

On Nagy-Foias Characteristic Function in Extensions Theory of Hermitian Operators

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Abstract. For a densely defined in a Hilbert space closed Hermitian operator with infinite defect numbers its maximal extensions are discussed. The Nagy-Foias characteristic function of an arbitrary maximal dissipative extension is derived. Mutually complementary classes of such extensions, referred to as inherited and acquired are introduced, and the peculiarity of characteristic function, as determining the class of extensions it corresponds to, is noted. In the setting of Calkin's abstract boundary conditions theory abstract analogs of Nagy-Foias and Weyl functions are presented in similar manner, as operator functions involved in boundary operators, describing the class of inherited extensions. Existence and analyticity of these functions are proved.

Key Words: Hermitian operator, maximal extensions, reduction operator of Calkin, Nagy-Foias characteristic function, Weyl function.

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Introduction

A closed densely defined Hermitian operator T in a Hilbert space \mathfrak{H} is considered. Maximal extensions of T can be described in frames of both the classical theory of von Neumann and the theory of abstract boundary conditions of Calkin, based on the concept of a reduction operator for T^* .

Here these theories are separately applied to the study of Nagy-Foias characteristic functions of maximal dissipative and Weyl functions of self-adjoint extensions of T . It turned out that the natural classification of maximal extensions as to whether the extension is inherited from T or not, and the use of a special reduction operator for a description of maximal extensions yields closely connected results.

The paper is composed of three sections.

In Sec. 1 notations and some known notions and statements that will be needed in the sequel are collected. For the most part they are referred to the theory of Calkin elaborated in [2], since it is not widespread. The basic concept of reduction operator for T^* as a tool for defining linear extensions of T via abstract boundary conditions is discussed. More detailed presentation of principle aspects of this subject, reviewed in a terminology of Kreĭn space, one can find in [10].

In Sec. 2 we complement the study in [17], presenting characteristic function (ch.f.) of an arbitrary maximal dissipative extension (m.d.ext.) of T with the help of Nagy-Foias definition ([10], VI. 1) for its Cayley transform. The formula derived with the results of [17], where ch.f. was presented for a fixed inherited m.d.ext. allowed to determine the class of considered extension by means of its ch.f. The Weyl function of an arbitrary self-adjoint extension of T here is defined with the use of von Neumann formulas only, in addition to that in [17].

In Sec. 3 the machinery of Calkin's theory is utilized to reveal abstract analogues of ch.f. and Weyl function on the base of their properties, studied in previous section. A close analogy with a discussion there is obtained by employing the canonical reduction operator for T^* built in [18].

The literature on characteristic and Weyl function (also Titchmarsh-Weyl, M -, Q -function) is extensive. The following will give, to some extent, a sufficient information. The monograph [15], devoted to the theory of characteristic functions of various classes of linear operators, is presented also its development, involving the contributions of many workers. On connections between some diverse approaches to this notion one can find in [12], [17], [21]. Concerning to the Weyl functions discussed here we refer to [3], [4], [5], [7], [19] and references therein.

1 Preliminaries

1.1. In a Hilbert space \mathfrak{H} with an inner product $\langle \cdot, \cdot \rangle$ consider a closed Hermitian operator T with the domain $\mathcal{D}(T)$ dense in \mathfrak{H} , so T^* exists and $\mathcal{D}(T^*) \supset \mathcal{D}(T)$. Open upper, lower half-planes of a complex plane C will be denoted C^\pm .

Throughout this paper some complex number $\gamma = \alpha + i\beta \in C^+$ will be fixed and the case $\dim \mathfrak{N}_\gamma = \dim \mathfrak{N}_{\bar{\gamma}} = \infty$, where $\mathfrak{N}_\gamma = \text{Ker}(T^* - \gamma I)$, $\mathfrak{N}_{\bar{\gamma}} = \text{Ker}(T^* - \bar{\gamma} I)$ will be discussed.

The direct-sum decomposition

$$\mathcal{D}(T^*) = \mathcal{D}(T) \dot{+} \mathfrak{N}_\gamma \dot{+} \mathfrak{N}_{\bar{\gamma}} \quad (1)$$

defines oblique projections $\mathcal{Q}_\gamma, \mathcal{Q}_{\bar{\gamma}}$ in $\mathcal{D}(T^*)$ onto $\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}$ respectively, and for arbitrary $f, g \in \mathcal{D}(T^*)$ the following identity holds

$$\langle T^* f, g \rangle - \langle f, T^* g \rangle = 2i\beta [\langle \mathcal{Q}_\gamma f, \mathcal{Q}_\gamma g \rangle - \langle \mathcal{Q}_{\bar{\gamma}} f, \mathcal{Q}_{\bar{\gamma}} g \rangle]. \quad (2)$$

Introduce the operator $T(\gamma) = \beta^{-1}(T - \alpha I)$, which clearly is Hermitian, $\mathcal{D}(T(\gamma)) = \mathcal{D}(T)$, $\mathcal{D}(T^*(\gamma)) = \mathcal{D}(T^*)$, and such that the defect subspaces $\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}$ of T are defect subspaces of $T(\gamma)$ at the points $i, -i$, hence (see [6], XII. 4) the linear manifold $\mathcal{D}(T^*(\gamma))$ with the inner product

$$\langle f, g \rangle_\gamma = \langle f, g \rangle + \langle T^*(\gamma)f, T^*(\gamma)g \rangle \quad (3)$$

is a Hilbert space, denoted \mathfrak{D}_γ , and now one has the orthogonal decomposition

$$\mathfrak{D}_\gamma = \mathcal{D}(T(\gamma)) \oplus \mathfrak{N}_\gamma \oplus \mathfrak{N}_{\bar{\gamma}}. \quad (4)$$

Corresponding notations for $\gamma = i$ ($T_i = T$) shall be $\mathfrak{D} := \mathfrak{D}_i$, $\mathfrak{N}_\pm := \mathfrak{N}_{\pm i}$, so

$$\mathfrak{D} = \mathcal{D}(T) \oplus \mathfrak{N}_+ \oplus \mathfrak{N}_-.$$

If \mathcal{P}_\pm are orthogonal projections in \mathfrak{D} onto \mathfrak{N}_\pm , then

$$\langle T^*f, g \rangle - \langle f, T^*g \rangle = i[\langle \mathcal{P}_+f, \mathcal{P}_+g \rangle_i - \langle \mathcal{P}_-f, \mathcal{P}_-g \rangle_i], \quad (5)$$

since $\langle \mathcal{P}_\pm f, \mathcal{P}_\pm g \rangle_i = 2\langle \mathcal{Q}_\pm f, \mathcal{Q}_\pm g \rangle$, $\mathcal{Q}_\pm := \mathcal{Q}_{\pm i}$.

It is also obvious that

$$\langle T^*f, g \rangle - \langle f, T^*g \rangle = \beta[\langle T^*(\gamma)f, g \rangle - \langle f, T^*(\gamma)g \rangle]. \quad (6)$$

For $\mathfrak{G}_1, \mathfrak{G}_2$ being Hilbert spaces, the Banach space of all bounded linear operators from \mathfrak{G}_1 to \mathfrak{G}_2 is denoted by $[\mathfrak{G}_1, \mathfrak{G}_2]$, and the algebra of all bounded linear operators in \mathfrak{G}_1 by $[\mathfrak{G}_1]$.

1.2. The basic concept of Calkin's theory is presented in Definition 1.1 of [2], where the graph of T^*

$$\text{Gr}T^* = \{(f, T^*f) \in \mathfrak{H} \oplus \mathfrak{H}; f \in \mathcal{D}(T^*)\}$$

is employed. From (3) it is obvious that Hilbert spaces $\text{Gr}T^* \subset \mathfrak{H} \oplus \mathfrak{H}$ and \mathfrak{D} can be identified by the map $\text{Gr}T^* \ni (f, T^*f) \leftrightarrow f \in \mathfrak{D}$, and in the following definition the Hilbert space $\text{Gr}T^*$ of [2] is replaced by that \mathfrak{D} .¹

Definition 1 *Let T be a closed linear operator in a Hilbert space \mathfrak{H} , and let T^* exist. Let \mathfrak{G} be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{G}}$. A closed linear operator Γ with the domain $\mathcal{D}(\Gamma)$ dense in \mathfrak{D} , and the range $\text{Ran}\Gamma \subset \mathfrak{G}$ is said to be a reduction operator for T^* , if there exists an unitary operator $W \in [\mathfrak{G}]$ such that for all $f, g \in \mathcal{D}(\Gamma)$ it holds the identity*

$$\langle T^*f, g \rangle - \langle f, T^*g \rangle = -\langle \Gamma f, W\Gamma g \rangle_{\mathfrak{G}}. \quad (7)$$

¹In this subsection we refer only to [2].

The significance of this definition is specified by its following corollaries (Th. 1.1, Th. 1.2, Th. 3.7).

C1. The operator T is Hermitian, $\mathcal{D}(T) \subset \mathcal{D}(\Gamma)$, $\text{Ker}\Gamma = \mathcal{D}(T)$, $\overline{\text{Ran}\Gamma} = \mathfrak{G}$;

C2. The unitary operator W is such that $W^2 = -I_{\mathfrak{G}}$, and $\dim \mathfrak{G}_{\pm} = \dim \mathfrak{N}_{\pm}$, where \mathfrak{G}_{\pm} are eigensubspaces of W corresponding to its eigenvalues $\pm i$.

Linear extensions of T are defined by means of Γ as follows (Def. 1.2, Th. 1.4).

An arbitrary linear manifold $\mathfrak{L} \subset \mathfrak{G}$ defines the operator

$$T_{\mathfrak{L}} = T^*|_{\mathcal{D}_{\mathfrak{L}}}, \quad \text{where } \mathcal{D}_{\mathfrak{L}} = \{f \in \mathcal{D}(\Gamma); \Gamma f \in \mathfrak{L}\} =: \mathcal{D}(T_{\mathfrak{L}}),$$

which is a linear extension of T , since $\mathcal{D}(T) \subset \mathcal{D}(T_{\mathfrak{L}})$. Conversely, any extension T_{γ} of T can be presented on this way, since $T_{\gamma} = T_{\mathfrak{L}_{\gamma}}$, where $\mathfrak{L}_{\gamma} = \{h \in \mathfrak{G}; h = \Gamma f, f \in \mathcal{D}(T_{\gamma})\}$. The Hilbert space \mathfrak{G} is called a space of abstract boundary values, and the condition $\Gamma f \in \mathfrak{L}$ is called the boundary condition defining $T_{\mathfrak{L}}$.

To determine properties of \mathfrak{L} , providing symmetric and self-adjoint extensions of T , notions of W -symmetric and hypermaximal W -symmetric manifolds in \mathfrak{G} are introduced by the relations $W\mathfrak{L} \subset \mathfrak{G} \ominus \mathfrak{L}$ and $W\mathfrak{L} = \mathfrak{G} \ominus \mathfrak{L}$ respectively (Def. 1.3, Def. 2.2).

For the case $\dim \mathfrak{N}_{+} = \dim \mathfrak{N}_{-}$ we are dealing with, from corollary C.2 it follows that there exists an isometry $V \in [\mathfrak{G}_{+}, \mathfrak{G}_{-}]$ ($V^*V = I_{\mathfrak{G}_{+}}$, $VV^* = I_{\mathfrak{G}_{-}}$), and it is proved (Th. 2.2) that the formula

$$\mathfrak{L}_V = \{h \in \mathfrak{G}; h = h_{+} - Vh_{+}, h_{+} \in \mathfrak{G}_{+}\} \quad (8)$$

establishes an one-to-one correspondence between the set of all hypermaximal W -symmetric subspaces in \mathfrak{G} and the set of all such isometries, thus describing the set of all self-adjoint extensions of T (Th. 2.7).

The set of all reduction operators for T^* is described by Stone (Th. 3.3). In particular, the following assertion is true.

Let Γ be a reduction operator for T^ with the associated unitary operator W . Let $\tilde{\mathfrak{G}}$ be an arbitrary Hilbert space ($\dim \tilde{\mathfrak{G}} = \dim \mathfrak{G}$), and $\tilde{U} \in [\mathfrak{G}, \tilde{\mathfrak{G}}]$ be any isometry. Then the operator $\tilde{\Gamma} = \tilde{U}\Gamma$ is a reduction operator for T^* with the associated unitary operator $\tilde{W} = \tilde{U}W\tilde{U}^* \in [\tilde{\mathfrak{G}}]$.*

A reduction operator for T^* may be either bounded or unbounded, and in what follows we shall deal only with a bounded operator Γ , that is with the case (Th. 3.2)

$$\mathcal{D}(\Gamma) = \mathfrak{D}, \quad \text{Ran}\Gamma = \mathfrak{G}. \quad (9)$$

Denoting $\mathfrak{N} = \mathfrak{N}_{+} \oplus \mathfrak{N}_{-}$, $\mathcal{P} = \mathcal{P}_{+} \oplus \mathcal{P}_{-}$, formula (5) can be presented as

$$\langle T^*f, g \rangle - \langle f, T^*g \rangle = \langle W\mathcal{P}f, \mathcal{P}g \rangle;$$

where $W = i(\mathcal{P}_{+} - \mathcal{P}_{-})|_{\mathfrak{N}}$, thus one has an example of a reduction operator for T^* .

1.3. Corollary C.2 infers that the operator $\mathcal{J} = -iW$ ($\mathcal{J}^* = \mathcal{J}$, $\mathcal{J}^2 = I_{\mathfrak{G}}$) is a signature operator, turning \mathfrak{G} into a Kreĭn space with an indefinite inner product $[h_1, h_2] = \langle \mathcal{J}h_1, h_2 \rangle_{\mathfrak{G}}$. Introducing orthogonal projections $P_{\pm} = \frac{1}{2}(I_{\mathfrak{G}} \pm \mathcal{J})$ and denoting $\mathfrak{G}_{\pm} = P_{\pm}\mathfrak{G}$, $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$, consider the operators

$$\Gamma_{\pm} = P_{\pm}\Gamma \in [\mathfrak{D}, \mathfrak{G}_{\pm}], \quad \text{Ran}\Gamma_{\pm} = \mathfrak{G}_{\pm}$$

and present identity (7) in the form

$$\langle T^*f, g \rangle - \langle f, T^*g \rangle = i\langle \mathcal{J}\Gamma f, \Gamma g \rangle_{\mathfrak{G}} = i[\langle \Gamma_+f, \Gamma_+g \rangle_{\mathfrak{G}} - \langle \Gamma_-f, \Gamma_-g \rangle_{\mathfrak{G}}], \quad (10)$$

we shall use henceforth. For reduction operator Γ the notation $\{\Gamma, \mathfrak{G}\} = \{\Gamma_{\pm}, \mathfrak{G}_{\pm}\}$ is applied.

A subspace $\mathfrak{L}_+ \subset \mathfrak{G}$ such that $\mathfrak{L}_+ = \{h \in \mathfrak{G}; \langle \mathcal{J}h, h \rangle_{\mathfrak{G}} \geq 0\}$ is called nonnegative (see [11], Sec. 1). From (10) one has

$$2\text{Im}\langle T^*f, f \rangle = \langle \mathcal{J}h, h \rangle_{\mathfrak{G}}, \quad h = \Gamma f, \quad f \in \mathcal{D}(T^*),$$

where vectors Γf cover \mathfrak{G} when f runs over $\mathcal{D}(T^*)$. Clearly, the operator $T_{\mathfrak{L}_+} = T^*|_{\mathcal{D}(T_{\mathfrak{L}_+})}$, $\mathcal{D}(T_{\mathfrak{L}_+}) = \{f \in \mathcal{D}(T^*); \Gamma f \in \mathfrak{L}_+\}$, is a dissipative extension of T , that is $\text{Im}\langle T_{\mathfrak{L}_+}f, f \rangle \geq 0$ for arbitrary $f \in \mathcal{D}(T_{\mathfrak{L}_+})$. In ([13], Sec. 1) the set of all maximal nonnegative subspaces in \mathfrak{G} is parameterized by the formula

$$\mathfrak{L}_K = \{h \in \mathfrak{G}; h = h_+ + Kh_+, h_+ \in \mathfrak{G}_+, K \in [\mathfrak{G}_+, \mathfrak{G}_-], \|K\| \leq 1\},$$

where the contractive operator K is called the angular operator of \mathfrak{L}_K .

Thus the above formula describes also the set of all m.d.exts. of T . In view of (10) the boundary condition defining the m.d.ext. T_K can be presented as

$$f \in \text{Ker}(\Gamma_- - K\Gamma_+), \quad \text{or} \quad (\Gamma_- - K\Gamma_+)f = 0,$$

since, evidently, $\mathfrak{L}_K = \text{Ker}(\Gamma_- - K\Gamma_+)$. An operator of the form $\Gamma_- - K\Gamma_+$ shall be referred to as a boundary operator.

Clearly, hypermaximal W -symmetric subspaces in \mathfrak{G} are hypermaximal neutral subspaces in terminology of Kreĭn spaces, and the boundary operator defining a self-adjoint extension of T is $\Gamma_- - V\Gamma_+$, where $V \in [\mathfrak{G}_+, \mathfrak{G}_-]$ is an isometry.

We complete this section with following remarks.

Remark 2 For the case of a bounded reduction operator $\{\Gamma, \mathfrak{G}\}$ the Stone's characterization of reduction operators can be modified as follows. Let the operator $\tilde{U} \in [\mathfrak{G}, \tilde{\mathfrak{G}}]$ be a $(\mathcal{J}, \tilde{\mathcal{J}})$ -isometry, that is $\tilde{U}\mathcal{J}\tilde{U}^* = \tilde{\mathcal{J}}$, where $\tilde{\mathcal{J}}^* = \tilde{\mathcal{J}}$, $\tilde{\mathcal{J}}^2 = I_{\tilde{\mathfrak{G}}}$. Then \tilde{U} is bounded invertible, $\tilde{U}^{-1} = \mathcal{J}\tilde{U}^*\tilde{\mathcal{J}}$, hence $\{\tilde{\Gamma}, \tilde{\mathfrak{G}}\}$, where $\tilde{\Gamma} = \tilde{U}^{-*}\Gamma$, is a bounded reduction operator for T^* as well.

Remark 3 In [18] it is shown that there exists a unitary operator $U \in [\mathfrak{G}]$ such that with the use of (10) the reduction operator $\{U\Gamma, \mathfrak{G}\}$ turns to the widely known concept of a boundary triplet, presented in [1], [11], [12], [8; 3.4].

2 Characteristic functions of maximal dissipative extensions

Denote $\mathcal{T}_+(T)$ the set of all m.d.exts. of T . If $\mathcal{K}(\gamma)$ is the set of all contractive operators $K \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$, $\|K\| \leq 1$, then the relation

$$\mathcal{T}_+(T) \ni T_K = T^*|_{\text{Ker}(\mathcal{Q}_{\bar{\gamma}} - K\mathcal{Q}_\gamma)} \leftrightarrow K \in \mathcal{K}(\gamma)$$

establishes an one-to-one correspondence between the sets $\mathcal{T}_+(T)$ and $\mathcal{K}(\gamma)$.

The initial purpose here is to present ch.f. of arbitrary m.d.ext. T_K . The cases $\|K\| < 1$ and $\|K\| = 1$ are diverse in the sense that defect operators $I_\gamma - K^*K$, $I_{\bar{\gamma}} - KK^*$ are bounded invertible in the first case, that cannot be occurred in the other case.

2.1. Throughout this subsection we assume that K is a strict contractions, $\|K\| < 1$. Decomposition (1) can be presented for an arbitrary nonreal φ , defining oblique projections \mathcal{Q}_φ , $\mathcal{Q}_{\bar{\varphi}}$ in $\mathcal{D}(T^*)$ onto \mathfrak{N}_φ , $\mathfrak{N}_{\bar{\varphi}}$ respectively. For arbitrary nonreal φ , ψ introduce the operators

$$\Theta(\varphi, \psi) = \mathcal{Q}_\varphi|_{\mathfrak{N}_\psi} \in [\mathfrak{N}_\psi, \mathfrak{N}_\varphi], \quad \text{so } \Theta(\varphi, \varphi) = I_\varphi, \quad \Theta(\varphi, \bar{\varphi}) = 0.$$

In [17] their following properties are proved:

- a) $\Theta(\varphi, \phi)\Theta(\phi, \psi) + \Theta(\varphi, \bar{\phi})\Theta(\bar{\phi}, \psi) = \Theta(\varphi, \psi)$,
- b) $\Theta^*(\varphi, \psi) = \frac{Im\psi}{Im\varphi}\Theta(\psi, \varphi)$,
- c) if $Im\varphi \cdot Im\psi > 0$, then $\Theta(\varphi, \psi)$ is boundedly invertible.

From now on $\gamma \in C^+$ be fixed, and λ varies on C^+ . Then:

- d) the operator function

$$\Theta_\gamma(\lambda) = \Theta(\bar{\gamma}, \lambda)\Theta^{-1}(\gamma, \lambda) \in [\mathfrak{N}_\gamma; \mathfrak{N}_{\bar{\gamma}}] \quad (1)$$

is a strict contractive analytic function on C^+ , and it holds that

$$\Theta_\gamma^*(\lambda) = \Theta(\gamma, \bar{\lambda})\Theta^{-1}(\bar{\gamma}, \bar{\lambda}) = \Theta_{\bar{\gamma}}(\bar{\lambda}),$$

e) the operator $\Theta_\gamma(\lambda)$ is such that an arbitrary $f_\lambda \in \mathfrak{N}_\lambda$ admits the unique presentation

$$f_\lambda = f_0 + f_\gamma + \Theta_\gamma(\lambda)f_\gamma, \quad f_0 \in \mathcal{D}(T), \quad (2)$$

and f_γ varies over the entire \mathfrak{N}_γ , when f_λ runs on \mathfrak{N}_λ .

Consider maximal dissipative and accumulative extensions

$$T_\gamma = T^*|_{Ker \mathcal{Q}_{\bar{\gamma}}}, \quad T_{\bar{\gamma}} = T^*|_{Ker \mathcal{Q}_\gamma} \quad (3)$$

of T . Since they are maximal extensions, and, evidently,

$$\langle T_\gamma f, g \rangle = \langle f, T_{\bar{\gamma}} g \rangle, \quad f \in Ker \mathcal{Q}_{\bar{\gamma}}, \quad g \in Ker \mathcal{Q}_\gamma,$$

hence $T_{\bar{\gamma}} = T_\gamma^*$.

Now consider an arbitrary strict contraction $K \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$ and introduce dissipative and accumulative extensions of T , defined as

$$T_K = T^*|_{Ker(\mathcal{Q}_{\bar{\gamma}} - K\mathcal{Q}_\gamma)}, \quad T_{K^*} = T^*|_{Ker(\mathcal{Q}_\gamma - K^*\mathcal{Q}_{\bar{\gamma}})}. \quad (4)$$

Again, maximality of T_K, T_{K^*} yields $T_K^* = T_{K^*}$.

Consider the Cayley transforms

$$C_\gamma = (T_\gamma - \gamma I)(T_\gamma - \bar{\gamma} I)^{-1} = I - (\gamma - \bar{\gamma})(T_\gamma - \bar{\gamma} I)^{-1}, \quad (5)$$

$$C_K = (T_K - \gamma I)(T_K - \bar{\gamma} I)^{-1} = I - (\gamma - \bar{\gamma})(T_K - \bar{\gamma} I)^{-1} \quad (6)$$

of T_γ, T_K , which are contractions in \mathfrak{H} .

Proposition 1 *Cayley transforms C_K, C_γ of T_K, T_γ are related by the formula*

$$C_K = C_\gamma - KP_\gamma = C_\gamma - P_{\bar{\gamma}}KP_\gamma, \quad (7)$$

where $P_\gamma, P_{\bar{\gamma}}$ are orthogonal projections in \mathfrak{H} onto $\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}$ respectively.

Proof. Let $f \in \mathfrak{H}$ and $g_K = (T_K - \bar{\gamma} I)^{-1} f \in \mathcal{D}(T_K)$, so $g_K = g_0 + g_\gamma + Kg_\gamma$. Then

$$f = (T_K - \bar{\gamma} I)g_K = Tg_0 + \gamma g_\gamma + \bar{\gamma}Kg_\gamma - \bar{\gamma}g_0 - \bar{\gamma}g_\gamma - \bar{\gamma}Kg_\gamma = (T_\gamma - \bar{\gamma} I)g,$$

where $g = g_0 + g_\gamma \in \mathcal{D}(T_\gamma)$. In view of $(T_\gamma - \bar{\gamma} I)^{-1} f = g$ and $g_K = g + Kg_\gamma$ we have

$$g_K = (T_K - \bar{\gamma} I)^{-1} f = (T_\gamma - \bar{\gamma} I)^{-1} f + K\hat{\mathcal{Q}}_\gamma(T_\gamma - \bar{\gamma} I)^{-1} f,$$

where $\hat{\mathcal{Q}}_\gamma$ is the projection in $\mathcal{D}(T_\gamma)$ onto \mathfrak{N}_γ . In [17] it is shown, that

$$\hat{\mathcal{Q}}_\gamma = \frac{1}{\gamma - \bar{\gamma}} P_\gamma (T_\gamma - \gamma I),$$

hence

$$(T_K - \bar{\gamma} I)^{-1} f = (T_\gamma - \bar{\gamma} I)^{-1} f + \frac{1}{\gamma - \bar{\gamma}} KP_\gamma f,$$

and (6) now takes the form

$$C_K = I - (\gamma - \bar{\gamma}) \left[(T_\gamma - \bar{\gamma}I)^{-1} + \frac{1}{\gamma - \bar{\gamma}} K P_\gamma \right] = C_\gamma - K P_\gamma$$

on account of (5). The proof is finished.

Clearly, $C_\gamma^* = C_{\bar{\gamma}}$, $C_K^* = C_{K^*}$ are Cayley transforms of $T_\gamma^* = T_{\bar{\gamma}}$, $T_K^* = T_{K^*}$ respectively, and

$$C_K^* = C_\gamma^* - P_\gamma K^* P_{\bar{\gamma}} = C_\gamma^* - K^* P_{\bar{\gamma}}. \quad (8)$$

The Nagy-Foias ch.f. of a contraction $C \in [\mathfrak{H}]$ is an analytic operator function, defined by (see [20], VI. 1)

$$\Theta_C(\omega) = [-C + D_* \omega (I - \omega C^*)^{-1} D] | \overline{D\mathfrak{H}}, \quad |\omega| < 1, \quad (9)$$

where $D = (I - C^*C)^{\frac{1}{2}}$, $D_* = (I - CC^*)^{\frac{1}{2}}$ are defect operators of C , and

$$CD = D_*C, \quad C^*D_* = DC^*. \quad (10)$$

In [17] it is proved that

$$I - C_\gamma^* C_\gamma = P_\gamma, \quad I - C_\gamma C_\gamma^* = P_{\bar{\gamma}}; \quad C_\gamma P_\gamma = P_{\bar{\gamma}} C_\gamma = 0, \quad C_\gamma^* P_{\bar{\gamma}} = P_\gamma C_\gamma^* = 0, \quad (11)$$

hence from (7) and (8) we get

$$I - C_K^* C_K = I - (C_\gamma^* - P_\gamma K^* P_{\bar{\gamma}}) (C_\gamma - P_{\bar{\gamma}} K P_\gamma) = P_\gamma (I_\gamma - K^* K) P_\gamma,$$

and, similarly, $I - C_K C_K^* = P_{\bar{\gamma}} (I_{\bar{\gamma}} - K K^*) P_{\bar{\gamma}}$.

Thus, for defect operators of contraction C_K one has

$$D_K = P_\gamma (I_\gamma - K^* K)^{\frac{1}{2}} P_\gamma, \quad D_{K^*} = P_{\bar{\gamma}} (I_{\bar{\gamma}} - K K^*)^{\frac{1}{2}} P_{\bar{\gamma}}. \quad (12)$$

Theorem 2 *Let C_K, C_γ be Cayley transforms of T_K, T_γ respectively. Then the Nagy-Foias ch.f. of contraction C_K is determined by*

$$\Theta_{C_K}(\omega) = (I_{\bar{\gamma}} - K K^*)^{-\frac{1}{2}} [K + \Theta_{C_\gamma}(\omega)] [I_\gamma + K^* \Theta_{C_\gamma}(\omega)]^{-1} (I_\gamma - K^* K)^{\frac{1}{2}}, \quad (13)$$

where $\Theta_{C_\gamma}(\omega)$ is the Nagy-Foias ch.f. of contraction C_γ .

Proof. Since $\|K\| < 1$, hence the operators $(I_\gamma - K^* K)^{-\frac{1}{2}}$, $(I_{\bar{\gamma}} - K K^*)^{-\frac{1}{2}}$ exist and are bounded, so $D_K \mathfrak{H} = \mathfrak{N}_\gamma$ is closed. On account of (11), from (9) it follows that we have to compute the operator

$$D_{K^*} [-C_K + \omega (I - \omega C_K^*)^{-1}] D_K^2$$

and then restrict it on $D_{\mathbf{K}}\mathfrak{H}$.

The use of relations (11) and (8) lead to

$$-D_{\mathbf{K}^*}C_{\mathbf{K}} = -P_{\bar{\gamma}}(I_{\bar{\gamma}} - \mathbf{K}\mathbf{K}^*)^{-\frac{1}{2}}P_{\bar{\gamma}}(C_{\gamma} - \mathbf{K}P_{\gamma}) = D_{\mathbf{K}^*}\mathbf{K}P_{\gamma},$$

thus the operator to deal with is

$$D_{\mathbf{K}^*} [P_{\bar{\gamma}}\mathbf{K}P_{\gamma} + P_{\bar{\gamma}}\omega(I - \omega C_{\mathbf{K}}^*)^{-1}P_{\gamma}(I_{\gamma} - \mathbf{K}^*\mathbf{K})P_{\gamma}]. \quad (14)$$

Taking into account (8) we derive

$$\begin{aligned} P_{\bar{\gamma}}\omega(I - \omega C_{\mathbf{K}}^*)^{-1}P_{\gamma} &= P_{\bar{\gamma}}\omega [(I - \omega C_{\gamma}^*) + \omega P_{\gamma}\mathbf{K}^*P_{\bar{\gamma}}]^{-1}P_{\gamma} = \\ &= P_{\bar{\gamma}}\omega(I - \omega C_{\mathbf{K}}^*)^{-1} [I + P_{\gamma}\mathbf{K}^*P_{\bar{\gamma}}\omega(I - \omega C_{\gamma}^*)^{-1}]^{-1}P_{\gamma}. \end{aligned} \quad (15)$$

Relative to the orthogonal decomposition $\mathfrak{H} = \mathfrak{N}_{\gamma} \oplus (\mathfrak{N}_{\gamma})^{\perp}$, a block-matrix presentation of the operator in square brackets above is of the following prototype

$$\begin{bmatrix} I_{\gamma} + A & B \\ 0 & I|(\mathfrak{N}_{\gamma})^{\perp} \end{bmatrix}^{-1} = \begin{bmatrix} (I_{\gamma} + A)^{-1} & -(I_{\gamma} + A)^{-1}B \\ 0 & I|(\mathfrak{N}_{\gamma})^{\perp} \end{bmatrix},$$

and since $P_{\gamma} = \begin{bmatrix} I_{\gamma} & 0 \\ 0 & 0 \end{bmatrix}$, hence

$$\begin{bmatrix} I_{\gamma} + A & B \\ 0 & I|(\mathfrak{N}_{\gamma})^{\perp} \end{bmatrix}^{-1} \begin{bmatrix} I_{\gamma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (I_{\gamma} + A)^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

which means that

$$\begin{aligned} [I + P_{\gamma}\mathbf{K}^*P_{\bar{\gamma}}\omega(I - \omega C_{\gamma}^*)^{-1}]^{-1}P_{\gamma} &= \\ &= P_{\gamma} [I_{\gamma} + \mathbf{K}^*P_{\bar{\gamma}}\omega(I - \omega C_{\gamma}^*)^{-1}P_{\gamma}]^{-1}P_{\gamma}. \end{aligned}$$

Thus formula (15) can be written as

$$\begin{aligned} P_{\bar{\gamma}}\omega(I - \omega C_{\gamma}^*)^{-1}P_{\gamma} [I_{\gamma} + \mathbf{K}^*P_{\bar{\gamma}}\omega(I - \omega C_{\gamma}^*)^{-1}P_{\gamma}]^{-1}P_{\gamma} &= \\ &= \Theta_{C_{\gamma}}(\omega) (I_{\gamma} + \mathbf{K}^*\Theta_{C_{\gamma}}(\omega))^{-1}P_{\gamma}, \end{aligned}$$

where $\Theta_{C_{\gamma}}(\omega) = P_{\bar{\gamma}}\omega(I - \omega C_{\gamma}^*)^{-1}P_{\gamma}$ is the Nagy-Foias ch.f. of contraction C_{γ} (see [17]).

Now, omitting the notation of projections P_{γ} , $P_{\bar{\gamma}}$ where they act as identity operators, formula (14) can be transformed as follows

$$\begin{aligned} D_{\mathbf{K}^*}P_{\bar{\gamma}} [\mathbf{K}P_{\gamma} + \Theta_{C_{\gamma}}(\omega) (I_{\gamma} + \mathbf{K}^*\Theta_{C_{\gamma}}(\omega))^{-1} (I_{\gamma} - \mathbf{K}^*\mathbf{K})] P_{\gamma} &= \\ D_{\mathbf{K}^*} [\mathbf{K} (I_{\gamma} - \mathbf{K}^*\mathbf{K})^{-1} + \Theta_{C_{\gamma}}(\omega) (I_{\gamma} + \mathbf{K}^*\Theta_{C_{\gamma}}(\omega))^{-1}] (I_{\gamma} - \mathbf{K}^*\mathbf{K}) P_{\gamma} &= \\ = (I_{\bar{\gamma}} - \mathbf{K}\mathbf{K}^*)^{-\frac{1}{2}} [\mathbf{K} + (I_{\gamma} - \mathbf{K}^*\mathbf{K}) \Theta_{C_{\gamma}}(\omega) (I_{\gamma} + \mathbf{K}^*\Theta_{C_{\gamma}}(\omega))^{-1}] (I_{\gamma} - \mathbf{K}^*\mathbf{K}), \end{aligned}$$

since, evidently, $K(I_\gamma - K^*K)^{-1} = (I_{\bar{\gamma}} - KK^*)^{-1}K$.

The operator in square brackets is

$$\begin{aligned} & [K(I_\gamma + K^*\Theta_{C_\gamma}(\omega)) + (I_\gamma - K^*K)\Theta_{C_\gamma}(\omega)](I_\gamma + K^*\Theta_{C_\gamma}(\omega))^{-1} = \\ & = (K + \Theta_{C_\gamma}(\omega))(I_\gamma + K^*\Theta_{C_\gamma}(\omega))^{-1} \end{aligned}$$

and, finally, restricting the operator derived to $D_K\mathfrak{H}$, we obtain formula (13). The proof is complete.

Characteristic functions of dissipative operators T_K, T_γ are defined as

$$\Theta_{T_K}(\lambda) = \Theta_{C_K} \left(\frac{\lambda - \gamma}{\lambda - \bar{\gamma}} \right), \quad \Theta_{T_\gamma}(\lambda) = \Theta_{C_\gamma} \left(\frac{\lambda - \gamma}{\lambda - \bar{\gamma}} \right),$$

where the fractional linear function $\lambda = \frac{\gamma - \omega\bar{\gamma}}{1 - \omega}$ maps the unit disk $|\omega| < 1$ onto the open half-plane C^+ (see [8], Sec. 28.12). In [17] it is proved that $\Theta_{T_\gamma}(\lambda) = -\Theta_\gamma(\lambda)$, where $\Theta_\gamma(\lambda)$ is defined by (1). Thus one has the following corollary of a previous theorem.

Theorem 3 *Let maximal dissipative extension T_K of T be determined by $T_K = T^*|Ker(\mathcal{Q}_{\bar{\gamma}} - K\mathcal{Q}_\gamma)$, $K \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$, $\|K\| < 1$. Then its Nagy-Foias characteristic function is an analytic in C^+ operator function with strict contractive values in $[\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$, determined by the formula*

$$\Theta_{T_K}(\lambda) = (I_{\bar{\gamma}} - KK^*)^{-\frac{1}{2}}(K - \Theta_\gamma(\lambda))(I_\gamma - K^*\Theta_\gamma(\lambda))^{-1}(I_\gamma - K^*K)^{\frac{1}{2}}. \quad (16)$$

Proof. We have only to verify that $\|\Theta_{T_K}(\lambda)\| < 1$, $\lambda \in C^+$. Given Hilbert spaces $\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}$ as closed subspaces of \mathfrak{H} , introduce their direct sum $\mathfrak{N}_{\gamma\bar{\gamma}} = \mathfrak{N}_\gamma \oplus \mathfrak{N}_{\bar{\gamma}}$ of pairs $(f_\gamma, f_{\bar{\gamma}})$, $f_\gamma \in \mathfrak{N}_\gamma, f_{\bar{\gamma}} \in \mathfrak{N}_{\bar{\gamma}}$, and the following operators in $[\mathfrak{N}_{\gamma\bar{\gamma}}]$

$$U = \begin{bmatrix} (I - K^*K)^{-\frac{1}{2}} & (I - K^*K)^{-\frac{1}{2}}K^* \\ (I - KK^*)^{-\frac{1}{2}}K & (I - KK^*)^{-\frac{1}{2}} \end{bmatrix}, \quad \mathcal{J}_\gamma = \begin{bmatrix} I_\gamma & 0 \\ 0 & -I_{\bar{\gamma}} \end{bmatrix}.$$

Clearly, $U^* = U$, and U is \mathcal{J}_γ -unitary, that is $U\mathcal{J}_\gamma U = \mathcal{J}_\gamma$. Formula (6) can be written as $\Theta_{T_K}(\lambda) = \Phi_U(-\Theta_\gamma(\lambda))$, where $\Phi_U(\cdot)$ is the Kreĩn-Shmylyan fractional linear transformation, possessing an interspherical property (see [14], Th. 1.1), that is $\|\Theta_{T_K}(\lambda)\| < 1$, since $\|\Theta_\gamma(\lambda)\| < 1$. The proof is complete.

2.2. Let $v \in C^+$ be arbitrary but $v \neq \gamma$. Consider m.d.ext. $T_v = T^*|Ker\mathcal{Q}_{\bar{v}}$ and its ch.f.

$$\Theta_{T_v}(\lambda) = -\Theta_v(\lambda) = -\Theta(\bar{v}, \lambda)\Theta^{-1}(v, \lambda) \in [\mathfrak{N}_v, \mathfrak{N}_{\bar{v}}].$$

Note that factors of $\Theta_v(\lambda)$ are not analytic functions. The domain $\mathcal{D}(T_v) = \mathcal{D}(T) \dot{+} \mathfrak{N}_v$ can be presented by means of defect subspaces $\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}$ also, with the use of property e) in p. 2.2, namely, since

$$\mathfrak{N}_v = \{f \in \mathcal{D}(T^*), f = f_0 + f_\gamma + \Theta_\gamma(v)f_\gamma, f_0 \in \mathcal{D}(T), f_\gamma \in \mathfrak{N}_\gamma\},$$

hence the domain of T_v is $\mathcal{D}(T_v) = \text{Ker}(\mathcal{Q}_{\bar{\gamma}} - \Theta_\gamma(v)\mathcal{Q}_\gamma)$.

Set $K = \Theta_\gamma(v)$ and denote $\tilde{\Theta}_{T_v}(\lambda) \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$ the ch.f. of T_v , defined by formula (16). In the following its part

$$(\Theta_\gamma(v) - \Theta_\gamma(\lambda))(I_\gamma - \Theta_\gamma^*(v)\Theta_\gamma(\lambda))^{-1}, \quad \lambda \in C^+, \quad (17)$$

the factors now are analytic functions. In what follows the above function shall be referred to as a principle part of ch.f.

Recall that two contractive analytic functions $\Theta(\lambda) \in [\mathfrak{G}, \mathfrak{G}_*]$, $\tilde{\Theta}(\lambda) \in [\tilde{\mathfrak{G}}, \tilde{\mathfrak{G}}_*]$ are said to be coinciding in the sense of Nagy-Foias, if there exist isometries $Z \in [\mathfrak{G}, \tilde{\mathfrak{G}}]$, $Z_* \in [\mathfrak{G}_*, \tilde{\mathfrak{G}}_*]$ such that $\tilde{\Theta}(\lambda) = Z_*\Theta(\lambda)Z^{-1}$ (see [20], V. 2.4).

Theorem 4 *Characteristic functions*

$$\Theta_{T_v}(\lambda) \in [\mathfrak{N}_v, \mathfrak{N}_{\bar{v}}] \text{ and } \tilde{\Theta}_{T_v}(\lambda) \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$$

coincide in the sense of Nagy-Foias.

Proof. With the use of properties a) - c), the first factor of (17) can be transformed as follows

$$\begin{aligned} \Theta_\gamma(v) - \Theta_\gamma(\lambda) &= \Theta_\gamma^*(\bar{v}) - \Theta_\gamma(\lambda) = \Theta^{-*}(\bar{\gamma}, \bar{v})\Theta^*(\gamma, \bar{v}) - \Theta(\bar{\gamma}, \lambda)\Theta^{-1}(\gamma, \lambda) = \\ &= \Theta^{-*}(\bar{\gamma}, \bar{v})[\Theta^*(\gamma, \bar{v})\Theta(\gamma, \lambda) - \Theta^*(\bar{\gamma}, \bar{v})\Theta(\bar{\gamma}, \lambda)]\Theta^{-1}(\gamma, \lambda) = \\ &= -\frac{\text{Im}\bar{v}}{\text{Im}\bar{\gamma}}\Theta^{-*}(\bar{\gamma}, \bar{v})[\Theta(\bar{v}, \gamma)\Theta(\gamma, \lambda) + \Theta(\bar{v}, \bar{\gamma})\Theta(\bar{\gamma}, \lambda)]\Theta^{-1}(\gamma, \lambda) = \\ &= -\Theta^{-1}(\bar{v}, \bar{\gamma})\Theta(\bar{v}, \lambda)\Theta^{-1}(\gamma, \lambda). \end{aligned}$$

Similarly, for the second factor we obtain

$$\begin{aligned} I_\gamma - \Theta_\gamma^*(v)\Theta_\gamma(\lambda) &= \\ &= \frac{\text{Im}v}{\text{Im}\gamma}\Theta^{-*}(\gamma, v)[\Theta(v, \gamma)\Theta(\gamma, \lambda) + \Theta(v, \bar{\gamma})\Theta(\bar{\gamma}, \lambda)]\Theta^{-1}(\gamma, \lambda) = \\ &= \Theta^{-1}(v, \gamma)\Theta(v, \lambda)\Theta^{-1}(\gamma, \lambda), \end{aligned}$$

hence the principle part of $\tilde{\Theta}_{T_v}(\lambda)$ is

$$-\Theta^{-1}(\bar{v}, \bar{\gamma})\Theta_v(\lambda)\Theta(v, \gamma) = \Theta^{-1}(\bar{v}, \bar{\gamma})\Theta_{T_v}(\lambda)\Theta(v, \gamma).$$

With the boundedly invertible defect operators

$$D_v = [I_\gamma - \Theta_\gamma^*(v)\Theta_\gamma(v)]^{\frac{1}{2}}, \quad D_{v^*} = [I_{\bar{\gamma}} - \Theta_\gamma(v)\Theta_\gamma^*(v)]^{\frac{1}{2}}$$

the formula (16) for $\tilde{\Theta}_{T_v}(\lambda)$ takes the form

$$\tilde{\Theta}_{T_v}(\lambda) = [\Theta(\bar{v}, \bar{\gamma})D_{v^*}]^{-1} \Theta_{T_v}(\lambda) [\Theta(v, \gamma)D_v]. \quad (18)$$

It is not difficult to verify that

$$D_v^2 = [\Theta(\gamma, v)\Theta(v, \gamma)]^{-1}, \quad D_{v^*}^2 = [\Theta(\bar{\gamma}, \bar{v})\Theta(\bar{v}, \bar{\gamma})]^{-1}.$$

Denote $Z = \rho^{-\frac{1}{2}}\Theta(v, \gamma)D_v$, $Z_x = \rho^{-\frac{1}{2}}\Theta(\bar{v}, \bar{\gamma})D_{v^*}$, where $\rho = \frac{Im\gamma}{Imv}$. Then

$$\begin{aligned} Z^*Z &= \frac{1}{\rho} [\Theta(\gamma, v)\Theta(v, \gamma)]^{-\frac{1}{2}} \Theta^*(v, \gamma)\Theta(v, \gamma) [\Theta(\gamma, v)\Theta(v, \gamma)]^{-\frac{1}{2}} = \\ &= [\Theta(\gamma, v)\Theta(v, \gamma)]^{-\frac{1}{2}} \Theta(v, \gamma)\Theta(v, \gamma) [\Theta(\gamma, v)\Theta(v, \gamma)]^{-\frac{1}{2}} = \mathcal{I}_\gamma, \end{aligned}$$

since $\Theta^*(v, \gamma) = \rho\Theta(\gamma, v)$, and, analogously, $Z_x^*Z_x = \mathcal{I}_{\bar{\gamma}}$. Clearly, (18) is $\tilde{\Theta}_{T_v}(\lambda) = Z_x^{-1}\Theta_{T_v}(\lambda)Z$. This finishes the proof.

2.3. Here we assume that $\|K\| = 1$, hence the point 1 belongs to a spectrum of both K^*K and KK^* . Then, either it is in their continuous spectrum $\sigma_c(K^*K)$, $\sigma_c(KK^*)$, or else is in their point spectrum $\sigma_p(K^*K)$, $\sigma_p(KK^*)$.

Indeed, if $1 \in \sigma_c(K^*K)$ and $KK^*f_{\bar{\gamma}} = f_{\bar{\gamma}}$, $f_{\bar{\gamma}} \neq 0$ then $f_\gamma = K^*f_{\bar{\gamma}}$, $f_\gamma \neq 0$, and $K^*Kf_\gamma = f_\gamma$, which contradicts $1 \in \sigma_c(K^*K)$.

In the first case unbounded operators $(I_\gamma - K^*K)^{-1}$, $(I_{\bar{\gamma}} - KK^*)^{-1}$ exist simultaneously, and are defined densely. The second case is resumed hereinafter as the analog of a canonical decomposition for a contraction in a Hilbert space (see [20], Th. 3.2).

Proposition 5 *Let $K \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$ and $1 \in \sigma_p(K^*K)$, $1 \in \sigma_p(KK^*)$. Then the subspaces \mathfrak{N}_γ , $\mathfrak{N}_{\bar{\gamma}}$ admit orthogonal decompositions*

$$\mathfrak{N}_\gamma = \mathfrak{N}_\gamma^0 \oplus \mathfrak{N}_\gamma^1, \quad \mathfrak{N}_{\bar{\gamma}} = \mathfrak{N}_{\bar{\gamma}}^0 \oplus \mathfrak{N}_{\bar{\gamma}}^1; \quad \dim \mathfrak{N}_\gamma^0 = \dim \mathfrak{N}_{\bar{\gamma}}^0, \quad (19)$$

reducing K to $K_0 = K|_{\mathfrak{N}_\gamma^0} \in [\mathfrak{N}_\gamma^0, \mathfrak{N}_{\bar{\gamma}}^0]$, $K_1 = K|_{\mathfrak{N}_\gamma^1} \in [\mathfrak{N}_\gamma^1, \mathfrak{N}_{\bar{\gamma}}^1]$ such that $K_0^* = K^*|_{\mathfrak{N}_{\bar{\gamma}}^0}$, $K_1^* = K^*|_{\mathfrak{N}_{\bar{\gamma}}^1}$. The operator K_0 is an isometry, and operators $(I_\gamma^1 - K_1^*K_1)^{-1}$, $(I_{\bar{\gamma}}^1 - K_1K_1^*)^{-1}$ exist simultaneously either as bounded, or else as densely defined unbounded operators.

We shall only sketch the proof, since details are verified without difficulties.

Set $\mathfrak{N}_\gamma^0 = Ker(I_\gamma - K^*K)$, $\mathfrak{N}_{\bar{\gamma}}^0 = Ker(I_{\bar{\gamma}} - KK^*)$, $\mathfrak{N}_\gamma^1 = \mathfrak{N}_\gamma \ominus \mathfrak{N}_\gamma^0$, and $\mathfrak{N}_{\bar{\gamma}}^1 = \mathfrak{N}_{\bar{\gamma}} \ominus \mathfrak{N}_{\bar{\gamma}}^0$. We have seen earlier that if $f_\gamma \in \mathfrak{N}_\gamma^0$, then $Kf_\gamma \in \mathfrak{N}_{\bar{\gamma}}^0$. It is

also clear that $Kf_\gamma \neq Kg_\gamma$, if $f_\gamma \neq g_\gamma$, $f_\gamma, g_\gamma \in \mathfrak{N}_\gamma^0$. The same is true if \mathfrak{N}_γ^0 and K^* are considered, hence $\dim \mathfrak{N}_\gamma^0 = \dim \mathfrak{N}_\gamma^0$. The rest, concerning to the operators K_0, K_1 and their adjoints, is checked readily.

It is also clear that either the point 1 is in a resolvent set of each $K_1^*K_1, K_1K_1^*$, or else is in their continuous spectrum, which completes this rough draft.

Now we are back to Th. 3. Essentially, we have to specify it only for the case $1 \in \sigma_c(K^*K) \cap \sigma_c(KK^*)$.

Indeed, let $\dim \mathfrak{N}_\gamma^0 = \dim \mathfrak{N}_\gamma^0 > 0$ and $K_0 \in [\mathfrak{N}_\gamma^0, \mathfrak{N}_\gamma^0]$ be the isometry of Prop. 5. Then the operator $\hat{T} = T^*|_{\mathcal{D}(\hat{T})}$ with the domain

$$\mathcal{D}(\hat{T}) = \{f \in \mathcal{D}(T^*); f = f_0 + f_\gamma^0 + K_0f_\gamma^0, f_0 \in \mathcal{D}(T), f_\gamma^0 \in \mathfrak{N}_\gamma^0\}$$

is a symmetric extension of T , and one has the chain of inclusions

$$T \subset \hat{T} \subset \hat{T}^* \subset T^*.$$

Since $\mathfrak{N}_\gamma^0 \perp \mathfrak{N}_\gamma^1$, it is readily verified, that

$$\langle \hat{T}f, g_\gamma^1 \rangle = \langle f, \gamma g_\gamma^1 \rangle \text{ for arbitrary } f \in \mathcal{D}(\hat{T}), g \in \mathfrak{N}_\gamma^1,$$

hence $g_\gamma^1 \in \mathcal{D}(\hat{T}^*), \hat{T}^*g_\gamma^1 = \gamma g_\gamma^1$. Thus \mathfrak{N}_γ^1 is a defect subspace of \hat{T} , and, similarly, \mathfrak{N}_γ^1 is, hence

$$\mathcal{D}(\hat{T}^*) = \mathcal{D}(\hat{T}) \dot{+} \mathfrak{N}_\gamma^1 \dot{+} \mathfrak{N}_\gamma^1,$$

and the operator $\hat{T}_{K_1} = \hat{T}^*|_{\mathcal{D}(\hat{T}_{K_1})}$ with the domain

$$\mathcal{D}(\hat{T}_{K_1}) = \left\{ f \in \mathcal{D}(\hat{T}^*); f = f_0 + f_\gamma^1 + K_1f_\gamma^1, f_0 \in \mathcal{D}(\hat{T}), f_\gamma^1 \in \mathfrak{N}_\gamma^1 \right\}$$

is a m.d.ext. of \hat{T} .

Introducing orthogonal projections P_γ^0, P_γ^1 in \mathfrak{N}_γ onto $\mathfrak{N}_\gamma^0, \mathfrak{N}_\gamma^1$, from Prop. 5 we have $K_0 = KP_\gamma^0, K_1 = KP_\gamma^1$, and it is not difficult to obtain the expected result $\hat{T}_{K_1} = T_K$.

Obviously, this case is occurred if and only if extensions T_K, T_K^* are not relatively prime, that is $\mathcal{D}(T)$ is a proper submanifold of $\mathcal{D}(T_K) \cap \mathcal{D}(T_K^*)$.

If the operators $(I_\gamma^1 - K_1K_1^*)^{-1}, (I_\gamma^1 - K_1^*K_1)^{-1}$ are bounded, then to the Hermitian operator \hat{T} and its m.d.ext. $\hat{T}_{K_1} = T_K$ is directly applied Th. 3 and results ch.f. of T_K as the operator function $\Theta_{T_K}(\lambda)$ with values in $[\mathfrak{N}_\gamma^1, \mathfrak{N}_\gamma^1]$, and analytic in C^+ .

In particular, $\|K_1\| < 1$ if $\dim \mathfrak{N}_\gamma^1 = \dim \mathfrak{N}_\gamma^1 = n < \infty$, or the operator K_1 is absolutely continuous.

On account of stated above, consider the only case left, $(I_\gamma - KK^*)^{-1}, (I_\gamma - K^*K)^{-1}$ are densely defined unbounded operators.

To justify the validity of derivations in the proof of Th. 2, concerning to defect operators and their square roots in this case too, consider the Hilbert space $\mathfrak{N}_{\gamma\bar{\gamma}} = \mathfrak{N}_{\gamma} \oplus \mathfrak{N}_{\bar{\gamma}}$ in the proof of Th. 3, and acting there self-adjoint operators

$$\mathcal{K} = \begin{bmatrix} 0 & \mathbf{K}^* \\ \mathbf{K} & 0 \end{bmatrix}, (\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-1} = \begin{bmatrix} (I_{\gamma} - \mathbf{K}^*\mathbf{K})^{-1} & 0 \\ 0 & (I_{\bar{\gamma}} - \mathbf{K}\mathbf{K}^*)^{-1} \end{bmatrix},$$

$$\mathcal{D}(\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-1} = \text{Ran}(\mathcal{I}_{\gamma} - \mathcal{K}^2).$$

In [22] it is proved that if unbounded self-adjoint operator A in \mathfrak{G} is positive ($\langle Ax, x \rangle_{\mathfrak{G}} \geq 0$ for all $x \in \mathcal{D}(A)$), then it possesses a unique positive self-adjoint square root, which commutes with every bounded operator B that commutes with A ($BA \supset AB$), thus there exists the operator $(\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-\frac{1}{2}}$ with the domain $\text{Ran}(\mathcal{I}_{\gamma} - \mathcal{K}^2)$.

Obviously $\mathcal{K}(\mathcal{I}_{\gamma} - \mathcal{K}^2) = (\mathcal{I}_{\gamma} - \mathcal{K}^2)\mathcal{K}$, and

$$(\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-1}(\mathcal{I}_{\gamma} - \mathcal{K}^2) = \mathcal{I}_{\gamma}, (\mathcal{I}_{\gamma} - \mathcal{K}^2)(\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-1} = \mathcal{I}_{\gamma}|_{\text{Ran}(\mathcal{I}_{\gamma} - \mathcal{K}^2)}.$$

Then $\mathcal{K} = (\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-1}\mathcal{K}(\mathcal{I}_{\gamma} - \mathcal{K}^2)$, and for arbitrary $g = (\mathcal{I}_{\gamma} - \mathcal{K}^2)f$ it holds $(\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-1}\mathcal{K}g = \mathcal{K}(\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-1}g$, hence

$$(\mathcal{I}_{\gamma} - \mathcal{K}^2)\mathcal{K}g = \mathcal{K}(\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-\frac{1}{2}}g.$$

Thus on $\mathcal{D}\left((\mathcal{I}_{\gamma} - \mathcal{K}^2)^{-\frac{1}{2}}\right) = \text{Ran}(\mathcal{I}_{\gamma} - \mathcal{K}^2)$ one has

$$\begin{aligned} & \begin{bmatrix} 0 & (I_{\gamma} - \mathbf{K}^*\mathbf{K})^{-\frac{1}{2}}\mathbf{K}^* \\ (I_{\bar{\gamma}} - \mathbf{K}\mathbf{K}^*)^{-\frac{1}{2}}\mathbf{K} & 0 \end{bmatrix} = \\ & = \begin{bmatrix} 0 & \mathbf{K}^*(I_{\bar{\gamma}} - \mathbf{K}\mathbf{K}^*)^{-\frac{1}{2}} \\ \mathbf{K}(I_{\gamma} - \mathbf{K}^*\mathbf{K})^{-\frac{1}{2}} & 0 \end{bmatrix}, \end{aligned}$$

that is equalities used in the proof of Th. 2.

2.4. The content of p. 2.2 and p. 2.3 directs to take the following notice.

Given a symmetric operator T , we are given also its defect subspaces \mathfrak{N}_{λ} and $\mathfrak{N}_{\bar{\lambda}}$ for all $\lambda \in C^+$, hence, at first hand, we have the sets of maximal dissipative and accumulative extensions T_{λ} , $T_{\bar{\lambda}}$ of formula (3). If $\mathcal{T}_+(T)$ denotes the set of all m.d.exts. of T , it can be divided, on the course of nature, onto disjoint classes, $\mathcal{T}_+^i(T)$, $\mathcal{T}_+^a(T) = \mathcal{T}_+(T) \setminus \mathcal{T}_+^i(T)$, where the class

$$\mathcal{T}_+^i(T) = \{T_{\lambda} = T^*|_{\text{Ker}Q_{\bar{\lambda}}}, \lambda \in C^+\}$$

can be referred to as the class of m.d.exts., inherited from T , so $\mathcal{T}_+^a(T)$ – as the class of acquired m.d.exts.

Now summarizing the discussion of this section we can state the following.

Theorem 6 *Let the operator function $\Theta_\gamma(\lambda)$ be given by formula (1). Then the class of inherited m.d.exts. of a Hermitian operator T with infinite defect numbers admits parameterization*

$$\mathcal{T}_+^i(T) = \{T_\lambda = T^* | Ker(\mathcal{Q}_{\bar{\gamma}} - \Theta_\gamma(\lambda)\mathcal{Q}_{\bar{\gamma}}); \lambda \in C^+\} \quad (20)$$

via the range $Ran \Theta_\gamma(\lambda) = \{\Theta_\gamma(\lambda); \lambda \in C^+\} \subset [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$ of $\Theta_\gamma(\lambda)$.

In discussion above we have presented ch.f. of an arbitrary m.d.ext. T_K , $\|K\| \leq 1$. Now the class of a given T_K can be determined by its ch.f.

First, if $\Theta_{T_K}(\lambda)$ takes values from $[\mathfrak{N}'_\gamma, \mathfrak{N}'_{\bar{\gamma}}]$ where $\mathfrak{N}'_\gamma, \mathfrak{N}'_{\bar{\gamma}}$ are proper subspaces of $\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}$, then the extension T_K is acquired.

If this is not the case, then the principle part

$$[K - \Theta_\gamma(\lambda)] [I_\gamma - K^*\Theta_\gamma(\lambda)]^{-1}, \quad \|K\| \leq 1$$

of ch.f. $\Theta_{T_K}(\lambda)$ shows that either $\Theta_{T_K}(\lambda) \neq 0$ on C^+ , or there is only one point $\lambda_0 \in C^+$ such that $\Theta_{T_K}(\lambda_0) = 0$.

Theorem 7 *A m.d.ext. T_K is an inherited extension if and only if its ch.f. takes values in $[\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$ and vanishes at some point $\lambda_0 \in C^+$.*

In completion of this subsection notice that if $\|K\| < 1$, then maximal extensions T_K, T_K^* are relatively prime, that is $\mathcal{D}(T_K) \cap \mathcal{D}(T_K^*) = \mathcal{D}(T)$. In view of bounded invertibility of operators $I_\gamma - K^*K, I_{\bar{\gamma}} - KK^*$ and formulas in (4), their transversality $\mathcal{D}(T_K) + \mathcal{D}(T_K^*) = \mathcal{D}(T^*)$ now can be presented as

$$\begin{aligned} & \mathfrak{N}_\gamma \dot{+} \mathfrak{N}_{\bar{\gamma}} = \\ & = Ker \left[(\mathcal{Q}_{\bar{\gamma}} - K\mathcal{Q}_\gamma) | (\mathfrak{N}_\gamma \dot{+} \mathfrak{N}_{\bar{\gamma}}) \right] \dot{+} Ker \left[(\mathcal{Q}_\gamma - K^*\mathcal{Q}_{\bar{\gamma}}) | (\mathfrak{N}_\gamma \dot{+} \mathfrak{N}_{\bar{\gamma}}) \right], \end{aligned}$$

and referred to as a direct transversality .

2.5. Here we introduce the Weyl function of a self-adjoint extension of T similar to ch.f. of T_γ , determined by (1). Clearly, formula (1) is equivalent to

$$Ker(\mathcal{Q}_{\bar{\gamma}} - \Theta_\gamma(\lambda)\mathcal{Q}_\gamma) = Ker\mathcal{Q}_{\bar{\lambda}},$$

meaning that given directly transversal maximal extensions $T_\gamma, T_{\bar{\gamma}} = T_\gamma^*$ with the help of $\Theta_\gamma(\lambda)$ is described the class $\mathcal{T}_+^i(T)$.

Let $\mathcal{V}(\gamma) \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$ be an isometry. Consider the pair of self-adjoint extensions $T_{\mathcal{V}\pm}$, given by the formula of von Neumann

$$\begin{aligned} \mathcal{D}(T_{\mathcal{V}\pm}) &= \{f \in \mathcal{D}(T^*), f = f_0 + f_\gamma \pm \mathcal{V}(\gamma)f_\gamma; f_0 \in \mathcal{D}(T), f_\gamma \in \mathfrak{N}_\gamma\} = \\ &= Ker(\mathcal{Q}_{\bar{\gamma}} \mp \mathcal{V}(\gamma)\mathcal{Q}_\gamma), \quad (21) \end{aligned}$$

and denote $\mathcal{Q}_\pm(\gamma) = \mathcal{Q}_{\bar{\gamma}} \mp \mathcal{V}(\gamma)\mathcal{Q}_\gamma$. Obviously, $\mathcal{Q}_\pm^2(\gamma) = \mathcal{Q}_\pm(\gamma)$, so they are oblique projections in $\mathcal{D}(T^*)$ onto $\mathfrak{N}_{\bar{\gamma}}$. Extensions $T_{\mathcal{V}_\pm}$ are relatively prime, and their property to be directly transversal

$$\mathfrak{N}_\gamma \dot{+} \mathfrak{N}_{\bar{\gamma}} = Ker \mathcal{Q}_+(\gamma) | (\mathfrak{N}_\gamma \dot{+} \mathfrak{N}_{\bar{\gamma}}) \dot{+} Ker \mathcal{Q}_-(\gamma) | (\mathfrak{N}_\gamma \dot{+} \mathfrak{N}_{\bar{\gamma}})$$

follows readily, taking into account that $\mathcal{V}(\gamma)$ is an isometry. In the decomposition

$$\mathcal{D}(T^*) = \mathcal{D}(T) \dot{+} \mathfrak{N}_\lambda \dot{+} \mathfrak{N}_{\bar{\lambda}}, \quad \lambda \in C^+$$

the same extensions are determined by isometries $\mathcal{V}_\pm(\lambda) \in [\mathfrak{N}_\lambda, \mathfrak{N}_{\bar{\lambda}}]$, and projections $\mathcal{Q}_\pm(\gamma)$ applied to an arbitrary $f = f_0 + f_\lambda + f_{\bar{\lambda}} \in \mathcal{D}(T^*)$ yield

$$\mathcal{V}_\pm(\lambda) = - [\Theta(\bar{\gamma}, \bar{\lambda}) \mp \mathcal{V}(\gamma)\Theta(\gamma, \bar{\lambda})]^{-1} [\Theta(\bar{\gamma}, \lambda) \mp \mathcal{V}(\gamma)\Theta(\gamma, \lambda)]. \quad (22)$$

For the sake of symmetry set $\mathcal{V}_+(\gamma) = \mathcal{V}(\gamma)$, $\mathcal{V}_-(\gamma) = -\mathcal{V}(\gamma)$, and employ the analog of $\mathcal{V}_-(\gamma) = -\mathcal{V}_+(\gamma)$ to $\mathcal{V}_\pm(\lambda)$ for arbitrary $\lambda \in C^+$. With the use of (1) one can obtain the following connection

$$\mathcal{V}_-(\lambda) = [\Theta(\bar{\gamma}, \bar{\lambda}) + \mathcal{V}(\gamma)\Theta(\gamma, \bar{\lambda})]^{-1} \mathcal{M}_\mathcal{V}(\lambda) [\Theta(\bar{\gamma}, \bar{\lambda}) - \mathcal{V}(\gamma)\Theta(\gamma, \bar{\lambda})] \mathcal{V}_+(\lambda), \quad (23)$$

where

$$\begin{aligned} \mathcal{M}_\mathcal{V}(\lambda) &= [\Theta(\bar{\gamma}, \lambda) + \mathcal{V}(\gamma)\Theta(\gamma, \lambda)] [\Theta(\bar{\gamma}, \lambda) - \mathcal{V}(\gamma)\Theta(\gamma, \lambda)]^{-1} = \\ &= [\Theta_\gamma(\lambda) + \mathcal{V}(\gamma)] [\Theta_\gamma(\lambda) - \mathcal{V}(\gamma)]^{-1}, \quad \mathcal{M}_\mathcal{V}(\gamma) = -I_{\bar{\gamma}} \end{aligned} \quad (24)$$

is an analytic in C^+ operator function with values in $[\mathfrak{N}_{\bar{\gamma}}]$. Presenting (23) as

$$[\Theta(\bar{\gamma}, \bar{\lambda}) + \mathcal{V}(\gamma)\Theta(\gamma, \lambda)] \mathcal{V}_-(\lambda) = \mathcal{M}_\mathcal{V}(\lambda) [\Theta(\bar{\gamma}, \bar{\lambda}) - \mathcal{V}(\gamma)\Theta(\gamma, \bar{\lambda})] \mathcal{V}_+(\lambda)$$

we have

$$\mathcal{Q}_-(\gamma)\mathcal{V}_-(\lambda) = \mathcal{M}_\mathcal{V}(\lambda)\mathcal{Q}_+(\gamma)\mathcal{V}_+(\lambda).$$

Formulas (22) can be written as $\mathcal{V}_\pm(\lambda) = [\mathcal{Q}_\pm(\gamma)|\mathfrak{N}_{\bar{\lambda}}]^{-1} [\mathcal{Q}_\pm(\gamma)|\mathfrak{N}_\lambda]$, hence, finally, we obtain

$$\mathcal{Q}_-(\gamma)|\mathfrak{N}_\lambda = \mathcal{M}_\mathcal{V}(\lambda)\mathcal{Q}_+(\gamma)|\mathfrak{N}_\lambda, \quad (25)$$

or, equivalently,

$$Ker [\mathcal{Q}_-(\gamma) - \mathcal{M}_\mathcal{V}(\lambda)\mathcal{Q}_+(\gamma)] = Ker \mathcal{Q}_{\bar{\lambda}}.$$

The function $\mathcal{M}_\mathcal{V}(\lambda)$ defined by (25) is well known as the Weyl function of the self-adjoint operator $T_\mathcal{V}$.

Determining the Weyl function for $\xi \in C^-$ by the same formula (24) one has

$$\begin{aligned}\mathcal{M}_{\mathcal{V}}(\zeta) &= [\Theta(\bar{\gamma}, \zeta) + \mathcal{V}(\gamma)\Theta(\gamma, \zeta)] [\Theta(\gamma, \bar{\zeta}) - \mathcal{V}(\gamma)\Theta(\gamma, \zeta)]^{-1} = \\ &= [I_{\bar{\gamma}} + \mathcal{V}(\gamma)\Theta_{\bar{\gamma}}(\zeta)] [I_{\bar{\gamma}} - \mathcal{V}(\gamma)\Theta_{\bar{\gamma}}(\zeta)]^{-1} = \\ &= [I_{\bar{\gamma}} - \mathcal{V}(\gamma)\Theta_{\bar{\gamma}}(\zeta)]^{-1} [I_{\bar{\gamma}} + \mathcal{V}(\gamma)\Theta_{\bar{\gamma}}(\zeta)] = \\ &= [\mathcal{V}^*(\gamma) - \Theta_{\bar{\gamma}}(\zeta)]^{-1} [\mathcal{V}^*(\gamma)\Theta_{\bar{\gamma}}(\zeta)] = -\mathcal{M}_{\mathcal{V}}^*(\bar{\zeta}),\end{aligned}$$

since $\Theta_{\gamma}^*(\bar{\zeta}) = \Theta_{\bar{\gamma}}(\zeta)$.

3 Characteristic and Weyl functions in the setting of Calkin's theory

3.1. Here, along with a Calkin reduction operator, is utilized its special case, introduced in [18]. There the existence of a reduction operator $\{\Gamma(\gamma), \mathfrak{G}\} = \{\Gamma_{\pm}(\gamma), \mathfrak{G}_{\pm}\}$ for $T^*(\gamma)$ is proved, which has the properties

- I. $\Gamma_+(\gamma)\mathfrak{N}_{\gamma} = \mathfrak{G}_+$, $\Gamma_+(\gamma)\mathfrak{N}_{\bar{\gamma}} = \{0\}$; $\Gamma_-(\gamma)\mathfrak{N}_{\gamma} = \{0\}$, $\Gamma_-(\gamma)\mathfrak{N}_{\bar{\gamma}} = \mathfrak{G}_-$,
- II. $\Gamma^*(\gamma)\Gamma(\gamma) = \mathcal{P}_{\gamma\bar{\gamma}}$, $\Gamma_+^*(\gamma)\Gamma_+(\gamma) = \mathcal{P}_{\gamma}$, $\Gamma_-^*(\gamma)\Gamma_-(\gamma) = \mathcal{P}_{\bar{\gamma}}$,

where $\mathcal{P}_{\gamma\bar{\gamma}}$, \mathcal{P}_{γ} , $\mathcal{P}_{\bar{\gamma}}$ are orthogonal projections in \mathfrak{D}_{γ} onto $\mathfrak{N}_{\gamma\bar{\gamma}} = \mathfrak{N}_{\gamma} \oplus \mathfrak{N}_{\bar{\gamma}}$, \mathfrak{N}_{γ} , $\mathfrak{N}_{\bar{\gamma}}$ in decomposition (4). In view of property II it referred to as the canonical reduction operator.

Let $\{\Gamma_{\pm}(\gamma), \mathfrak{G}_{\pm}\}$ be the canonical reduction operator. Combining formulas (6) and (10) one has

$$\begin{aligned}\langle T^*f, g \rangle - \langle f, T^*g \rangle &= \beta [\langle T^*(\gamma)f, g \rangle - \langle f, T^*(\gamma)g \rangle] = \\ &= i\beta [\langle \Gamma_+(\gamma)f, \Gamma_+(\gamma)g \rangle_{\mathfrak{G}} - \langle \Gamma_-(\gamma)f, \Gamma_-(\gamma)g \rangle_{\mathfrak{G}}],\end{aligned}\quad (1)$$

hence, evidently, reduction operator $\{\Gamma_{\pm}(\gamma), \mathfrak{G}_{\pm}\}$ serves as that for T^* , defining extensions of $T(\gamma)$ and T of the same nature.

Properties I, II mean that the operator $\Gamma_{\gamma\bar{\gamma}} := \Gamma(\gamma)|_{\mathfrak{N}_{\gamma\bar{\gamma}}} \in [\mathfrak{N}_{\gamma\bar{\gamma}}, \mathfrak{G}]$ is an isometry, and its matrix representation relative to decompositions

$$\mathfrak{N}_{\gamma\bar{\gamma}} = \mathfrak{N}_{\gamma} \oplus \mathfrak{N}_{\bar{\gamma}}, \quad \mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{G}_-$$

is the block diagonal matrix

$$\Gamma_{\gamma\bar{\gamma}} = \begin{bmatrix} \Theta_{\gamma} & 0 \\ 0 & \Theta_{\bar{\gamma}} \end{bmatrix}, \quad (2)$$

where $\Theta_{\gamma} \in [\mathfrak{N}_{\gamma}, \mathfrak{G}_+]$, $\Theta_{\bar{\gamma}} \in [\mathfrak{N}_{\bar{\gamma}}, \mathfrak{G}_-]$ are isometries too (see [18]). From I it is clear that boundary conditions $\Gamma_-(\gamma)f = 0$, $\Gamma_+(\gamma)f = 0$ define maximal extensions T_{γ} , $T_{\bar{\gamma}}$ of (3).

Introduce the operators

$$\Theta_+[\gamma, \varphi] = \Gamma_+(\gamma)|\mathfrak{N}_\varphi, \quad \Theta_-[\gamma, \varphi] = \Gamma_-(\gamma)|\mathfrak{N}_\varphi, \quad \varphi \in C_+ \cup C^-.$$

Proposition 1 *Let $\lambda = \mu + i\nu$, $\nu > 0$. Then:*

- a) *the operator $\Theta_+[\gamma, \lambda] \in [\mathfrak{N}_\lambda, \mathfrak{G}_+]$ has bounded inverse, and the operator $\Theta_\gamma[\lambda] = \Theta_-[\gamma, \lambda]\Theta_+^{-1}[\gamma, \lambda] \in [\mathfrak{G}_+, \mathfrak{G}_-]$ is a strict contraction;*
b) *the operator function $\Theta_\gamma[\lambda]$ is analytic in C^+ .*

Proof. a) For arbitrary $f_\lambda, g_\lambda \in \mathfrak{N}_\lambda$ the identity (1) yields

$$(\lambda - \bar{\lambda})\langle f_\lambda, g_\lambda \rangle = i\beta \left[\langle \Theta_+[\gamma, \lambda]f_\lambda, \Theta_+[\gamma, \lambda]g_\lambda \rangle_{\mathfrak{G}} - \langle \Theta_-[\gamma, \lambda]f_\lambda, \Theta_-[\gamma, \lambda]g_\lambda \rangle_{\mathfrak{G}} \right]. \quad (3)$$

Thus

$$\|\Theta_+[\gamma, \lambda]f_\lambda\|_{\mathfrak{G}}^2 = \frac{2\nu}{\beta} \|f_\lambda\|^2 + \|\Theta_-[\gamma, \lambda]f_\lambda\|_{\mathfrak{G}}^2,$$

hence $\Theta_+[\gamma, \lambda]$ is bounded invertible.

Since f_λ, g_λ are arbitrary, from (3) it follows that

$$\begin{aligned} \Theta_+^*[\gamma, \lambda]\Theta_+[\gamma, \lambda] - \Theta_-^*[\gamma, \lambda]\Theta_-[\gamma, \lambda] &= \\ &= \Theta_+^*[\gamma, \lambda] \left[I_{\mathfrak{G}_+} - \Theta_\gamma^*[\lambda]\Theta_\gamma[\lambda] \right] \Theta_+[\gamma, \lambda] > 0, \end{aligned}$$

hence $I_{\mathfrak{G}_+} - \Theta_\gamma^*[\lambda]\Theta_\gamma[\lambda] > 0$, that is $\|\Theta[\lambda]\| < 1$.

To prove the part b) recall (1), that is $f_\lambda = f_0 + f_\gamma + \Theta_\gamma(\lambda)f_\gamma$, $f_0 \in \mathcal{D}(T)$. On account of I and (2) one has

$$\Theta_+[\gamma, \lambda]f_\lambda = \Gamma_+(\gamma)f_\lambda = \Gamma_+(\gamma)f_\gamma = \Theta_+[\gamma, \gamma]f_\gamma = \Theta_\gamma f_\gamma, \quad (3.4_+)$$

$$\Theta_-[\gamma, \lambda]f_\lambda = \Gamma_-(\gamma)f_\lambda = \Gamma_-(\gamma)\Theta_\gamma(\lambda)f_\gamma = \Theta_-[\gamma, \bar{\gamma}]\Theta_\gamma(\lambda)f_\gamma = \Theta_{\bar{\gamma}}\Theta_\gamma(\lambda)f_\gamma, \quad (3.4_-)$$

since it is understood that $\Theta_+[\gamma, \gamma] = \Theta_\gamma$, $\Theta_-[\gamma, \bar{\gamma}] = \Theta_{\bar{\gamma}}$ in (2).

From (3.4₊) we have $f_\lambda = \Theta_+^{-1}[\gamma, \lambda]\Theta_\gamma f_\gamma$, and (3.4₋) now turns to $\Theta_-[\gamma, \lambda]\Theta_+^{-1}[\gamma, \lambda]\Theta_\gamma f_\gamma = \Theta_{\bar{\gamma}}\Theta_\gamma(\lambda)f_\gamma$, thus we obtain

$$\Theta_\gamma[\lambda] = \Theta_{\bar{\gamma}}\Theta_\gamma(\lambda)\Theta_\gamma^*, \quad (5)$$

so $\Theta_\gamma[\lambda]$ is an analytic function, since $\Theta_\gamma(\lambda)$ is. The proof is complete.

Likewise, if $\zeta \in C^-$, then $\Theta_-[\gamma, \zeta]$ has bounded inverse, and

$$\Theta_\gamma[\zeta] := \Theta_+[\gamma, \zeta]\Theta_-^{-1}[\gamma, \zeta] = \Theta_\gamma\Theta_{\bar{\gamma}}(\zeta)\Theta_{\bar{\gamma}}^* \quad (6)$$

is a contractive analytic function in C^- with values in $[\mathfrak{G}_-, \mathfrak{G}_+]$. Clearly, $\Theta_\gamma^*[\bar{\lambda}] = \Theta_\gamma[\lambda]$.

Formulas (5), (6) mean that $\Theta_\gamma[\lambda]$, $\Theta_\gamma[\zeta]$ coincide with the Nagy-Foias ch.f. of maximal extensions T_γ , T_γ^* .

The following statement is an immediate corollary of Prop. 1.

Proposition 2 *Let $\mathcal{X}[\lambda]$ be a contractive function in C^+ with values in $[\mathfrak{G}_+, \mathfrak{G}_-]$. Then the boundary condition*

$$[\Gamma_-(\gamma) - \mathcal{X}[\lambda]\Gamma_+(\gamma)]f = 0 \quad (7)$$

defines m.d.ext. T_λ if and only if $\mathcal{X}[\lambda] = \Theta_\gamma[\lambda]$.

Indeed, if $\text{Ker} [\Gamma_-(\gamma) - \mathcal{X}[\lambda]\Gamma_+(\gamma)] = \mathcal{D}(T_\lambda)$ so $[\Gamma_-(\gamma) - \mathcal{X}[\lambda]\Gamma_+(\gamma)]\mathfrak{N}_\lambda = \{0\}$, hence $\mathcal{X}[\lambda] = \Theta_\gamma[\lambda]$. If $[\Gamma_-(\gamma) - \Theta_\gamma[\lambda]\Gamma_+(\gamma)]f = 0$, $f = f_0 + f_\gamma + f_{\bar{\gamma}}$, $f \neq f_0$, then from (5) it follows that $f_{\bar{\gamma}} = \Theta_\gamma(\lambda)f_\gamma$, so $f \in \mathfrak{N}_\lambda$ and $\text{Ker} [\Gamma_-(\gamma) - \Theta_\gamma[\lambda]\Gamma_+(\gamma)] = \mathcal{D}(T_\lambda)$.

3.2. Let $K \in [\mathfrak{G}_+, \mathfrak{G}_-]$, $\|K\| < 1$, and

$$\mathcal{D}_K = (I_+ - K^*K)^{\frac{1}{2}}, \quad \mathcal{D}_{K^*} = (I_- - KK^*)^{\frac{1}{2}}$$

be its defect operators.

As in the proof of Th. 3, introduce the \mathcal{J} -unitary operator

$$U_K = \begin{bmatrix} D_K^{-1} & -D_K^{-1}K^* \\ -D_{K^*}^{-1}K & D_{K^*}^{-1} \end{bmatrix}, \quad (8)$$

and, referring back to Remark 2 in Sec. 1, consider the reduction operator $\{\Gamma_K, \mathfrak{G}\} = \{\Gamma_{K_\pm}, \mathfrak{G}_\pm\}$ for $T^*(\gamma)$ (also for T^*), where $\Gamma_K = U_K\Gamma(\gamma)$, and $\Gamma_{K_\pm} = P_\pm\Gamma_K$ are

$$\Gamma_{K_+} = D_K^{-1}[\Gamma_+(\gamma) - K^*\Gamma_-(\gamma)], \quad \Gamma_{K_-} = D_{K^*}^{-1}[\Gamma_-(\gamma) - K\Gamma_+(\gamma)]. \quad (9)$$

Thus boundary conditions $\Gamma_{K_-}f = 0$, $\Gamma_{K_+}f = 0$ define m.d.exts. T_K and its adjoint $T_{K^*} = T_K^*$ respectively.

Definition 3 *Let $\{\Gamma_\pm(\gamma), \mathfrak{G}_\pm\}$ be the canonical reduction operator for T^* , and reduction operator $\{\Gamma_{K_\pm}, \mathfrak{G}_\pm\}$ be defined by (9).*

An operator function $\mathcal{X}_K[\lambda]$, $\lambda \in C^+$ with contractive values in $[\mathfrak{G}_+, \mathfrak{G}_-]$ is said to be the abstract Nagy-Foias ch.f. of m.d.ext. $T_K = T^|_{\text{Ker}\Gamma_{K_-}}$, if the boundary condition*

$$[\Gamma_{K_-} - \mathcal{X}_K[\lambda]\Gamma_{K_+}]f = 0, \quad f \in \mathfrak{D} \quad (10)$$

defines the inherited extension T_λ for every $\lambda \in C^+$.

Theorem 4 *The abstract Nagy-Foias ch.f. exists and it holds that*

$$\mathcal{X}_K[\lambda] = -\Phi_{U_K}(\Theta_\gamma[\lambda]), \quad (11)$$

where $\Phi_{U_K}(\cdot)$ is the Kreĩn-Shmulyan fractional linear transformation, associated with the matrix U_K of (8). The function $\mathcal{X}_K[\lambda]$ is analytic in C^+ , and $\|\mathcal{X}_K[\lambda]\| < 1$.

Proof. On account of (9), the operator in (10) is

$$\begin{aligned} & [D_{K^*}^{-1} + \mathcal{X}_K[\lambda]D_K^{-1}K^*] \Gamma_-(\gamma) - [D_{K^*}^{-1}K + \mathcal{X}_K[\lambda]D_K^{-1}] \Gamma_+(\gamma) = \\ & = [D_{K^*}^{-1} + \mathcal{X}_K[\lambda]K^*D_K^{-1}] \Gamma_-(\gamma) - [KD_{K^*}^{-1} + \mathcal{X}_K[\lambda]D_K^{-1}] \Gamma_+(\gamma). \end{aligned}$$

Since $\|\mathcal{X}_K[\lambda]K^*\| < 1$, then $D_{K^*}^{-1} + \mathcal{X}_K[\lambda]K^*D_K^{-1} = [I_- + \mathcal{X}_K[\lambda]K^*] D_{K^*}^{-1}$ has a bounded inverse, hence condition (10) is equivalent to

$$\left\{ \Gamma_-(\gamma) - [D_{K^*}^{-1} + \mathcal{X}_K[\lambda]K^*D_K^{-1}]^{-1} [KD_{K^*}^{-1} + \mathcal{X}_K[\lambda]D_K^{-1}] \Gamma_+(\gamma) \right\} f = 0.$$

From Prop. 2 it follows that

$$[D_{K^*}^{-1} + \mathcal{X}_K[\lambda]K^*D_K^{-1}]^{-1} [KD_{K^*}^{-1} + \mathcal{X}_K[\lambda]D_K^{-1}] = \Theta_\gamma[\lambda],$$

and not complicated derivations lead to

$$\begin{aligned} \mathcal{X}_K[\lambda] &= [D_{K^*}^{-1}\Theta_\gamma[\lambda]D_K - K] [I_- - K^*D_{K^*}^{-1}\Theta_\gamma[\lambda]D_K]^{-1} = \\ &= [D_{K^*}^{-1}\Theta_\gamma[\lambda] - D_{K^*}^{-1}K] [D_K^{-1} - D_K^{-1}K^*\Theta_\gamma[\lambda]]^{-1} = -\Phi_{U_K}(\Theta_\gamma[\lambda]). \end{aligned} \quad (12)$$

The formula above shows that $\mathcal{X}_K[\lambda]$ is analytic in C^+ , and interspherical property of $\Phi_{U_K}(\cdot)$ yields $\|\mathcal{X}_K(\lambda)\| < 1$, $\lambda \in C^+$. The proof is complete.

The formula similar to (12) was obtained in [12] by means of a boundary triplet, adjusted for ch.f of m.d.ext. in sense of A.V. Strauss [21]. For the case under consideration boundary operators introduced in [21] coincide with that Γ_\pm in (10) of a certain reduction operator. In this connection we refer also to [3], [4], [17].

3.3. In similar fashion can be presented also the Weyl function of a self-adjoint extension of T .

Let $V \in [\mathfrak{G}_+, \mathfrak{G}_-]$ be an isometry. Consider boundary operators

$$\Gamma_{V\pm} = \Gamma_\pm(\gamma) \mp V\Gamma_\pm(\gamma). \quad (13)$$

In [18] it is shown that self-adjoint extensions $T_{V\pm} = T^*|_{Ker\Gamma_{V\pm}}$ coincide with those defined by (21) with the isometry $\mathcal{V}(\gamma) = \Theta_\gamma^*V\Theta_\gamma \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$.

Introduce the unitary operator

$$U_V = \frac{1}{\sqrt{2}} \begin{bmatrix} I_+ & -V^* \\ V & I_- \end{bmatrix} \in [\mathfrak{G}] \quad (14)$$

such that $P_{V\pm} = U_V P_\pm U_V^*$ are orthogonal projections on hypermaximal neutral subspaces $\mathfrak{L}_{V\pm} = P_{V\pm}\mathfrak{G}$, and $\mathfrak{L}_{V+} \oplus \mathfrak{L}_{V-} = \mathfrak{G}$.²

It is clear that $Ker\Gamma_{V\pm} = KerP_{V\pm}\Gamma(\gamma)$.

²It should be noted that U_V is the unitary operator indicated in Remark 3.

If $\mathcal{M} \in [\mathfrak{L}_{V^+}, \mathfrak{L}_{V^-}]$, consider the projection $P_{V^-} - \mathcal{M}P_{V^+}$. From (14) it follows that

$$\mathcal{M} = \begin{bmatrix} -V^*MV & -V^*M \\ MV & M \end{bmatrix}, \quad M \in [\mathfrak{G}_-],$$

hence the boundary operator $(P_{V^-} - \mathcal{M}P_{V^+})\Gamma(\gamma)$ is such that

$$\text{Ker}(P_{V^-} - \mathcal{M}P_{V^+})\Gamma(\gamma) = \text{Ker}(\Gamma_{V^-} - M\Gamma_{V^+}).$$

Definition 5 Let $\{\Gamma_{\pm}(\gamma), \mathfrak{G}\}$ be the canonical reduction operator for T^* , and self-adjoint extensions $T_{V_{\pm}}$ of T be given by the boundary conditions $\Gamma_{V_{\pm}}f = 0$.

The operator function $M_V(\lambda)$, $\lambda \in C^+$ with values in $[\mathfrak{G}_-]$ is said to be an abstract Weyl function of T_{V^+} , if the boundary condition

$$[\Gamma_{V^+} - M_V(\lambda)\Gamma_{V^-}]f = 0, \quad f \in \mathfrak{D} \quad (15)$$

defines the inherited m.d.ext. T_{λ} for every $\lambda \in C^+$.

Theorem 6 An abstract Weyl function $M_{V^+}(\lambda)$ of a self-adjoint extension T_{V^+} exists, and

$$\begin{aligned} M_{V^+}(\lambda) &= -[I_- - \Theta_{\gamma}[\lambda]V^*]^{-1}[I_- + \Theta_{\gamma}[\lambda]V^*] = \\ &= -[I_- + \Theta_{\gamma}[\lambda]V^*][I_- - \Theta_{\gamma}[\lambda]V^*]^{-1}. \end{aligned} \quad (16)$$

The operator function $M_V(\lambda)$ is analytic in C^+ , and such that

$$\text{Re } M_{V^+}(\lambda) = \frac{1}{2} [M_{V^+}(\lambda) + M_{V^+}^*(\lambda)] < 0. \quad (17)$$

Proof. Rewrite formulas (13) as

$$\Gamma_-(\gamma) = \frac{1}{2} [\Gamma_{V^+} + \Gamma_{V^-}], \quad \Gamma_+(\gamma) = -\frac{1}{2} V^* [\Gamma_{V^+} - \Gamma_{V^-}].$$

From Prop. 2 it follows that for any $\lambda \in C^+$ the boundary condition

$$[(\Gamma_{V^+} + \Gamma_{V^-}) + \Theta_{\gamma}[\lambda]V^* (\Gamma_{V^+} - \Gamma_{V^-})]f = 0$$

is necessary and sufficient to define m.d.ext. T_{λ} . The operator above is

$$\begin{aligned} &[I_- + \Theta_{\gamma}[\lambda]V^*]\Gamma_{V^+} + [I_- - \Theta_{\gamma}[\lambda]V^*]\Gamma_{V^-} = \\ &= [I_- + \Theta_{\gamma}[\lambda]V^*] \{ \Gamma_{V^+} + [I_- + \Theta_{\gamma}[\lambda]V^*]^{-1} [I_- - \Theta_{\gamma}[\lambda]V^*] \Gamma_{V^-} \}, \end{aligned}$$

which proves (16).

Evidently, $M_{V^+}(\lambda)$ is analytic in C^+ .

Formula (16) can be presented also as

$$M_{V_+}(\lambda) = -[V + \Theta_\gamma[\lambda]][V - \Theta_\gamma[\lambda]]^{-1},$$

and it can be easily verified that

$$\frac{1}{2}(M_{V_+}^* + M_{V_+}) = -(V + \Theta_\gamma[\lambda])^{-*} [I_+ - \Theta_\gamma^*[\lambda]\Theta_\gamma[\lambda]] (V + \Theta_\gamma[\lambda])^{-1} < 0.$$

The proof is complete.

It is clear that the operator function $M_{V_-}(\lambda) = M_{V_+}^{-1}(\lambda)$ is the Weyl function of a self-adjoint extension T_{V_-} .

A Weyl function of T_{V_+} in lower half-plane C^- is determined similarly, considering boundary operators $\Gamma_{V_+} - M_{V_+}(\zeta)\Gamma_{V_-}$, $\Gamma_+(\gamma) - \Theta_\gamma[\zeta]\Gamma_-(\gamma)$ as defining maximal accumulative extension T_ζ for every $\zeta \in C^-$. The corresponding formula is

$$M_{V_+}(\zeta) = [I_- + V\Theta_\gamma[\zeta]]^{-1} [I_- - V\Theta_\gamma[\zeta]] = [I_- - V\Theta_\gamma[\zeta]] [I_- + V\Theta_\gamma[\zeta]]^{-1},$$

hence one has

$$M_{V_+}(\zeta) = -M_{V_+}^*(\bar{\zeta}). \quad (18)$$

An operator function with properties (17), (18) is obtained in [16], where the original method of Weyl for determining defect subspaces of a differential operator was applied to a symmetric canonical differential operator.

From (16) it follows that

$$[I_- - M_V(\lambda)]^{-1} = \frac{1}{2} [I_- - \Theta_\gamma[\lambda]V^*],$$

hence

$$\Theta_\gamma[\lambda] = [I_- - 2[I_- - M_V(\lambda)]^{-1}]V = -[I_- + M_V(\lambda)][I_- - M_V(\lambda)]^{-1}V.$$

Thus one has

$$[I_- + M_V(\lambda)][I_- - M_V(\lambda)]^{-1}V = [I_- + M_{V_1}(\lambda)][I_- - M_{V_1}(\lambda)]^{-1}V_1,$$

where $M_{V_1}(\lambda)$ is the Weyl function of an arbitrary other self-adjoint extension T_{V_1} , determined by the isometry $V_1 \in [\mathfrak{N}_\gamma, \mathfrak{N}_{\bar{\gamma}}]$. The formula above can be presented also as the fractional linear transformation

$$M_{V_1}(\lambda) = \Phi_W(M_V(\lambda)),$$

where W is a unitary and \mathcal{J}_1 -unitary operator in $\mathfrak{G}_- \oplus \mathfrak{G}_-$, given by

$$W = \frac{1}{2} \begin{bmatrix} I_- + V_1V^* & I_- - V_1V^* \\ I_- - V_1V^* & I_- + V_1V^* \end{bmatrix}, \quad \text{and} \quad \mathcal{J}_1 = \begin{bmatrix} 0 & I_- \\ I_- & 0 \end{bmatrix}.$$

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