Sigma)$ is an algebra, then the set

$$T_{\mathfrak{A}} = \{|A| \mid A \in \Sigma\} \subseteq \mathcal{N}$$

is called the arithmetic type of the algebra $\mathfrak{A}$.

The notion of the arithmetic type of a model (and an algebraic system) is defined in a similar way.

For example, the arithmetic type of the ring $Q(+, \cdot)$ is $T = \{2\}$, for the Boolean algebra $Q(+, \cdot, ', 0, 1)$ is $T = \{0, 1, 2\}$ .

A $T$-algebra is an algebra with arithmetic type $T \subseteq \mathcal{N}$. A class of algebras is called a class of $T$-algebras if every algebra in it is a $T$-algebra. A $T$-reduct is a reduct with arithmetic type $T \subseteq \mathcal{N}$.

We call the algebra $\mathfrak{A} = (Q; \Sigma)$ functional-trivial, if for every arity $n \in T_{\mathfrak{A}}$ the algebra $\mathfrak{A}$ possesses only one $n$-ary operation $A \in \Sigma$. Otherwise, the algebra is called functional-nontrivial.

Let $\mathfrak{A} = (Q; \Sigma)$ be a $T$-algebra and $\mathfrak{A}' = (Q'; \Sigma')$ be a $T'$-algebra, where $T \subseteq T'$. The pair $(\varphi, \tilde{\psi})$ of maps $\varphi : Q \to Q'$ and $\tilde{\psi} : \Sigma \to \Sigma'$ is called a bihomomorphism from $T$-algebra $\mathfrak{A}$ into $T'$-algebra $\mathfrak{A}'$ and is denoted by $(\varphi, \tilde{\psi}) : \mathfrak{A} \Rightarrow \mathfrak{A}'$ or $(\varphi, \tilde{\psi}) : \mathfrak{A} \rightrightarrows \mathfrak{A}'$, if the map $\tilde{\psi}$ preserves the arity of operations, i.e. $|\tilde{\psi}A| = |A|$ and for any operation $A \in \Sigma$, $|A| = n$, the equality

$$\varphi A(x_1, \ldots, x_n) = \left[\tilde{\psi}(A)\right](\varphi x_1, \ldots, \varphi x_n)$$

holds for any $x_1, \ldots, x_n \in Q$.

The introduced morphisms include the concept of semi-linear transformations of linear spaces, linear algebras and modules, the concept of a weak homomorphism, ascending to E. Marczewski ([122, 254]), the notion of a polymorphism introduced by N. Bourbaki ([30], p.153), and others.

The pair $(\varepsilon, \tilde{\varepsilon})$ of identical maps $\varepsilon : Q \to Q$ and $\tilde{\varepsilon} : \Sigma \to \Sigma$ is a bihomomorphism of $T$-algebra $(Q; \Sigma)$ into itself. If $(\varphi_1, \tilde{\psi}_1) : \mathfrak{A} \Rightarrow \mathfrak{A}_1$ and $(\varphi_2, \tilde{\psi}_2) : \mathfrak{A}_1 \Rightarrow \mathfrak{A}_2$ are bihomomorphisms of $T$-algebras, then the pair $(\varphi_1 \cdot \varphi_2, \tilde{\psi}_1 \cdot \tilde{\psi}_2) : \mathfrak{A} \Rightarrow \mathfrak{A}_2$ also is a bihomomorphism called the composition of bihomomorphisms $(\varphi_1, \tilde{\psi}_1)$ and $(\varphi_2, \tilde{\psi}_2)$:

$$\varphi_1\varphi_2\left(A(x_1, \ldots, x_n)\right) = \varphi_2\left(\varphi_1 A(x_1, \ldots, x_n)\right) =$$

$$= \varphi_2\left(\left[\tilde{\psi}_1(A)\right](\varphi_1 x_1, \ldots, \varphi_1 x_n)\right) =$$

$$= \left[\tilde{\psi}_2\left(\tilde{\psi}_1 A\right)\right](\varphi_2(\varphi_1 x_1), \ldots, \varphi_2(\varphi_1 x_n)) =$$

$$= \left[\tilde{\psi}_1\tilde{\psi}_2(A)\right](\varphi_1\varphi_2(x_1), \ldots, \varphi_1\varphi_2(x_n));$$

Thus, $T$-algebras and their bihomomorphisms $(\varphi, \tilde{\psi})$ (as morphisms) form a category, in which the product of objects is called the superproduct of algebras.

By analogy with bihomomorphisms $(\varphi, \tilde{\psi})$ we'll introduce the following concept of a mixed homomorphism.

Let $\mathfrak{A} = (Q; \Sigma)$ be a $T$-algebra and $\mathfrak{A}' = (Q'; \Sigma')$ be a $T'$-algebra, where $T' \subseteq T$. The pair $(\varphi, \tilde{\psi})$ of maps $\varphi : Q \to Q'$ and $\tilde{\psi} : \Sigma' \to \Sigma$ is called a mixed homomorphism from $T$-algebra $\mathfrak{A}$ into $T'$-algebra $\mathfrak{A}'$ and is denoted by $(\varphi, \tilde{\psi}) : \mathfrak{A} \leftrightarrows \mathfrak{A}'$, if the map $\tilde{\psi}$ preserves the arity of operations and

$$\varphi\left(\left[\tilde{\psi}(A')\right](x_1, \ldots, x_n)\right) = A'(\varphi x_1, \ldots, \varphi x_n)$$

for any operation $A' \in \Sigma'$, $|A'| = n$, and for any elements $x_1, \ldots, x_n \in Q$.

The pair $(\varepsilon, \tilde{\varepsilon})$ of identical maps is a mixed homomorphism of $T$-algebra $\mathfrak{A} = (Q; \Sigma)$ into itself. If $(\varphi_1, \tilde{\psi}_1) : \mathfrak{A} \leftrightarrows \mathfrak{A}'$ and $(\varphi_2, \tilde{\psi}_2) : \mathfrak{A}' \leftrightarrows \mathfrak{A}''$ are the mixed homomorphisms of $T$-algebras, then the pare $(\varphi_1 \cdot \varphi_2, \tilde{\psi}_2 \cdot \tilde{\psi}_1)$ also is a mixed homomorphism $\mathfrak{A} \leftrightarrows \mathfrak{A}''$ called the composition of the given mixed homomorphisms:

$$\varphi_1\varphi_2\left(\left[\tilde{\psi}_2\tilde{\psi}_1(A)\right](x_1, \ldots, x_n)\right) = \varphi_2\left(\varphi_1\left(\left[\tilde{\psi}_1\left(\tilde{\psi}_2 A'\right)\right](x_1, \ldots, x_n)\right)\right) =$$

$$= \varphi_2\left([\psi_2(A')](\varphi_1 x_1, \ldots, \varphi_1 x_n)\right) = A'\left(\varphi_2(\varphi_1 x_1), \ldots, \varphi_2(\varphi_1 x_n)\right) =$$

$$= A'\left((\varphi_1\varphi_2)x_1, \ldots, (\varphi_1\varphi_2)x_n\right);$$

So, $T$-algebras and their mixed homomorphisms $(\varphi, \tilde{\psi})$ (as morphisms) also form a category. Similarly one can introduce mixed homomorphisms for models and algebraic systems.

For bihomomorphisms $(\varphi, \tilde{\psi})$ one introduces in the standard manner the notions of biepimorphism, bimonomorphism, biisomorphism, biautomorphism and biendomorphism. A bihomomorphism $(\varphi, \tilde{\psi})$ is called a biepimorphism if the maps $\varphi$, $\tilde{\psi}$ are surjective, and a bimonomorphism if $\varphi$ and $\tilde{\psi}$ are injective. A biisomorphism is simultaneously a biepimorphism and a bimonomorphism. A bihomomorphism (resp. an biisomorphism) $(\varphi, \tilde{\psi})$ of a $T$-algebra into itself is called an biendomorphism (resp. an biautomorphism).

The set of all biendomorphisms (respectively biautomorphisms) $(\varphi, \tilde{\psi})$ of the same $T$-algebra forms a semigroup with identity (respectively a group) under the component-wise multiplication of pairs:

$$(\varphi_1, \tilde{\psi}_1) \cdot (\varphi_2, \tilde{\psi}_2) = (\varphi_1\varphi_2, \tilde{\psi}_1\tilde{\psi}_2).$$

We shall denote the group of all biautomorphisms $(\varphi, \tilde{\psi})$ of the same $T$-algebra $\mathfrak{A}$ by $Aut\mathfrak{A}$. The biautomorphisms of the kind $(\varphi, \tilde{\varepsilon})$, where $\tilde{\varepsilon}$ is the identical mapping, are ordinary automorphisms. Their set is a group, denoted here by $Aut^{(\circ)}\mathfrak{A}$, and it's obvious that $Aut^{(\circ)}\mathfrak{A} \trianglelefteq Aut\mathfrak{A}$, i.e. $Aut^{(\circ)}\mathfrak{A}$ is an invariant subgroup of the group $Aut\mathfrak{A}$.

**Theorem 19** *(*[137, 144]*) Let $G$ be an arbitrary group with any invariant subgroup $H$. There exists an algebra $\mathfrak{A}$, such that $G \simeq Aut\mathfrak{A}$ and $H \simeq Aut^{(\circ)}\mathfrak{A}$, where the last isomorphism is induced by the first (cf.* [225, 229]*).*

A similar question for the semigroup $End\mathfrak{A}$ of all biendomorphisms $(\varphi, \tilde{\psi})$ of algebra $\mathfrak{A}$ and the semigroup $End^{(\circ)}\mathfrak{A}$ of all ordinary endomorphisms $(\varphi, \tilde{\varepsilon})$ remains open.

Under specific choices of groups $H \trianglelefteq G$ the corresponding algebra $\mathfrak{A}$ will satisfy some additional conditions. In this direction we can formulate the following results.

**Theorem 20** ([137]). *1) If $H$ is the one-element group, then in Theorem 19 one can select the algebra $\mathfrak{A}$ with the non-trivial hyperidentity of associativity*

$$X\,(x,\, Y(y,\, z)) = Y\,(X(x,\, y),\, z)\,;$$

*2) If $G \simeq Hol(H)$, then in Theorem 19 one can select the algebra $\mathfrak{A}$ with the non-trivial hyperidentity of left distributivity*

$$X\,(x,\, Y(y,\, z)) = Y\,(X(x,\, y),\, X(x,\, z))\,.$$

Let $T \subseteq N$ and $T \neq \emptyset$. The hyperidentity (1) (coidentity (3)) is called a $T$-hyperidentity ($T$-coidentity), if $\{|X_1|, \ldots, |X_m|\} \subseteq T$. We'll say, that the $T$-hyperidentity (1) holds (is satisfied, valid, true) in the $T$-algebra $\mathfrak{A} = (Q; \Sigma)$, if the equality $\omega_1 = \omega_2$ is valid when every object variable and every functional variable in it is replaced respectively by an arbitrary element of $Q$ and any operation of the corresponding arity from $\Sigma$. Similarly, the $T$-coidentity (3) holds in the $T$-algebra $\mathfrak{A} = (Q; \Sigma)$, if there exist values for object variables $x_1, \ldots, x_n$ from $Q$, such that the equality $\omega_1 = \omega_2$ holds when every functional variable in it is replaced by any operation of the corresponding arity from $\Sigma$. In addition, the object variables in the coidentity $\omega_1 = \omega_2$ are replaced by the corresponding fixed values from $Q$.

**Examples.** 1) In any multioperator $\Omega$-group (B. I. Plotkin [194]) the following $\{n\}$-coidentity is valid:

$$X(0, \ldots 0) = 0,$$

where all object variables are replaced by the zero element of $\Omega$-group.

2) (J. von Neumann) Let $L(+, \cdot)$ be a modular lattice and $a, b, c \in L$. The sublattice of $L$, generated by the elements $a, b, c$, will be distributive iff the following $\{2\}$-coidentity of left distributivity

$$X\left(a, Y(b, c)\right) = Y\left(X(a, b), X(a, c)\right)$$

holds in $L(+, \cdot)$.

3) In any Boolean algebra $Q(+, \cdot, ', 0, 1)$ the following $\{1, 2\}$-hyperidentities are valid:

$$X\left(Y(x, y)', z\right)' = Y\left(X(x', z)', X(y', z)'\right),$$

$$X\left(x, Y(y, z)\right) = Y\left(X(x, y), Y(x, z)\right),$$

$$X\left(x, X(y, z)\right) = X\left(X(x, y), z\right),$$

$$X(x, y) = X(y, x),$$

$$X(x, x) = x.$$

The $T$-algebra of words (terms) is the free object in the category of $T$-algebras and their bihomomorphisms $(\varphi, \tilde{\psi})$. Let $\mathfrak{X}$ be an arbitrary nonempty set, elements of which are called object variables. Let $T \subseteq N$, $T \neq \emptyset$ and $U$ be an arbitrary $T$-set of symbol operations, i.e.

$$T = \{|\omega| \mid \omega \in U\},$$

and $\mathfrak{X} \bigcap U = \emptyset$. The elements of the set $U$ are also called functional variables. The notion of a $T$-word (or $T$-term) is defined inductively: first, every object variable is a $T$-word, and, second, if $\omega \in U$ is a functional variable with arity $m$ and $v_1, \ldots, v_m$ are $T$-words, then the expression $\omega(v_1, \ldots, v_m)$ is also a $T$-word. There are no other $T$-words. The equality of two $T$-words is defined as the equality of their graphs. We denote the set of all $T$-words by $(\mathfrak{X})U$. The corresponding $T$-algebra $((\mathfrak{X})U; U) = U(\mathfrak{X})$ is the $T$-algebra of words, where if $\omega \in U$, $|\omega| = m$, then

$$\omega : (v_1, \ldots, v_m) \to \omega(v_1, \ldots, v_m)$$

for any $v_1, \ldots, v_m \in (\mathfrak{X})U$.

A hyperidentity $\omega_1 = \omega_2$ is a $T$-hyperidentity if $\omega_1$, $\omega_2$ are elements of a $T$-algebra of words, or if $\omega_1$ and $\omega_2$ are $T$-words.

A system of $T$-hyperidentities is said to be true (valid) in a $T$-algebra $\mathfrak{A}$ if every hyperidentity of this family is valid in $\mathfrak{A}$; a $T$-hyperidentity $\omega_1 = \omega_2$ is valid in a class of $T$-algebras if it is valid in any algebra of this class. In particular, one can consider hyperidentities of varieties: a hyperidentity $\omega_1 = \omega_2$ is called a hyperidentity of the variety $V$ if it is valid in every algebra $\mathfrak{A} \in V$.

A $T$-hyperidentity $\omega_1 = \omega_2$ is called a consequence of a system $\mathcal{L}$ of $T$-hyperidentities and is denoted by $\mathcal{L} \Rightarrow (\omega_1 = \omega_2)$ if the system $\mathcal{L}$ is valid in a $T$-algebra, then the hyperidentity $\omega_1 = \omega_2$ is also valid in it, that is, for any $T$-algebra $\mathfrak{A}$:

$$\mathfrak{A} \models \mathcal{L} \Rightarrow \mathfrak{A} \models (\omega_1 = \omega_2)$$

(the notation $\mathfrak{A} \models \mathcal{L}$ means that any hyperidentity from $\mathcal{L}$ is valid in the algebra $\mathfrak{A}$).

**Proposition 13** *Let $U(\mathfrak{X}) = ((\mathfrak{X})U; U)$ be a $T$-algebra of words and let $\mathfrak{A} = (Q; \Sigma)$ be an arbitrary $T$-algebra. If $\tilde{\psi} : U \to \Sigma$ is an arbitrary map that preserves the arity of operations, then every map $\varphi_0 : \mathfrak{X} \to Q$ can be extended to a map $\varphi : (\mathfrak{X})U \to Q$ so that the pair $(\varphi, \tilde{\psi})$ is a bihomomorphism from the $T$-algebra of words $U(\mathfrak{X})$ into the $T$-algebra $\mathfrak{A}$. Consequently, the $T$-hyperidentity $\omega_1 = \omega_2$ (where $\omega_1$, $\omega_2 \in (\mathfrak{X})U$) holds in the $T$-algebra $\mathfrak{A} = (Q; \Sigma)$ iff the equality $\varphi(\omega_1) = \varphi(\omega_2)$ is valid for every bihomomorphism $(\varphi, \tilde{\psi}) : U(\mathfrak{X}) \Rightarrow \mathfrak{A}$. Similarly, the $T$-coidentity $\omega_1 = \omega_2$ is valid in a $T$-algebra $\mathfrak{A} = (Q; \Sigma)$, iff there exists a map $\varphi_0 : \mathfrak{X} \to Q$, such that for every bihomomorphism $(\varphi, \tilde{\psi}) : U(\mathfrak{X}) \Rightarrow \mathfrak{A}$, $\varphi|_{\mathfrak{X}} = \varphi_0$, the equality $\varphi(\omega_1) = \varphi(\omega_2)$ is valid. An Analogous result is also valid for the $\exists\forall(\forall)$-identities. (For the categorical definition of $\forall\exists(\forall)$-identities the concept of second order algebra has been considered* ([135])*.)*

Let $\mathcal{L}$ be some non-empty set of $T$-hyperidentities, and let $\mathfrak{M}^T_{\mathcal{L}}$ be the class of all $T$-algebras in which every hyperidentity from $\mathcal{L}$ is valid. A class of $T$-algebras $\mathfrak{N}$ is called a hypervariety of $T$-algebras if there exists a system $\mathcal{L}$ of $T$-hyperidentities with the property

$$\mathfrak{N} = \mathfrak{M}^T_{\mathcal{L}}.$$

In this case $\mathcal{L}$ is called a defining system of hyperidentities for $\mathfrak{N}$.

Two systems of $T$-hyperidentities $\mathcal{L}_1$ and $\mathcal{L}_2$ are said to be equivalent if

$$\mathfrak{M}^T_{\mathcal{L}_1} = \mathfrak{M}^T_{\mathcal{L}_2},$$

that is, if every $T$-hyperidentity from $\mathcal{L}_2$ (resp. from $\mathcal{L}_1$) is a consequence of $\mathcal{L}_1$ (resp. from $\mathcal{L}_2$). We say that a system of $T$-hyperidentities $\mathcal{L}$ has a finite base (or is finitely based) if it is equivalent to a finite system of $T$-hyperidentities $\mathcal{L}_0$.

Every class of $T$-algebras $\mathfrak{M}$ corresponds to the class $\Lambda^T_{\mathfrak{M}}$ of all $T$-hyperidentities, each of which is valid in the class $\mathfrak{M}$. The pair of maps

$$\mathcal{L} \to \mathfrak{M}^T_{\mathcal{L}} \quad \text{and} \quad \mathfrak{M} \to \Lambda^T_{\mathfrak{M}}$$

forms a Galois correspondence.

An intersection of hypervarieties of $T$-algebras is also a hypervariety of $T$-algebras, namely:

$$\bigcap_{i \in I} \mathfrak{M}^T_{\mathcal{L}_i} = \mathfrak{M}^T_{\bigcup_{i \in I} \mathcal{L}_i},$$

so that for any class of $T$-algebras $\mathfrak{M}$ there exists a smallest hypervariety (relative to set-theoretic inclusion) of $T$-algebras $\mathfrak{M}^* \supseteq \mathfrak{M}$ called the hypervariety of $T$-algebras generated by $\mathfrak{M}$. It is clear that $\mathfrak{M}^*$ is defined by the system of $T$-hyperidentities $\Lambda^T_{\mathfrak{M}}$, so that the characterization of the hypervariety $\mathfrak{M}^*$ is equivalent to the characterization of all the hyperidentities of the class of $T$-algebras $\mathfrak{M}$.

A hyperidentity $\omega_1 = \omega_2$ is called a termal hyperidentity of the algebra $\mathfrak{A}$ if it is valid in the termal algebra $\mathcal{F}(\mathfrak{A})$. Let $V$ be a class of algebras. A hyperidentity $\omega_1 = \omega_2$ is called a termal hyperidentity of $V$ if it is a termal hyperidentity for any algebra $\mathfrak{A} \in V$. Termal hyperidentities of a class of varieties $V_i$, $i \in I$, are defined similarly. We shall say that a hyperidentity $\omega_1 = \omega_2$ is termally valid in an algebra $\mathfrak{A}$ (or a class $V$) if $\omega_1 = \omega_2$ is a termal hyperidentity of $\mathfrak{A}$ (resp. of $V$). In this case we shall also say that the algebra $\mathfrak{A}$ (or the class $V$) termally satisfies the hyperidentity $\omega_1 = \omega_2$. For the termal hyperidentities of varieties see ([248, 25, 182, 183, 221, 62, 140, 142, 145, 76, 78, 79, 118, 47, 56, 57, 58, 59, 177, 193, 206, 230, 243, 262, 263, 89]).

**Examples.** 1) The following hyperidentities

$$X(x,\, x) = x,$$

$$Y(y,\, x) = Y(y,\, X(x,\, Y(y,\, x))),$$

$$X(x,\, X(y,\, z)) = X(X(x,\, y), z),$$

$$X(X(x,\, y), X(u,\, v)) = X(X(x,\, u), X(y,\, v)),$$

$$X(Y(X(x,\, y),\, z),\, Y(y,\, z)) = Y(X(x,\, y),\, z),$$

$$X(x, X(x, y)) = X(x, y),$$

$$X(X(x, y), y) = X(x, y),$$

$$X(x, Y(x, y)) = Y(x, X(x, y)),$$

$$X(Y(x, y), y) = Y(X(x, y), y),$$

$$X(x, X(Y(x,y),y)) = X(x,y),$$
$$X(X(x, Y(x,y)),y) = X(x,y),$$
$$X(x, Y(y, X(z, X(u,v)))) = X(x, Y(y, X(Y(y,u), X(z, X(u,v))))$$

are termal hyperidentities of the variety of lattices.

2) The hyperidentity

$$X\left(Y\left(Z_{11}(x), \ldots, Z_{1m}(x)\right), \ldots, Y\left(Z_{n1}(x), \ldots, Z_{nm}(x)\right)\right) =$$
$$= Y\left(X\left(Z_{11}(x), \ldots, Z_{n1}(x)\right), \ldots, X\left(Z_{1m}(x), \ldots, Z_{nm}(x)\right)\right)$$

is a termal hyperidentity of the variety of groups (semigroups, Moufang loops) for arbitrary $n, m \in \mathcal{N}$. (To check the validity of this hyperidentity in the termal algebra of a Moufang loop we need to use the classical theorem of R. Moufang ([134, 37]): in a Moufang loop the subloop generated by any two elements is a group.)

3) For any $n, m \in \mathcal{N}$ the hyperidentity

$$X\left(Y\left(x_{11}, \ldots, x_{1m}\right), \ldots, Y\left(x_{n1}, \ldots, x_{nm}\right)\right) =$$
$$= Y\left(X\left(x_{11}, \ldots, x_{n1}\right), \ldots, X\left(x_{1m}, \ldots, x_{nm}\right)\right)$$

is a termal hyperidentity of the variety of commutative groups (semigroups).

The following result is the reformulation of Theorem 18.

**Theorem 21** *If the bimonoid of variety $V$ is a non-trivial bigroup, then the variety $V$ termally satisfies the hyperidentity of idempotency: $X(x, x) = x$.*

There exists a one-to-one correspondence between the termal hyperidentities of algebra $\mathfrak{A}$ and clone-identities of clone $Cl(\mathfrak{A})$. For example, to the termal hyperidentity of distributive lattice $\mathfrak{A}$

$$X\left(x, Y(y, z)\right) = Y\left(X(x, y), X(x, z)\right)$$

corresponds the clone-identity

$$\mu_2^3\left(X, \delta_3^1, \mu_2^3(Y, \delta_3^2, \delta_3^3)\right) =$$
$$= \mu_2^3\left(Y, \mu_2^3\left(X, \delta_3^1, \delta_3^2\right), \mu_2^3\left(X, \delta_3^1, \delta_3^3\right)\right)$$

of $Cl(\mathfrak{A})$.

A hyperidentity $\omega_1 = \omega_2$ is called a termal consequence (briefly $t$-consequence) of a system of hyperidentities $\mathcal{L}$ and is denoted by $\mathcal{L} \Rightarrow_t (\omega_1 = \omega_2)$ if for any algebra $\mathfrak{A}$

$$\mathcal{F}(\mathfrak{A}) \models \mathcal{L} \Rightarrow \mathcal{F}(\mathfrak{A}) \models (\omega_1 = \omega_2);$$

In this connection the concept of termal equivalence ($t$-equivalence) of two systems of hyperidentities, the concept of termal basis ($t$-basis) arises in a natural way.

The following result shows that the concept of hyperidentity arises naturally in algebraic researches and applications.

**Proposition 14** *Let $(Q;\Sigma)$ be a binary algebra and $S = Q \times \Sigma$. Let us define the following binary multiplication on the set $S$:*

$$(a, X) \cdot (b, Y) = (X(a, b), Y).$$

1) *$S(\cdot)$ is a semigroup iff the algebra $(Q;\Sigma)$ satisfies the following hyperidentity:*
$$X(x, Y(y, z)) = Y(X(x, y), z);$$

2) *$S(\cdot)$ is idempotent (i.e. satisfies the identity $x \cdot x = x$) iff the algebra $(Q;\Sigma)$ is idempotent, i.e. the algebra $(Q;\Sigma)$ satisfies the hyperidentity of idempotency:*
$$X(x, x) = x;$$

3) *$S(\cdot)$ is left distributive (i.e. satisfies the identity $x(yz) = (xy)(xz)$) iff the algebra $(Q;\Sigma)$ satisfies the following hyperidentity:*
$$X(x, Y(y, z)) = Y(X(x, y), X(x, z));$$

4) *$S(\cdot)$ is right distributive (i.e. satisfies the identity $(xy)z = (xz)(yz)$) iff the algebra $(Q;\Sigma)$ satisfies the following hyperidentity:*
$$X(Y(x, y), z) = Y(X(x, z), X(y, z));$$

5) *$S(\cdot)$ is medial (i.e. satisfies the identity $(xy)(zt) = (xz)(yt)$) iff the algebra $(Q;\Sigma)$ satisfies the following hyperidentity:*
$$Y(X(x, y), Z(z, t)) = Z(X(x, z), Y(y, t));$$

6) *$S(\cdot)$ is transitive (i.e. satisfies the Kolmogoroff identity $xy \cdot yz = xz$) iff the algebra $(Q;\Sigma)$ satisfies the following hyperidentity:*
$$Y(X(x, y), Y(y, z)) = X(x, z)).$$

***Proof.*** 6) If $x = (a, X)$, $y = (b, Y)$, $z = (c, Z)$ then:

$$\begin{aligned}
xy \cdot yz = xz \longleftrightarrow \\
&((a, X)(b, Y))((b, Y), (c, Z)) = (a, X)(c, Z) \longleftrightarrow \\
&(X(a, b), Y)(Y(b, c), Z) = (X(a, c), Z) \longleftrightarrow \\
&(Y(X(a, b), Y(b, c)), Z) = (X(a, c), Z) \longleftrightarrow \\
&Y(X(a, b), Y(b, c)) = X(a, c).
\end{aligned}$$

□

The following algebraic problem is classical: what are all (idempotent) varieties of algebras that do not contain finitely generated infinite algebras? Such varieties are called Burnside varieties of algebras (W. Burnside). This is an unsolved hard problem even for varieties of classical algebraic structures (see S. I. Adian [5, 6], E. I. Zel'manov [270, 271, 272, 273]). For instance,

1) A finitely generated distributive lattice is finite [83, 195];

2) A finitely generated Boolean algebra is finite [83, 195];

3) A finitely generated De Morgan algebra is finite [156];

4) A finitely generated Boole-De Morgan algebra is finite [157];

5) A finitely generated algebra with two binary, one unary and two nullary operations, satisfying the hyperidentities of the variety of Boolean algebras is finite [158];

6) A finitely generated algebra with two binary and one unary operations, satisfying the hyperidentities of the variety of De Morgan algebras is finite [159];

7) A finitely generated idempotent semigroup is finite [34, 35, 84, 105, 106, 107, 128, 217].

As a consequence from the last proposition we obtain an infinite number of new idempotent varieties of binary algebras with the following hyperidentity of associativity

$$X(x, Y(y, z)) = Y(X(x, y), z),$$

in which every finitely generated algebra is finite.

The binary algebra $(Q; \Sigma)$ is called functionally non-trivial, if the cardinality of $\Sigma$ is $|\Sigma| > 1$. If $(Q; \Sigma)$ is a functionally non-trivial idempotent binary algebra with the mentioned hyperidentity of associativity, then the cardinality of $Q$ is $|Q| \geq 4$. An example of an idempotent functionally non-trivial algebra $Q(+, \cdot)$ with the distinguished hyperidentity of associativity is given by the Cayley tables of their operations $+$ and $\cdot$ as shown below

| $+$ | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 |
| **2** | 1 | 2 | 1 | 4 |
| **3** | 3 | 3 | 3 | 3 |
| **4** | 4 | 4 | 4 | 4 |

| $\cdot$ | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 |
| **2** | 1 | 2 | 4 | 4 |
| **3** | 3 | 3 | 3 | 3 |
| **4** | 4 | 4 | 4 | 4 |

Moreover, there exist 24 functionally non-trivial idempotent binary algebras $Q(+,\cdot)$, $|Q|=4$, with this hyperidentity of associativity. Further, the superproduct of two such algebras is an idempotent algebra with four binary operations, satisfying the given hyperidentity of associativity, etc [137, 139, 144, 149, 176].

**Problem 21** *Characterize the Burnside varieties of (idempotent) quasigroups.*

Let $\Omega$ be a signature with arithmetical type $T_\Omega$, i.e.

$$T_\Omega = \{|\omega| \mid \omega \in \Omega\},$$

and let $\mathcal{L}$ be some non-empty set of $T_\Omega$-hyperidentities, and let $\mathfrak{N}^\Omega_{\mathcal{L}}$ be the class of all $\Omega$-algebras, in which every hyperidentity from $\mathcal{L}$ is valid. It's easy to prove that $\mathfrak{N}^\Omega_{\mathcal{L}}$ is the variety for every $\Omega$ and $\mathcal{L} \neq \emptyset$. The variety $V$ of $\Omega$-algebras is called hypervariety of $\Omega$-algebras, if there exists a system $\mathcal{L}$ of $T_\Omega$-hyperidentities, such that

$$V = \mathfrak{N}^\Omega_{\mathcal{L}}.$$

Let $L$ be some non-empty set of $T$-hyperidentities, where $T = \mathcal{N}$, and $S^\Omega_L$ is the class of all $\Omega$-algebras, in which every hyperidentity from $L$ is termally valid. It's easy to note that $S^\Omega_L$ is a variety for any $\Omega$ and $L \neq \emptyset$. The variety $V$ of $\Omega$-algebras is said to be solid ([78]), if

$$V = S^\Omega_L$$

for some $L \neq \emptyset$. For characterization of all solid varieties of semigroups see [197].

The concept of solid hypervariety is defined analogously. Let $\mathcal{Z}$ be some non-empty set of hyperidentities ($\mathcal{N}$-hyperidentities), and $P^T_{\mathcal{Z}}$ be the class of all $T$-algebras, in which every hyperidentity from $\mathcal{Z}$ is termally valid. It is easy to note that the class $P^T_{\mathcal{Z}}$ is the hypervariety of $T$-algebras for every $\mathcal{Z} \neq \emptyset$ and $T \subseteq \mathcal{N}$. The hypervarieties $\mathcal{W}$ of $T$-algebras are called solid, if

$$\mathcal{W} = P^T_{\mathcal{Z}}$$

for some $\mathcal{Z} \neq \emptyset$ [145].

Hypervarieties, solid varieties and solid hypervarieties are characterized in the second part of the paper.

## Funding

This research is supported by the State Committee of Science of the Republic of Armenia, grants: 10-3/1-41 and 15T-1A258.

# References


[1] J. Aczél, *Forlesungen über Functionalgleichungen und ihre Anwendungem*, Berlin, VEB Deutsch. Verl. Wiss., 1961.

[2] J. Aczél, *Quelques problemes non resolus dans la theorie des equations fonctionnelles*. Matematika Biblioteka **25**(1963), pp.149-152.

[3] J. Aczél, Proof of a theorem of distributive type hyperidentities. Algebra Universalis **1**(1971), pp. 1-6.

[4] J. Aczél, J. Dhombres, *Functional equations in several variables with applications to mathematics, information theory and to the natural and social sciences*, Cambridge University Press, Cambridge, 1991.

[5] S. I. Adian, The Burnside Problem and Identities in Groups, Springer, Berlin/ New York, 1979.

[6] S. I. Adian, The Burnside Problem and related topics, Russian Math. Surveys, **65(5)**(2010), pp. 805-855.

[7] M. Aguiar, M. Livernet, The associative operad and the weak order on the symmetric groups. J. Homotopy Relat. Struct., **2**(2007), pp.57-84.

[8] A. A. Albert, Quasigroup I. Trans. Amer. Math. Soc. **54**(1943), pp. 507-519.

[9] A. A. Albert, Quasigroup II. Trans. Amer. Math. Soc. **55**(1944), pp.401-419.

[10] A. D. Anosov, On homomorphisms of many-sorted algebraic systems in connection with cryptographic applications. Discrete Math. Appl. **17(4)**(2007), pp. 331-347.

[11] B. H. Arnold, Distributive lattices with a third operation defined. Pacific J. Math. **1**(1951), pp. 33-41.

[12] V. A. Artamonov, V. N. Salii, L. A. Skornyakov, L. N. Shevrin, E. G. Shulgeifer, *General Algebra*, Vol.2. Nauka, Moscow, 1991. (in Russian)

[13] A. Avron, The structure of interlaced bilattices. Math. Struct. in Comp. Science **6**(1996), pp. 287-299.

[14] R. Balbes, Ph. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[15] W. E. Barnes, On Γ-rings of Nobusawa. Pacific J. Math. **3**(1966), pp. 411-422.

[16] N. D. Belnap, A useful for valued logic. in: G. Epstein, J.M. Dunn (Eds)., Modern Uses of Multiple-Valued Logic, Reidel Publishing Comnpany, Boston, (1977), pp.7-77.

[17] N. D. Belnap, *How computer should think*. in: G. Ryle (Ed.), Contemporary Aspects of Philosophy, Oriel Press, (1977), pp. 30-56.

[18] V. D. Belousov, Systems of quasigroups with generalized identities. Russian Math. Surveys, **20**(1965), pp. 73-143.

[19] V. D. Belousov, Gratzer algebra and S-systems of quasigroups. Mat.Issled. Akad. Nauk. Mold. **1**(1966), pp. 55-81. (in Russian)

[20] V. D. Belousov, *Foundations of the theory of quasigroups and loops*, Nauka, Moscow, 1967. (in Russian)

[21] V. D. Belousov, Yu. M. Movsisyan, On the rank of generalized identities, Izv. Akad. Nauk. Arm., ser. Mat. **9**(1974), pp. 135-142. (in Russian)

[22] G. B. Belyavskaya, A. M. Cheban, S-systems of arbitrary index I. Mat.Issled. Akad. Nauk. Mold. **7(1(23))**(1972), pp. 23-43. (in Russian)

[23] G. B. Belyavskaya, A. M. Cheban, S-systems of arbitrary index II. Mat.Issled. Akad. Nauk. Mold. **7(2(24))**(1972), pp. 3-13. (in Russian)

[24] G. M. Bergman, Hyperidentities of groups and semigroups. Aequationes Math., **23**(1981), pp. 50-65.

[25] G. M. Bergman, *An invitation on general algebra and universal constructions*, Second edition, Springer, 2015.

[26] J. Berman, Distributive latticec with an additional operation. Aequationes Math. **16**(1977), pp. 165-171.

[27] J. Berman, A proof of Lyndon's finite basis theorem. Discrete Math., **52**(1980), pp. 229-233.

[28] A. Bialynicki-Birula, H. Rasiowa, On the representation of quasi-Boolean algebras. Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys. **5**(1957), pp. 259-261.

[29] G. Birkhoff, On the structure of abstract algebras. Proc. Cambridge Phil. Soc. **31**(1953), pp. 433-454.

[30] G. Birkhoff, *Lattice Theory.* 3rd edn., American Mathematical Society, Providence, Rhode Island, 1979.

[31] G. Birkhoff, J. D. Lipson, Heterogeneous algebras. J. Combinat. Theory **8**(1970), pp. 115-133.

[32] S. L. Bloom, Z. Ésik, E. G. Manes, A Cayley theorem for Boolean algebras. Amer. Math. Monthly **97**(1990), pp. 831-833.

[33] F. Bou, R. Jansana, U. Rivieccio, Varieties of interlaced bilattices. Algebra Universalis **66**(2011), pp. 115-141.

[34] T. C. Brown, On the finiteness of semigroups in which $x^r = x$. *Proc. Camb. Philos. Soc.*, **60**(1964), pp. 1028–1029.

[35] T. Brown, E. Lazerson, On finitely generated idempotent semigroups. Semigroup Forum, **78**(2009), pp. 183-186.

[36] R. H. Bruck, Contributions to the theory of loops. Trans. Amer. Math. Soc. **60**(1946), pp. 245-354.

[37] R. H. Bruck, *A survey of binary systems.* Berlin, Heidelberg, Göttingen, Springer, 1971.

[38] J. A. Brzozowski, De Morgan bisemilattices. in: The 30th IEEE International Sympozium on Multiple-Valued Logic (Portland, Oregon, May 2000), IEEE Press, Los Alamitos, California, (2000), pp.23-25.

[39] J. A. Brzozowski, A characterization of De Morgan algebras. Int. J. Algebra Comput. **11(5)**(2001), pp. 525-537.

[40] J. A. Brzozowski, C.-J. H. Seger, *Asynchronous Circuits*, Springer - Verlag, 1995.

[41] S. Burris, H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, 1981.

[42] C. C. Chang, H. J. Keisler, *Model Theory*, Nort-Holland, Amsterdam, 1973.

[43] O. Chein, H. O. Pflugfelder, J. D. H. Smith, *Quasigroups and Loops: Theory and Applications*, Sigma Series in Pure Math. 9, Heldermann Verlag, 1990.

[44] G. Chupona, Za finitarnite operazii. Godishen zbornik univerzitetot Skopje, Prirodno-Matematichki Fakultet **1(12)**(1959-1961), pp. 7-49.

[45] A. Church, *Introduction to mathematical logic*, Vol. 1. Princeton University Press, Princeton, NJ 1956.

[46] P. M. Cohn, *Universal Algebra*, Harper & Row Publishers, New York, Evanston and London, 1965.

[47] D. Cowan, S. L. Wismath, *Unari hyperidentities for varieties of inverse semigroups*, Preprint, 1996.

[48] Y. Crama, P. L. Hammer, *Boolean Functions: Theory, Algorithms, and Applications*, Cambridge University Press, 2011, New York.

[49] S. S. Davidov, On invertible $T$-algebras. J. Math. Sci. (N.Y.) **193:3**(2013), pp.394-400.

[50] S. S. Davidov, A characterization of binary invertible algebras linear over a group. Quasigroups and Related Systems **19**(2011), pp. 207-222.

[51] S. S. Davidov, A characterization of binary invertible algebras of various type of linearity. Quasigroups and Related Systems **20**, 169-176(2012).

[52] S. S. Davidov, On regular medial division algebras. Quasigroups and Related Systems **21**(2013), pp. 157-166.

[53] S. S. Davidov, A characterization of invertible algebras linear over a group by second-order formulas. J. Algebra Appl. 2017. DOI:10.1142/S0219498817502383.

[54] S. S. Davidov, A. Krapež, Yu. M. Movsisyan, Functional Equations with Division and regular Operations. Asian-Eur. J. Math., 2(2018).

[55] R. Dedekind, Uber Zerlegungen von Zahlen durch ihre grossten gemeinsamen Teiler. Festschrift der Techn. Hochsch. Braunschweig bei Gelegenheit der 69. Versammlung deutscher und Arzte 1-40(1987).

[56] K. Denecke, Hyperequational theory, Math. Japonica **47**(1998), pp. 333-356.

[57] K. Denecke, J. Koppitz, Presolid varieties of commutative semigroups. Tatra Mt. Math. Publ. **5**(1995), pp. 41-48.

[58] K. Denecke, J. Koppitz, Hyperassociative Varieties of Semigroups. Semigroup Forum **49**(1994), pp. 41-48.

[59] K. Denecke, D. Lau, R. Pöschel, D. Schweigert, Hyperidentities, hyperequational classes, and clone congruences. Contribution to General Algebra 7, Verlag Hölder-Pichler-Tempsky, Wien, (1991), 97-118.

[60] K. Denecke, R. Pöschel, A characterization of Sheffer functions by hyperidentities. Semigroup Forum **37**(1988), 351-362.

[61] K. Denecke, R. Pöschel, The characterization of primal algebras by hyperidentities. Contributions to General Algebra 6, Wien, (1988), pp.67-87.

[62] K. Denecke, S. L. Wismath, Hyperidentities and Clones. Gordon and Breach Science Publishers, 2000.

[63] J. Denes, A. D. Keedwell, *Lattin squares and their applications*, Akademiai Kiado, Budapest, 1974 (also Academic Press, New York,1974, and English Universities Press, London, 1974).

[64] L. E. Dickson, Definition of a group and a field by independent postulates. Trans. Amer. Math. Soc., **6**(1905), pp. 198-204.

[65] Z. Ésik, A Cayley theorem for ternary algebras. Int. J. Algebra Comput. **8(3)**(1998), pp. 311-316.

[66] T. Evans, Universal algebra and Euler's officer problem. Amer.Math.Monthly **86**(1979), 466-473.

[67] M. C. Fitting, Bilattices in logic programming. in: G. Epstein ed., Proc. 20th Internat. Symp. on Multi-Valued Logic, IEEE, New York, (1990), pp. 63-70.

[68] L. Fuchs, *Infinite Abelian Groups*, Volume II, Academic Press, New York and London, 1973.

[69] B. Ganter, H. Werner, Equational classes of Steiner systems. Algebra Universalis. **5**(1975), pp. 125-140.

[70] G. Gargov, Knowledge, Uncertainty and ignorance in logic: Bilattices and beyond. J. Appl. Non-Classical Logics **9(2-3)**(1999), pp. 195-283.

[71] P. S. Gevorgyan, On binary G-spaces. Math.Notes **96(4)**(2014), pp. 600-602.

[72] M. L. Ginsberg, Bilattices. Preprint 1986.

[73] M. L. Ginsberg, Multi-valued logics. Preprint 1986.

[74] M. L. Ginsberg, Multi-valued inference. Preprint 1986.

[75] M. L. Ginsberg, Multi-valued logics: a uniform approach to reasoning in artificial intelligence. Computational Intelligence **4**, 265-316(1988), pp. 265-316.

[76] E. Graczynska, *Algebras of M-solid quasivarieties*, Siatras International Bookshop, Athens, 2014.

[77] E. Graczynska, On the problem on M-hyperquasivarieties, arXiv:math/06096000v3 [math.GM], 2007.

[78] E. Graczynska, D. Schweigert, Hypervarieties of a given type. Algebra Universalis **27**(1990), pp. 305-318.

[79] E. Graczynska, R. Pöschel, M. V. Volkov, Solid pseudovarieties. In.: General algebra and applications in discrete mathematics, Shaker-Verlag, Aachen, (1997), 93-100.

[80] G. Grätzer, A theorem on doubly transitive permutation groups with aplication to universal algebra. Fund. Math. **53**(1963), pp. 25-41.

[81] G. Grätzer, Two Mal'tsev-type theorems in universal algebra. J. Combinatorial theory **8**(1970), pp. 334-342.

[82] G. Grätzer, *Universal Algebra*, 2nd ed., Springer-Verlag, 1979.

[83] G. Grätzer, *Lattice Theory: Foundation*, Springer Basel AG, 2011.

[84] J. A. Green, D. Rees, On semi-groups in which $x^r = x$. Proc. Camb. Philos. Soc., **48**(1952), pp.35–40.

[85] M. Hall (Jr.), *The theory of groups*, New York, 1959.

[86] M. Hall, On a theorem of Jordan. Pacific J. Math. **4:2**(1954), pp. 291-226.

[87] P. J. Higgins, Algebras with a scheme of operators. Math. Nachr., **27**(1963), pp. 115-132.

[88] L. Humberstone, Note on Extending Congruential Modal Logics. Notre Dame Journal of Formal Logic **57(1)**(2016), pp. 95-100.

[89] J. Hyndman, R. McKenzie, W. Taylor, k-ary monoids of term operations. Semigroup Forum **44**(1992), pp. 31-52.

[90] J. Jakubik, M. Kolibiar, Lattices with a third distributive operation. Math. Slovaca **27**(1977), pp. 287-292.

[91] B. Jonsson, Congruence distributive varieties. Math. Japonica **42**(1995), pp. 353-401.

[92] C. Jordan, Recherches sur les substitutions. J. Math. Pures Appl. **2(17)**(1872), pp. 351-363.

[93] J. A. Kalman, Lattices with involution. Trans. Amer. Math. Soc. **87**(1958), pp. 485-491.

[94] M. I. Kargapolov, Yu. I. Merzlyakov, *Foundation of group theory*, Nauka, Moscow, 1977. (in Russian)

[95] H. Karzel, Inzidenzgruppen. Lecture Notes by Pieper, I. and Sorensen, K., University of Hamburg, (1965), pp.123-135.

[96] T. Kepka, Distributive Steiner quasigroups of order $3^5$. Comment. Math. Univ. Carolinae **195**, (1978), pp. 389-401.

[97] W. Kerby, Uber eine scharf 3-fach transitiven Gruppen zugeordnete algebraische Structur. Abh. Math. Sem. Univ. Hamburg **37**, (1972), pp. 225-235.

[98] A. Knoebel, A. Romanowska, Distributive Multisemilattices. Dissertationes Mathematicae, *Rozprawy Marhematyczne*, 1991, Warszawa.

[99] S. R. Kogalovskii, On multiplicative semigroups of rings. Dokl. Akad. Nauk SSSR, **140**(1961), pp. 1005-1007. (in Russian)

[100] J. Koppitz, K. Denecke, *M-Solid Varieties of Algebras*, Springer, 2006.

[101] A. D. Korshunov, Monotone boolean functions. Russian Math. Surveys **58(5)**(2003), pp. 928-1001.

[102] A. Krapež, Generalized associativity on groupoids. Publ. Inst. Math. (Beograd) **28**(1980), pp. 105-112.

[103] A. G. Kurosh, *Lectures on General Algebra*, Chelsea, New York, 1963.

[104] A. G. Kurosh, *General Algebra*, Nauka, Moscow, 1974. (in Russian)

[105] G. Lallement, Semigroups and Combinatorial Applications, *Wiley, New York*, 1979.

[106] E. Lazerson, Idempotent semigroups. *Working Paper No. 38, Communications Research Division*, Institute for Defense Analyses, von Neumann Hall, Princeton, NJ, 1961.

[107] M. Lothaire, Combinatorics on Words. *Cambridge University Press, Cambridge*, 1997.

[108] D. Lau, *Functional algebras on finite sets, a basic course on many-valued logic and clone theory*, Springer-Verlag, Berlin Heidelberg, 2006.

[109] F. W. Lawvere, Functorial semantics of algebraic theories. Proc. Nat. Acad. Sci. **50**(1963), pp. 869-872.

[110] W. Leissner, Ein Stufenaufbau der Fasthereiche, Fastkorper and Korper aus ihrer multiplikativen Gruppe. Abh. Math. Sem. Univ. **46**, 55-89(1977).

[111] J.-L. Loday, M. O. Ronco, *Order structure on the algebra of permutations and of planar binary trees*, J. Algebraic Combin., 15(2002), 253-270.

[112] J. Luh, On the theory of simple Γ-rings. Michigan Math. J. **16**, 65-75(1969).

[113] A. I. Mal'tsev, On the general theory of algebraic systems. Amer.Math.Soc. Translations **(2)27**(1963), pp. 125-142.

[114] A. I. Mal'tsev, Model correspondences. AMS Selected Translations **23**(1959), pp. 313-336.

[115] A. I. Mal'tsev, Some questions of the theory of classes of models. Proceedings of the IVth All-Union Mathematical Congress, **1**(1963), pp. 169-198. (in Russian)

[116] A. I. Mal'tsev, *Algebraic systems*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[117] A. I. Mal'tsev, I. A. Mal'tsev, *Iterative Post algebras*, Nauka, Moscow, 2012. (in Russian)

[118] I. A. Mal'tsev, D. Schveigert, Hyperidentitäten von AZ-algebren. Preprint, Fachbereich Mathematik, Universität Kaiserslautern, 1989.

[119] M. Mando, Regular $\forall\exists(\forall)$-identities of length four in algebras. Ph.D. Thesis, 1990. Yerevan State University, Yerevan, Armenia.

[120] Yu. I. Manin, *Cubic Forms*, North-Holland Mathematical Library, Vol 4, 2nd edn. North-Holland, Amsterdam, 1986.

[121] H. B. Mann, The construction of orthogonal Latin squares. Ann. Math. Statist. **13**(1942), pp. 418-423.

[122] E. Marczewski, Independence and homomorphisms in abstract algebras. Fund. Math. **50**(1961), pp.45-61.

[123] A. A. Markov, Constructive logic. Uspekhi Mat. Nauk, **5**(1950), p.187. (in Russian)

[124] S. V. Matveev, Distributive groupoids in knot theory. Mat. Sb. **119(1)**(1982), pp. 78-88.

[125] V. D. Mazurov, On faithful doubly transitive groups. Problems of Algebra and Logic, Procedeengs of the Siberian Branch of the RAS, **30**(1996), pp. 114-118.

[126] V. D. Mazurov, On doubly transitive permutation groups. Siberian Math. J., **31(4)**(1990), pp. 778-780.

[127] R. N. McKenzie, G. F. McNulty, W. T. Taylor, *Algebras, lattices, varieties, Volume I*, Wadsworth, 1983.

[128] D. McLean, Idempotent semigroups. *Am. Math. Mon.*, 61(1954), pp. 110–113.

[129] V. G. Melkonian, Quasi-identities and conditional hyperidentities in distributive lattices. Dokl. Akad. Nauk Armenii **94**(1993) pp. 195-199. (in Russian)

[130] V. G. Melkonian, Circuit integrating through lattice hyperterms. Discrete Mathematics, Algorithms and Applications, **3**(2011), pp. 101-119.

[131] B. Mobasher, D. Pigozzi, G. Slutzki, G. Voutsadakis, A duality theory for bilattices. Algebra Universalis **43**(2000), pp. 109-125.

[132] G. C. Moisil, Recherches sur l'algebre de la logique. Annales scientifiques de l'universite de Jassy **22**(1935), pp.1-117.

[133] C. Moor, Quasi-Linear Cellular Automata. Physica **D103**(1997), pp. 100-132. Proceedings of the International Workshop on Lattice Dynamics.

[134] R. Moufang, Zur structure von Alternative Körpern. Math. Ann. **110**(1935), pp. 416-430.

[135] Yu. M. Movsisyan, Biprimitive classes of algebras of second degree. Matematicheskie Issledovania **9**(1974), pp. 70-84. (in Russian)

[136] Yu. M. Movsisyan, Coidentities in algebras. Dokl. Akad. Nauk Arm. SSR **77**(1983), pp. 51-54. (in Russian)

[137] Yu. M. Movsisyan, *Introduction to the Theory of Algebras with Hyperidentities*, Yerevan State University Press, Yerevan, 1986 (in Russian).

[138] Yu. M. Movsisyan, The multiplicative group of a field and hyperidentities. Math. USSR Izv., **35(2)**(1990), pp. 377-391.

[139] Yu. M. Movsisyan, *Hyperidentities and hypervarieties in algebras*, Yerevan State University Press, Yerevan, 1990. (in Russian)

[140] Yu. M. Movsisyan, Hyperidentities of Boolean algebras. Russ. Acad. Sci. Izv. Math. **40(3)**(1993), pp. 607-622.

[141] Yu. M. Movsisyan, On a theorem of Schauffler. Math. Notes, **53**(1993), pp.172-179.

[142] Yu. M. Movsisyan, Hyperidentities in the variety of lattices. Math. Notes **59(6)**(1996),pp. 686-688.

[143] Yu. M. Movsisyan, Algebras with hyperidentities of the variety of Boolean algebras. Russ. Acad. Sci. Izv. Math. **60(6)**(1996), pp. 1219-1260.

[144] Yu. M. Movsisyan, Hyperidentities in algebras and varieties. Russian Math. Surveys, **53(1)**(1998), pp,57-108.

[145] Yu. M. Movsisyan, Hyperidentities and hypervarieties. Sci. Math. Jpn. **54**(2001),pp. 595-640.

[146] Yu. M. Movsisyan, Boolean Bisemigroups, Bigroups and Local Bigroups. Computer Science and Information Technologies, Proceedings of the Conference, (2005), pp. 97-104.

[147] Yu. M. Movsisyan, On the representations of De Morgan algebras. Trends in logic III, Studialogica, Warsaw, 2005, http://www.ifispan.waw.pl/studialogica/Movsisyan.pdf

[148] Yu. M. Movsisyan, Binary representation of algebras with at most two binary operations. A Cayley theorem for distributive lattices. Internat. J. Algebra Comput. **19(1)**(2009), pp. 97-106.

[149] Yu. M. Movsisyan, Bilattices and hyperidentities. Proc. Steklov Inst. Math. **274(1)**(2011), pp. 174-192.

[150] Yu. M. Movsisyan, Interlaced, modular, distributive and Boolean bilattices. Armen. J. Math. **1(3)**(2008), pp. 7-13.

[151] Yu. M. Movsisyan, Hyperidentities and Related Concepts, II. Armen. J. Math. **1**(2018).

[152] Yu. M. Movsisyan, V. A. Aslanyan, Hyperidentities of De Morgan algebras. Log. J. IGPL **20**(2012), pp. 1153-1174.

[153] Yu. M. Movsisyan, V. A. Aslanyan, Algebras with hyperidentities of the variety of De Morgan algebras. J. Contemp. Math. Anal. **5**(2013), pp. 233-240.

[154] Yu. M. Movsisyan, V. A. Aslanyan, Subdirectly irreducible algebras with hyperidentities of the variety of De Morgan algebras. J. Contemp. Math. Anal. **6**(2013), pp. 241-246.

[155] Yu. M. Movsisyan, V. A. Aslanyan, A functional completeness theorem for De Morgan functions. Discrete Appl. Math. **162**(2014), pp.1-16.

[156] Yu. M. Movsisyan, V. A. Aslanyan, De Morgan Functions and Free De Morgan Algebras. Demonstratio Math. **2**(2014), pp. 271-283.

[157] Yu. M. Movsisyan, V. A. Aslanyan, Boole-De Morgan algebras and quasi-De Morgan functions. Comm. Algebra **42**(2014), pp. 4757-4777.

[158] Yu. M. Movsisyan, V. A. Aslanyan, Super-Boolean functions and free Boolean quasilattices. Discrete Math. Algorithms Appl. **6**(2014), pp. 1450024(13 pages).

[159] Yu. M. Movsisyan, V. A. Aslanyan, Super-De Morgan functions and free De Morgan quasilattices. Cent. Eur. J. Math. **12**(2014), pp. 1749-1761.

[160] Yu. M. Movsisyan, A. G. Barkhudaryan, On hypervarieties of $QB$-algebras. Proceedings of the YSU, Phisics and Mathematics **2**(1996), pp. 16-24. (in Russian)

[161] Yu. M. Movsisyan, L. Budaghyan, The elementary characterization of algebras with hyperidentities of Boolean algebras. International conference "Mathematical Logic, Algebra And Set Theory" dedicated to the 100th anniversary of P.S.Novikov, August 27-31, 2001.

[162] Yu. M. Movsisyan, L. Budaghyan, On elementary decidability of quasi-Boolean algebras. International Conference "Mathematics in Armenia, Advances and Perspectives", September 30 - October 7, Yerevan, (2003), pp. 65-67.

[163] Yu. M. Movsisyan, S. S. Davidov, *Algebras that are nearly quasigroups*, Moscow, LENAND, 2018. (in Russian)

[164] Yu. M. Movsisyan, S. S. Davidov, M. Safaryan, Constraction of free g-dimonoids. Algebra Discrete Math., **18**(2014), pp. 140-150.

[165] Yu. M. Movsisyan, D. S. Davidova, Representation theorem for interlaced q-bilattices. Logic, Algebra and Truth Degrees, 10-14 September 2012, Kanazawa, Ishikawa, Japan, JAIST, (2012), pp. 110-113.

[166] Yu. M. Movsisyan, D. S. Davidova, Hyperidentities of weakly idempotent lattices. J. Contemp. Math. Anal. **6**(2015), pp. 259-264.

[167] Yu. M. Movsisyan , D. S. Davidova, A set-theoretical representation for weakly idempotent lattices and interlaced weakly idempotent bilattices. European Journal of Mathematics **2**(2016), pp. 863-873.

[168] Yu. M. Movsisyan, T. A. Hakobyan, Associative nontrivial hyperidentities in semigroups. J. Contemp. Math. Anal. **46(3)**(2011), pp. 121-130.

[169] Yu. M. Movsisyan, T. A. Hakobyan, Distributive hyperidentities in semigroups. J. Contemp. Math. Anal. **46(6)**(2011), pp.293-298.

[170] Yu. M. Movsisyan, T. A. Hakobyan, Artin theorem for semigroups. J. Algebra Appl. **16(2)**(2017), pp. 1750034(16 pages).

[171] Yu. M. Movsisyan, V. G. Melkonian, On hyperidentities of distributive lattices. Dokl. Akad. Nauk Armenii **95**(1995), pp. 75-78. (in Russian)

[172] Yu. M. Movsisyan, E. Nazari, A Cayley theorem for the multiplicative semigroup of a field. J. Algebra Appl. **11(2)**(2012), pp. 1-12.

[173] Yu. M. Movsisyan, E. Nazari, Transitive Modes. Demonstratio Math. **44(3)**(2011), pp. 511-522.

[174] Yu. M. Movsisyan, J. Pashazadeh, Characterization of binary polynomials of idempotent algebras. J. Math. Sci. (N.Y.) **186(5)**(2012), pp. 802-804.

[175] Yu. M. Movsisyan, A. B. Romanowska, J. D. H. Smith, Hyperidentities and Ginsberg-Fitting type theorems. Eighteenth Midwest Conference on Combinatorics, Cryptography, and Computing, October 28-30, 2004, Rochester, New York.

[176] Yu. M. Movsisyan, A. B. Romanowska, J. D. H. Smith, Superproducts, hyperidentities, and algebraic structures of logic programming. J. Combin. Math. Combin. Comput., **58**(2006), pp. 101-111.

[177] Yu. M. Movsisyan, I. R. Simonyan, Hyperidentities of distributivity in varieties of semigroups. Computer Science and Infofmation Technologies, Yerevan, (1999), pp.24-26.

[178] Yu. M. Movsisyan, V. Toghanyan, On t-stochastic algebras. Aequationes Math. **76**(2008), pp. 241-248.

[179] Yu. M. Movsisyan, V. Toghanyan, On De Morgan Bimonoids. Computer Science and Information Technologies, Proceedings of the Conference,(2007), pp. 57-57.

[180] W. D. Neumann, Malcev conditions, spectra and Kronecker product. J. Austral. Math. Soc. **25**(1978), pp.103-117.

[181] N. Nobusawa, On a generalization of the ring theory. Osaka J. Math. **1**(1964), pp.81-89.

[182] R. Padmanabhan, P. Penner, Bases of hyperidentities of lattices and semilattices. C.R. Math. Rep. Acad. Sci. Canada **4**(1982), pp.9-14.

[183] R. Padmanabhan, P. Penner, Binary hyperidentities of lattices. Aequationes Math. **44**(1992), pp. 154-167.

[184] R. Padmanabhann, P. Penner, Lattice ordered polynomial algebras. Order **15**(1998), pp.75-86.

[185] V. Pambuccian, Euclidean geometry problems rephrased in terms of midpoints and point-reflections, Elem. Math. **60** (2005), pp. 19-24.

[186] G. Paseman, A small basis for Hyperassociativity. Preprint, Berkley, 1993.

[187] G. Paseman, On Two Problems From "Hyperidentities and Clones". 2014, arxiv.org/abs/1408.XXXX.

[188] J. Pashazadeh, A characterization of De Morgan bisemigroup of binary functions. Internat. J. Algebra Comput. **18(5)**(2008), pp.951-956.

[189] P. Penner, Hyperidentities of semilattices. Houston J. Math. **10**(1984), pp. 81-108.

[190] T. Pirashvili, Sets with two associative operations, Cent. Eur. J. Math. **2**(2003), pp. 169-183.

[191] H. Pflugfelder, *Quasigroups and Loops, Introduction*, Heldermann Verlag, Berlin, 1990.

[192] A. G. Pinus, *Foundations of universal algebra*, Novosibirsk State Technical University, Novosibirsk, 2005. (Russian)

[193] J. Plonka, On hyperidentities of some varieties. General Algebra and Discrete Mathematics. V.21. Lemgo: Helderman Verlag. Res. Exposition Math.(1995). pp. 199-213.

[194] B. I. Plotkin, *Automorphisms groups of algebraic systems*, Nauka, Moscow, 1966. (in Russian)

[195] B. I. Plotkin, *Universal algebra, algebraic logic, and databases*, Kluwer Academic Publisher, 1994.

[196] L. Polák, On hyperassociativity. Algebra Universalis **36**(1996), pp. 293-300.

[197] L. Polák, All Solid Varieties of Semigroups. J. Algebra **219**(1999), pp. 421-436.

[198] L. S. Pontryagin, *Continuous groups*, Nauka, Moscow, 1973. (Russian)

[199] E. Post, Introduction to a general theory of elementary propositions. Amer. J. Math. **43**(1921), pp. 163-185.

[200] E. Post, Two-valued iterative systems of mathematical logic. Annals of Math. Studies, **5**(1941), Prinseton Univ. Press, Princeton-London.

[201] H. A. Priestley, Distributive bilattices and their cousins: representation via natural dualities. Research Workshop on Duality Theory in Algebra, Logic and Computer Science, University of Oxford, June 13-14, (2012), pp.18-18.

[202] J. H. Przytycki, Knot and distributive homology: from arc colorings to Yang-Baxter homology. In: New Ideas in Low Dimensional Topology, (2015), pp. 413-488.

[203] A. Pynko, Regular bilattices. J. Applied Non-Classical Logics **10**(2000), pp. 93-111.

[204] R. W. Quackenbush, Varieties of Steiner loops and Steiner quasigroups. Can.J.Math. **28**(1976), pp.1187-1198.

[205] H. Rasiowa, *An algebraic approach to non-classical logics*, Nort-Holland Publ. Co., Amsterdam, 1974.

[206] M. Reichel, Bi-homomorphismen und Hyperidentitaten. Dissertation, University Potsdam, 1994.

[207] B. Richter, Dialgebren, Doppelalgebren und ihre Homologie, Diplomarbeit, Universitat Bonn (1997). http:/www.math.uni-hamburg.de/home/richter/publications.html.

[208] A. B. Romanowska, J. D. H. Smith, *Modal theory*, Helderman-Verlag, Berlin, 1985.

[209] A. B. Romanowska, J. D. H. Smith, *Modes*, World Scientific, Singapore, 2002.

[210] A. B. Romanowska, A. Trakul, On the structure of some bilattices. Universal and Applied Algebra (K.Hatkowska, B.Stawski, eds.) World Scientific (1989), pp. 235-253.

[211] I. G. Rosenberg, La structure des fonctions de plusieurs variables sur un ensemble fini. C.R. Acad. Sci. Paris, Ser. A.B. **260**(1965), pp. 3817-3819.

[212] G. Sabbagh, How not to characterize the multiplicative groups of fields. J. London Math. Soc. **1**(1969), pp. 369-370.

[213] A. Sade, Theorie des systemes demosiens de groupoides. Pacific Journal of Mathematics **10**(1960), pp. 625-660.

[214] A. Sade, Groupoides en relation associative et semigroupes mutuellements associatifs. Ann. Soc. Sci. Bruxelies **1(75)**(1961), pp. 52-57.

[215] H. P. Sankappanavar, A characterization of principal congruences of De Morgan algebras and its applications. Math Logic in Latin America, Proc. IV Latin Amer.Symp.Math.Logic, Santiago, (1978), pp. 341-349. North-Holland Pub. Co., Amsterdam, 1980.

[216] H. P. Sankappanavar, Decision Problems: History and Methods. In A.I.Arruda, N.C.A.da Costa, and R. Chuaqui, eds., Mathematical logic: Proceedings of the First Brazilian Conference, Marcel Dekker, New York, (1978), pp. 241-290.

[217] M. Sapir, Combinatorial algebra: Syntax and Semantix. *Springer*, 2014.

[218] R. Schauffler, Eine Anwendung zyklischer Permutationen and ihre Theorie. Ph.D. Thesis, Marburg University, 1948.

[219] R. Schauffler, Uber die Bildung von Codewörtern. Arch. elekt. Uebertragung **10**(1956), pp. 303-314.

[220] R. Schauffler, Die Associativität im Ganzen, besonders bei Quasigruppen. Math. Z., **67**(1957), pp. 428-435.

[221] D. Schweigert, Hyperidentities. Algebras and Orders, Kluwer Academic Publishers, Dordrecht Boston, London, (1993), pp. 405-506.

[222] V. Shcherbacov, Elements of Quasigroup Theory and Applications, Taylor & Francis Group, 2017.

[223] V. A. Shcherbakov, Elements of quasigroup theory and some its applications in code theory and cryptology. 2003. (http://www.karlin.mff.cuni.cz/ drapal/speccurs.pdf)

[224] M. K. Sen, On Γ-semigroup. In: Algebra and Its Applications (New Delhi, 1981). Lecture Notes in Pure and Appl. Math., **91**(1984), pp. 301-308. Dekker, New York.

[225] J. R. Senft, On weak automorphisms of universal algebras. Dissertationes Math., Rozprawy Mat., **74**(1970).

[226] M. K. Sen, N. K. Saha, On Γ-semigroup-I. Bull. Calcutta Math. Soc. **78**(1986), pp. 180-186.

[227] A. Seth, Γ-group congruences on regular Γ-semigroups. Int. J. Math. Math. Sci. **15(1)**(1992), pp. 103-106.

[228] I. P. Shestakov, A. I. Shirshov, A. M. Slińko, K. A. Zhevlakov, Rings that are nearly associative, Academic Press, New York, 1982.

[229] J. Sichler, Weak automorphisms of universal algebras. Algebra Universalis **3**(1973), pp. 1-7.

[230] I. R. Simonyan, Lattices of distributive and hyperdistributive varieties of semigroups. Proceedings of the YSU, Phisics and Mathematics, **2**(2004), pp. 7-19. (in Russian)

[231] L. A. Skornyakov, Stochastic algebra. Soviet Math. (Iz. Vuz. Matematika), **29(7)**(1985), pp. 1-12.

[232] J. Slominski, On the satisfiabilities and varieties for abstract algebras induced by the cones and functor dinamics. Demonstratio Math., **26**(1993), pp. 11-22.

[233] D. M. Smirnov, *Varieties of algebras*, Nauka, Novosibirsk, 1992 (in Russian).

[234] J. D. H. Smith, *Malcev varieties*, Lecture notes in Mathematics **554**(1976), Springer Verlag, Berlin New-York.

[235] J. D. H. Smith, Finite codes and groupoid words. European J. Combin., **12**(1991), pp. 331-339.

[236] J. D. H. Smith, *An Introduction to Quasigroups and Their Representations*, Chapman & Hall/Crc, 2006.

[237] J. D. H. Smith, On groups of hypersubstitutions, Algebra Universalis, **64**(2010), pp. 39-48.

[238] J. D. H. Smith, A. B. Romanowska, *Post-Modern Algerbra*, Wiley, New York, 1999.

[239] E. Sperner, Ein Satz uber Untermengen einer endlichen Menge. Math.Z. **27**(1928), pp. 544-548.

[240] S. K. Stein, On the foundation of quasigroups. Trans. Amer. Math. Soc. **85**(1957), pp. 228-256.

[241] J. Steiner, Combinatorische aufgabe. J. Reine Angew. Math. **45**(1853), pp. 181-182.

[242] A. Szendrei, *Clones in universal algebra*, Les presses de L'universite de Montreal, 1986.

[243] Z. Szylicka, Hyperidentities of some generalizations of lattices. Algebra Universalis **39**(1998), pp. 1-29.

[244] A. Tarski, On the calculus of binary relations. J. Symbolic Logic **6**(1941), pp. 73-89.

[245] A. Tarski, A. Mostowski, R. Robinson, *Undecidable Theories*, North-Holland, Amsterdam, 1953.

[246] W. Taylor, Characterizing Mal'tsev conditions. Algebra Universalis **3**, 351-397(1973), pp. 351-397.

[247] W. Taylor, Mal'tsev conditions and spectra. J. Austral. Math. Soc. **29**(1980), pp. 143-152.

[248] W. Taylor, Hyperidentities and hypervarieties. Aequationes Math., **23**(1981), pp. 30-49.

[249] W. Taylor, Abstract clone theory. Algebras and Orders, Kluwer Academic Publishers, Dordrecht Boston, London,(1993), pp. 507-530.

[250] K. Tent, Sharply 3-transitive groups, Adv. in Math., **286** (2016), pp. 722 - 728.

[251] K. Tent, M. Ziegler, Sharply 2-transitive groups. Adv. Geom., **16(1)** (2016), pp. 131-134.

[252] J. Tits, Sur les groupen doublement transitif continus. Comment. Math. Helv. **26**(1952), pp. 203-224.

[253] V. H. Toghanyan, Subdirectly irreducible algebras with various equations. Aequationes Math. **68**(2004), pp. 98-107.

[254] T. Traczyk, Weak isomorphisms of Boolean and Post algebras. Colloq. Math. **13(2)**(1965), pp. 159-164.

[255] V. Trnkova, Amazingly extensive use of Cook continuum. Math. Japonica **51**(2000), pp. 499-549.

[256] K. Urbanik, On algebraic operations in idempotent algebras. Colloquium Math. **XIII**(1965), pp. 129-157.

[257] J. Ušan, Globally associative systems of $n$-ary quasigroups. Publ. Inst. Math. (Beograd) **19**(1975), pp. 155-165.

[258] J. Ušan, Description of Super Associative Algebras with n-Quasigroup Operations. Math. Morav. **5**(2001), pp. 129-157.

[259] J. Ušan, *n-Groups in the light of the neutral operations*, Electronic version (2006).

[260] J. Ušan, M. Žižović, $n$-ary analogy of Schauffler theorem. Publ. Inst. Math. (Beograd) **19**(1975), pp. 167-172.

[261] F. W. Wilke, Pseudo-fields and doubly transitive groups. Bull. Austral. Math. Soc. **7**(1972), pp. 163-168.

[262] S. L. Wismath, Hyperidentity bases for rectangular bands and other semigroup varieties. J. Austrral. Math. Soc. (Ser. A) **55**(1993), pp. 270-286.

[263] S. L. Wismath, A basis for the type $< n >$ hyperidentities of the medial variety of semigroups. Aequationes Math. **49**(1995), pp. 205-213.

[264] Yu. I. Yanov, A. A. Muchnik, On the existence of k-valued closed classes withoth a finite basis. Dokl. Acad. Nauk SSSR **127**(1959), pp. 44-46. (in Russian)

[265] Yu. L. Yershov, *Decidability problems and cunstractive models*, Nauka, Moscow,1980. (in Russian)

[266] K. A. Zaretskii, An abstract characterization of the bisemigroup of binary operations. Math. Notes, **1(5)**(1967), pp. 347-350.

[267] D. Žarkov, A remark on globally associative systems of $n$-quasigroups over $Q = 1, 2, 3, 4$. Publ. Inst. Math. (Beograd) **21**(1977), pp. 207-211.

[268] H. Zassenhaus, Uber endliche Fastkorper. Abh. Math. Sem. Univ. Hamburg **11**(1935/36), pp. 187-220.

[269] H. Zassenhaus, Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen. Abh. Math. Sem. Univ. Hamburg **11**(1936), pp. 17-40.

[270] E. I. Zel'manov, The solution of the restricted Burnside problem for groups of odd exponent. Izv. Akad. Nauk. SSSR. Ser. Mat. **54(1)**(1990), pp. 42-59 ; trans. in Math. USSR-Izv. **36(1)**(1991), pp. 41-60.

[271] E. I. Zel'manov, The solution of the restricted Burnside problem for 2-groups. Mat. Sb. (N.S.), **182(4)**(1991), pp. 568-592.

[272] E. I. Zel'manov, On additional laws in the Burnside problem for periodic groups. Int. J. Algebra Comput. **3(4)**(1993), pp. 583-600.

[273] E. I. Zel'manov, Some open problems in the theory of infinite dimensional algebras. J. Korean Math. Soc. **44(5)**(2007), pp. 1185-1195.

[274] J. L. Zemmer, Near-fields, planar and non-planar. Math. Stud. **31**(1964), pp. 145-150.

[275] M. R. *Ž*ižović, Associative systems of topological $n$-quasigroups. Algebraic Conference, Novi Sad (1981), pp. 55-60.

[276] A. V. Zhuchok, Free products of doppelsemigroups. Algebra Universalis **76**(2016), pp. 355-366.

[277] A. V. Zhuchok, M. Demko, Free $n$-dinilpotent doppelsemigroups. Algebra Discrete Math., **22**(2016), pp. 304-316.

Yu. M. Movsisyan
*Yerevan State University,*
*Alex Manoogian 1, 0025 Yerevan, Armenia*
*movsisyan@ysu.am*

**Please, cite to this paper as published in**
Armen. J. Math., V. **9**, N. 2(2017), pp. 146–222