Armenian Journal of Mathematics
Volume 7, Number 2, 2015, 121–145

Exhaustive Weakly Wandering Sequences and Alpha-type Transformations

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Abstract. An increasing sequence of integers, $\mathbb{B}$, is given for which there exists a family of ergodic, infinite measure preserving transformations T_α, $0 \leq \alpha \leq 1$ so that (1) T_α is of α-type and (2) $\mathbb{B}$ is an exhaustive weakly wandering sequence for each T_α.

Key Words: exhaustive weakly wandering, alpha-type, ergodic, infinite measure preserving
Mathematics Subject Classification 2010: 37A40, 28D05

Introduction

Two properties, first introduced in 1970 by Hajian and Kakutani [8], that differentiate infinite measure preserving transformations from finite measure preserving transformations are exhaustive weakly wandering sequences and α-type, $0 \leq \alpha < 1$. That is, a transformation which has an exhaustive weakly wandering sequence cannot preserve a finite measure; and a transformation which is α-type for $0 \leq \alpha < 1$, cannot preserve a finite measure.

These two properties are usually studied separately. In particular, the only published example for which the α-type is presented and an explicit exhaustive weakly wandering sequence is given is the Hajian-Kakutani transformation (see [8] and Example 3 of [2]).

This example has $\alpha = \frac{1}{2}$ and an exhaustive weakly wandering sequence given as a direct sum of integers $\oplus_{i=1}^{\infty}\{0, 2^{2i-1}\}$. It is easily generalized to α-type transformations with $\alpha = \frac{p-1}{p}$ for any integer $p > 1$ and exhaustive weakly wandering sequence $\oplus_{i=1}^{\infty}\{0, 2^{i-1}p^i\}$ (see Section 3).

For all other $\alpha \in [0, 1)$, explicit exhaustive weakly wandering sequences are not known. This includes $\alpha = 0, \frac{1}{3}$ and α irrational to name a few (see Section 4 for a simple example with $\alpha = \frac{1}{3}$). In addition, it was not known if a sequence $\mathbb{B}$ can be exhaustive weakly wandering for transformations of different α-type.

We settle these with the following.

Theorem 1 *There exists an increasing sequence of positive integers*

$$\mathbb{B} = \{0 = b_0 < b_1 < b_2 < \cdots\}$$

and there exist ergodic infinite measure preserving maps T_α, $0 \leq \alpha \leq 1$ so that T_α is of α-type, $\mathbb{B}$ is an Exhaustive Weakly Wandering sequence for each T_α, and the associated exhaustive weakly wandering sets have measure one.

The proof is by a cutting-and-stacking construction and generalizes the Hajian-Kakutani transformation

1 Preliminaries and Definitions

Throughout this paper, all transformations will be, by assumption or construction invertible, ergodic, infinite measure preserving on a non-atomic, Lebesgue space with a sigma-finite measure $(X, \mathcal{B}, \mu)$. As usual, statements and equalities are to be understood as "modulo sets of measure zero".

General definitions in ergodic theory can be found in [2] and the books [1, 5, 6].

The following definitions are included for easy reference.

Definition 1 *The transformation T is said to be of α-type for a fixed $\alpha \in [0, 1]$ if*

$$\limsup_{n\to\infty} \mu(T^n A \cap A) = \alpha \cdot \mu(A)$$

for all A satisfying $\mu(A) < \infty$.

For finite measure preserving transformations $\alpha = 1$ is possible. However, no finite measure preserving transformation can be of α-type for $0 \leq \alpha < 1$. In [9] Hamachi and Osikawa showed that for every $0 \leq \alpha \leq 1$ there exist an infinite measure preserving invertible ergodic transformation of α-type - but they gave no indication of the exhaustive weakly wandering sequences.

Definition 2 *An infinite set of integers $\mathbb{B} = \{b_i\}$ is* **exhaustive weakly wandering** *for the transformation T, if there exists a set of positive measure W satisfying the two conditions*

1. *$T^{b_i} W \cap T^{b_j} W = \emptyset$, $i \neq j$ (weakly wandering),*

2. *$X = \cup_{b_i \in \mathbb{B}} T^{b_i} W$, (exhaustive).*

Note that each sequence $\mathbb{B}$ may have more than one set W (for example if the transformation S commutes with T then the set SW is also an exhaustive weakly wandering set for the sequence $\mathbb{B}$). Similarly, each set W is exhaustive weakly wandering for more than one sequence (for example the shifted sequence $\{b + k : b \in \mathbb{B}\}$)

It is also possible for the the set W to be of finite or infinite measure. However, in this paper we will only work with exhaustive weakly wandering sets of finite measure - in which case, the measure is considered normalized with $\mu(W) = 1$.

It follows from Jones and Krengel [10] that exhaustive weakly wandering sequences exist for all infinite measure preserving ergodic transformations. Despite this, there are very few transformations for which explicit exhaustive weakly wandering sequences are known. The Jones and Krengel result is an existence proof - it gave no explicit exhaustive weakly wandering sequence and did not touch upon α-type.

There is not a great amount of restrictions in order for a sequence of integers to be exhaustive weakly wandering for some transformation, but there are some [3]. However, the Hajian-Kakutani example and the examples presented here will have strong arithmetic and combinatorial properties and the results obtained here will depend upon these properties. Specifically, the exhaustive weakly wandering sequence will be a direct sum of integers which "aligns" with the construction of the transformations.

Definition 3 *Let $\mathbb{B}_i$ be a finite set of integers with $0 \in \mathbb{B}_i$ for all $i \geq 1$ and the cardinality of each $\mathbb{B}_i$ greater than 1. The* **Direct Sum** *of the $\mathbb{B}_i$ is $\oplus_{i=1}^{\infty}\mathbb{B}_i = \{b = \sum_{i=1}^{\infty} b_i\}$ where $b_i \in \mathbb{B}_i$ and $b_i = 0$ for all but finitely many of the i.*

Finite direct sums will also be used, and are defined in the obvious manner. We will be using them for factorizations of finite cyclic groups. The technique connects factorizations of finite abelian groups to the the cutting-and-stacking construction and it is this connection which supplies the proof of exhaustiveness for the more complex examples. The technique is general enough that it allows one to simultaneously control the α-type and in particular allows one to obtain any α-type for $\alpha \in [0, 1]$. This, in turn, further shows that non-isomorphic transformations of different α-type can have the same exhaustive weakly wandering sequence.

Definition 4 *A* **Finite Cyclic Group** *for an integer $N > 1$, as used here, will simply be $\mathbb{G}_{\mathbb{N}} = \{0, 1, 2, \dots N - 1\}$ with addition Modulo N.*

Definition 5 *$\mathbb{G} = \mathbb{A} \oplus \mathbb{B}$ is a* **factorization** *of the finite cyclic group $\mathbb{G}$ if $\mathbb{A}$ and $\mathbb{B}$ are non-empty subsets of $\mathbb{G}$ and every element $g \in \mathbb{G}$ is a unique sum $g = a + b$ (Mod N) where $a \in \mathbb{A}$ and $b \in \mathbb{B}$.*

For the examples presented here, both $\mathbb{A}$ and $\mathbb{B}$ will be direct sums.

We will also be interested in **factorizations of the nonnegative integers**

$$\mathbb{N} = \mathbb{A} \oplus \mathbb{B}$$

where both $\mathbb{A}$ and $\mathbb{B}$ have infinite cardinality.

2 The Hajian-Kakutani Transformation

In this section we review the Hajian-Kakutani transformation (see [8] and Example 3 in [2]). It was presented originally as a skyscraper construction, however for generalization purposes we present it here via an alternate cutting-and-stacking rank-one construction (rank-one simply means that at each step of the construction there is only one column). In either case, the transformation is piecewise linear ($T' = 1$ a.e.) on a countable collection of intervals whose union has infinite Lebesgue measure.

We denote this as $(X, \mathcal{B}, \mu)$.

The result of the construction will be the following.

Theorem 2 *The Hajian-Kakutani transformation is ergodic, infinite measure preserving. It is of α-type with $\alpha = \frac{1}{2}$. There is a set W with $\mu(W) = 1$ which is exhaustive weakly wandering under the sequence of integers*

$$\begin{aligned} \mathbb{B} &= \bigoplus_{k=1}^{\infty} \{0,\ 2^{2k-1}\} \qquad (1) \\ &= \{0,\ 2,\ 8,\ 10,\ 32,\ 34,\ \cdots\} \end{aligned}$$

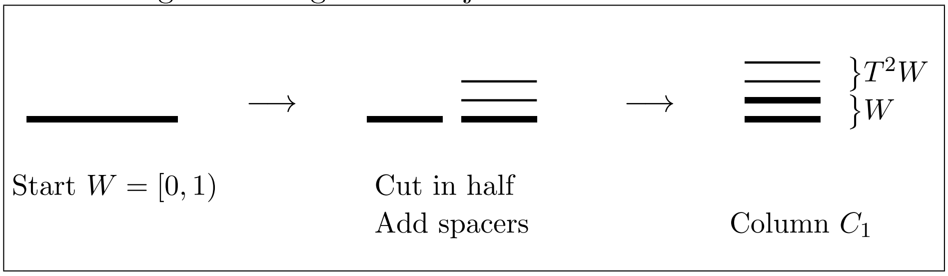
2.1 Construction

Step 0: Start with $W = [0, 1)$. Consider this as a column C_0 of height $h_0 = 1$ and width $w_0 = 1$. The transformation is not defined anywhere yet.

Step 1: Begin by cutting C_0 into two equal width subcolumns (*i.e.* W is divided into two equal width subintervals). Place two spacer intervals of width 1/2 above the right subcolumn. Next stack the right hand column with the spacers above the left hand subcolumn.

This yields column C_1 of height $h_1 = 4$ and width $w_1 = 1/2$ (see Figure 1). The transformation is defined as moving linearly up the column. It is not yet defined on the top level.

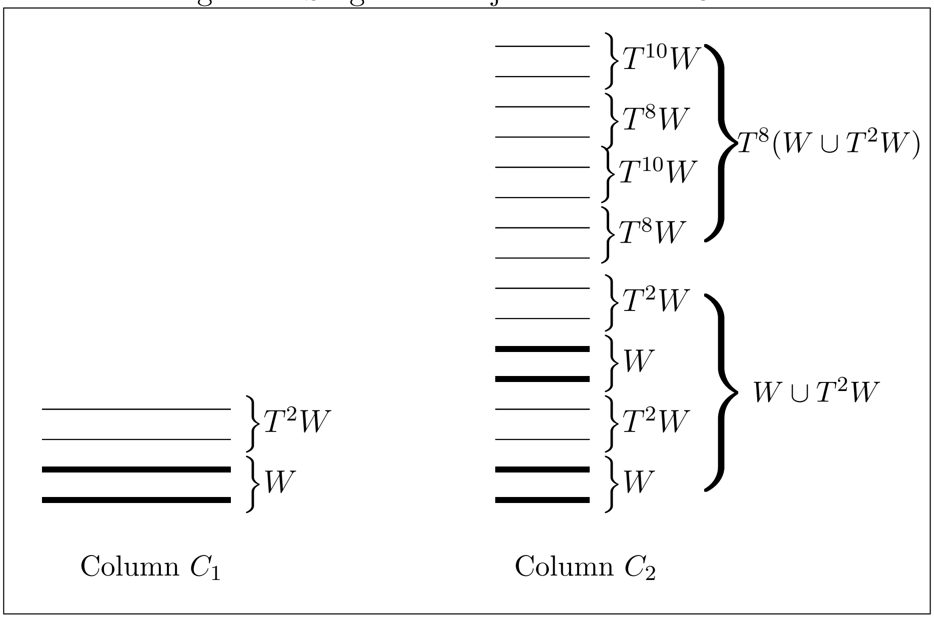
Figure 1: Stage 1 for Hajian-Kakutani Transformation T



Step 2: Cut the column C_1 into two equal width subcolumns. Place $8 = 2 \cdot 2^2 = 2h_1$ spacer intervals above the right side column. Stack the right side subcolumn with its spacers above the left side subcolumn (see Figure 2).

This results in column C_2 of height $h_2 = 4^2$ and width $w_2 = 1/2^2$. The transformation is defined as mapping each level to the level immediately above it. This matches the previous definition wherever it was defined. The transformation is not yet defined on the top level.

Figure 2: Stage 2 of Hajian-Kakutani Construction



Step n: Cut column C_{n-1} into two equal width subcolumns. Above the right side subcolumn place $2h_{n-1}$ spacer intervals of width w_{n-1}.

This results in column C_n of height $h_n = 4h_{n-1} = 4^n$ and width $w_n = w_{n-1}/2 = 1/2^n$. The transformation is again defined as moving up the levels linearly.

Remark 1 *As a cutting-and-stacking construction, it is well known that the resulting transformation is ergodic ([6]). Infinite measure preserving is also clear.*

Remark 2 *Let $\mathcal{L}$ denote the countable collection of all the levels at each step in the construction. Then it is well known that finite disjoint unions of members of $\mathcal{L}$ generate the the sigma-algebra $\mathcal{B}$.*

This will be used in proving the α-type for transformations.

2.2 Observations

The proof of Theorem 2 follows from the following easy observations.

2.2.1 Exhaustive Weakly Wandering Sequence

The exhaustive weakly wandering sequence for the Hajian-Kakutani transformation appears automatically as a direct sum from the construction.

Define $\mathbb{B}_i = \{0, 2^{2i-1}\} = \{0, 2h_{i-1}\}$. $i \geq 1$. From the construction we have inductively

$$\begin{aligned} C_1 &= W \dot{\cup} T^2 W (disjoint) \\ &= \dot{\cup} T^b W, \quad b \in \mathbb{B}_1 \end{aligned}$$

and

$$\begin{aligned} C_2 &= C_1 \dot{\cup} T^8 C_1 \\ &= \big(W \dot{\cup} T^2 W\big) \dot{\cup} T^8 \big(W \dot{\cup} T^2 W\big) \\ &= \dot{\cup} T^b W, \quad b \in \mathbb{B}_1 \oplus \mathbb{B}_2 \end{aligned}$$

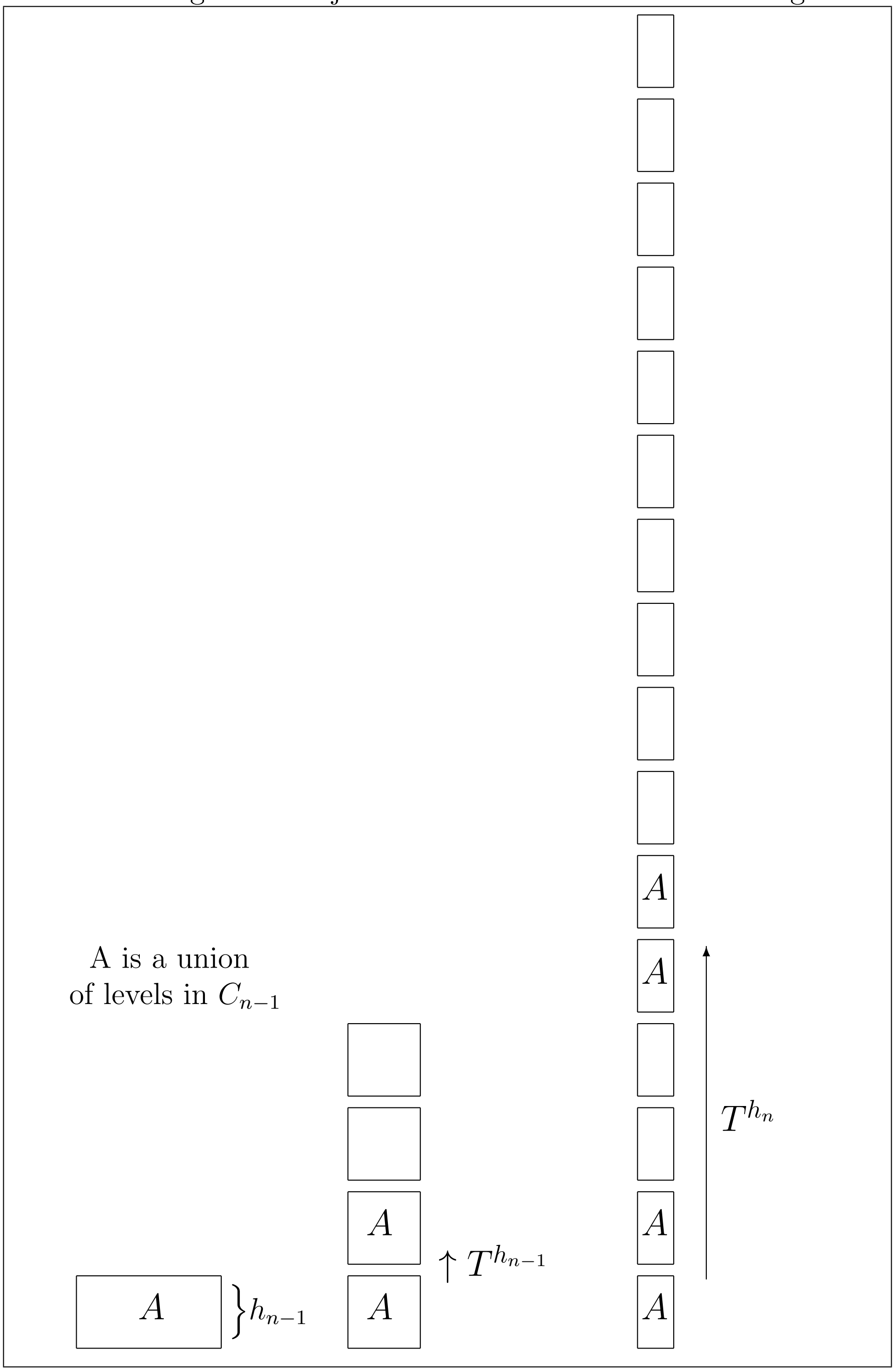
and in general

$$\begin{aligned} C_n &= C_{n-1} \dot{\cup} T^{2^{2n-1}} C_{n-1} \\ &= \dot{\cup} T^b W, \quad b \in \oplus_{i=1}^n \mathbb{B}_i \end{aligned}$$

Taking the limit we conclude that $\mathbb{B} = \oplus_{i=1}^\infty \mathbb{B}_i$ is an exhaustive weakly wandering sequence with exhaustive weakly wandering set $W = [0, 1)$. That is

$$X = \dot{\cup}_{b \in \mathbb{B}} T^b W \ (disjoint)$$

Figure 3: Hajian-Kakutani Construction - 3 Stages



2.2.2 α-Type

To see that the α-type is $\alpha = 1/2$, let A denote any set which is a union of disjoint levels in C_{n-1} for some n (see Figure 3).

From the construction, the set A is also a union of disjoint levels in C_n. Viewing A sitting inside C_n and applying $T^{h_{n-1}}$ we see that half of A moves up into the spacer section, while the other half moves onto the top half of A. Therefore $\mu(T^{h_{n-1}}A \cap A) = \frac{1}{2}\mu(A)$.

The set A is also a union of disjoint levels in C_{n+1} and the same argument gives $\mu(T^{h_n}A \cap A) = \frac{1}{2}\mu(A)$.

Suppose $h_{n-1} < k < h_n$. Viewing A as a union of disjoint levels of C_{n+1} we see that at least half of T^kA is mapped upward into spacers and so $\mu(T^kA \cap A) \leq \frac{1}{2}\mu(A)$. Since the set A is also a disjoint union of levels in C_m for all $m > n+1$ we have $\limsup_{k\to\infty}\mu(T^kA \cap A) = \frac{1}{2}\mu(A)$.

Given an arbitrary set B of finite measure, we can approximate it by a set A which is a disjoint union of levels in some column and the result follows.

This completes the proof of Theorem 2.

2.3 A Direct Sum of Integers Associated with W

We have seen that the exhaustive weakly wandering sequence mirrors the construction and appears naturally as a direct sum (Section 2.2.1). For this example, a similar result is true for the set W.

Observe that in Column C_1, the set W consists of the two intervals, Level 0 and Level 1 (see Figure 1). In Column C_2, W consists of the four intervals, Level 0, Level 1, Level 4 and Level 5 (see Figure 2).

Putting $\mathbb{A}_1 = \{0, 1\}$ and $\mathbb{A}_2 = \{0, 4\}$, we have that W is associated to the levels defined by $\mathbb{A}_1 \oplus \mathbb{A}_2$.

This holds for higher columns. Suppose W contains Level k of Column C_{n-1}. At the next step, Column C_{n-1} of height h_{n-1} is divided in two. This means that W will contain Level k and Level $k + h_{n-1}$ of Column C_n.

Hence if we put $\mathbb{A}_n = \{0, h_{n-1}\}$, then W consists exactly of Levels $a \in \oplus_1^n \mathbb{A}_i$ in Column C_n.

We will return to this idea when we discuss factorizations of cyclic groups.

3 Simple Generalization $\alpha = \frac{p-1}{p}$

In this section we present a simple generalization of the Hajian-Kakutani Transformation. The change is to the number of cuts. The generalization is well known but has never appeared in print. The construction, and hence proofs, are nearly identical to that for the Hajian-Kakutani transformation.

For a fixed integer $p > 1$ we define a transformation V_p yielding the following

Theorem 3 *The transformation V_p is ergodic, infinite measure preserving. It is of α-type with $\alpha = \frac{p-1}{p}$. The set W is exhaustive weakly wandering under the direct sum sequence $\oplus_{k=1}^{\infty}\{0, 2^{k-1}p^k\}$.*

3.0.1 Construction

Start with $W = [0, 1) = C_0$.

Step 1: Cut W into p equal width subintervals. Place p spacer intervals over the right-most subinterval. Stack these, right-on-left, *i.e.* each interval is placed above the one to its immediate left (See Figure 4.)

This results in column C_1 of height $h_1 = 2p$ and width $w_1 = 1/p$.

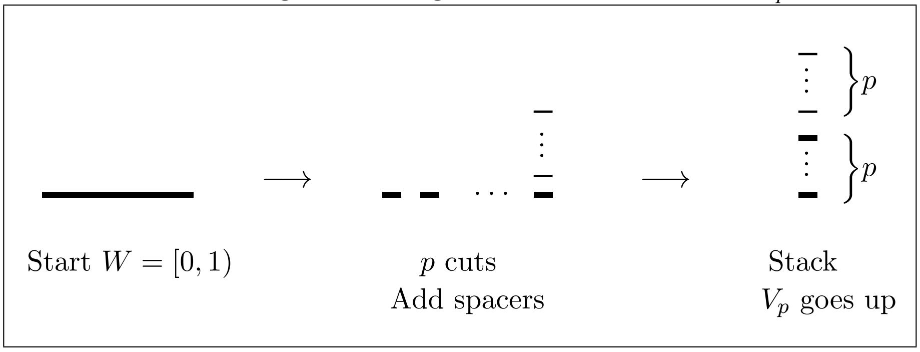
Step 2: Divide the column C_1 into p equal width subcolumns. Above the right-most subcolumn, place $2p^2$ spacer intervals. Stack each subcolumn onto the one to its immediate left. (See Figure 5.)

This results in column C_2 of height $h_2 = 2^2p^2$ and width $w_2 = 1/p^2$.

Step n: At step $n - 1$ we have column C_{n-1} of height $h_{n-1} = (2p)^{p-1}$ and width $w_{n-1} = 1/p^{n-1}$. Cut column C_{n-1} into p equal width subcolumns. Above the right-most subcolumn place $2^{n-1}p^n$ spacer intervals of width $1/p^n$. Stack each subcolumn above the one to its immediate left.

This results in column C_n of height $h_n = (2p)^n$ and width $w_n = 1/p^n$.

Figure 4: Stage 1 for construction of V_p



3.0.2 Observations

To see the exhaustive weakly wandering sequence observe that $C_1 = W \dot{\cup} V^p W$; $C_2 = C_1 \dot{\cup} V^{2p^2} C_1$; and in general $C_n = C_{n-1} \dot{\cup} V^{2^{n-1}p^n} C_{n-1}$. Observe that $2^{n-1}p^n = p \cdot (2p)^{n-1} = p \cdot h_{n-1}$ which is the height of the (previous) column multiplied by the number of cuts made to the (previous) column. That is define

$$\mathbb{B}_k = \{0, 2^{k-1}p^k\} = \{0, p \cdot h_{k-1}\} \tag{2}$$

and conclude the exhaustive weakly wandering sequence is $\oplus_1^\infty \mathbb{B}_k$.

To see that $\alpha = \frac{p-1}{p}$ first examine $W = C_0$ as part of column C_1. W consists of the lower p levels. Applying T, $p-1$ levels of W move up to another level of W while the very top level of W moves into the spacers. Applying T^2 we see two levels of W move into the spacers resulting in a smaller intersection. The same holds for T^k for $h_0 < k < h_1$.

Similarly the column C_n consists of p consecutive blocks of height h_n inside of C_{n+1}. Therefore

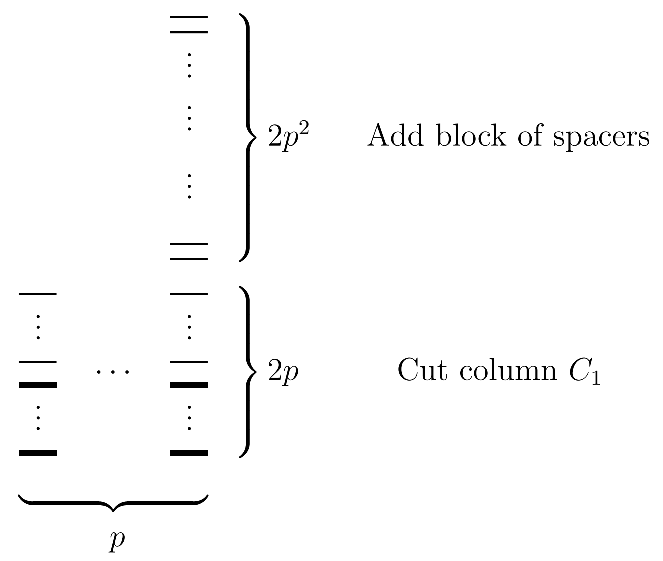
$$\mu(T^{h_n} C_n \cap C_n) = \frac{p-1}{p} \mu(C_n)$$

for all n.

If $h_n < k < h_{n+1}$ then $\mu(T^k C_n \cap C_n) < \frac{p-1}{p} \mu(C_n)$ because more and more of the pieces of the blocks move into the spacers. From this one duplicates the analysis for a set A consisting of disjoint unions of levels of C_n and then use an approximation argument for an arbitrary set of finite measure.

This completes the proof of Theorem 3.

Figure 5: Stage 2 for construction of V_p



4 A Variation to get $\alpha = 1/3$ and $\alpha = 2/3$

In this section we construct two transformations. They will both have the same exhaustive weakly wandering sequence. The first $T_{\{0,0,3\}}$ will be of α-type for $\alpha = 2/3$. The second $T_{\{0,1,2\}}$ is a modification of the first and will be of α-type for $\alpha = 1/3$.

The analysis of modifying the first to the second will use factorizations of cyclic groups in order to show that the exhaustive weakly wandering sequence of the first is also the exhaustive weakly wandering sequence for the second.

As in the Hajian-Kakutani transformation, the exhaustive weakly wandering sequence $\oplus\mathbb{B}_i$ arises automatically as a direct sum. As pointed out in Section 2.3, there is also a direct sum of integers associated with the set W. The relation between the two is that for each n, $\left(\oplus_1^n \mathbb{A}_i\right) \oplus \left(\oplus_1^n \mathbb{B}_i\right)$ will be a factorization of the finite cyclic group $\{0, 1, \ldots, h_n - 1\}$ which corresponds to the height of column C_n.

Technically these results are subsumed in Theorem 1. However it will be far simpler to analyze the issues in these two examples before getting to the larger theorem.

4.1 Transformation $T_{\{0,0,3\}}$ with $\alpha = 2/3$

This example is a straightforward variation of everything that has gone before.

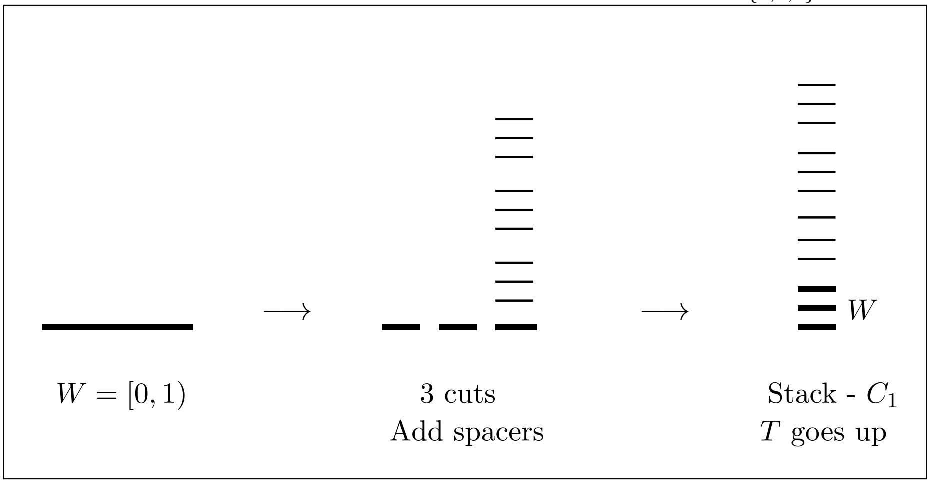
4.1.1 Construction

Step 0: Start with $W = [0, 1)$ and consider this a column C_0 of height $h_0 = 1$ and width $w_0 = 1$.

Step 1: Cut C_0 into three equal width subintervals. As before we put all the spacers above the rightmost subcolumn. In this case, add 3 blocks of $3 \cdot h_0$ spacer intervals for a total of 9 spacers. As usual, stack each subcolumn above the one to its immediate left. See Figure 6.

This results in column C_1 of height $h_1 = 4 \cdot 3$.

Figure 6: Stage 1 for construction of $T_{\{0,0,3\}}$



Remark 3 *We are writing the height as $4 \cdot 3$ because we are viewing it as 4 "blocks" of 3 intervals (see Figure 6). Observe that this corresponds to the fact that C_1 is the disjoint union of 4 images of $C_0 = W$, that is*

$$C_1 = \dot{\cup}_{k=0}^{3} T^{3k} W \quad (disjoint).$$

Hence define $\mathbb{B}_1 = \{0, 3, 2 \cdot 3, 3 \cdot 3\} = \{0, 3, 6, 9\}$. Observe that W consists of levels $0, 1, 2$ of column C_1. Using this we define $\mathbb{A}_1 = \{0, 1, 2\}$.

This gives the factorization

$$\begin{aligned}\{0, 1, \ldots, h_1 - 1\} &= \{0, 1, 2\} \oplus \{0, 3, 6, 9\} \\ &= \mathbb{A}_1 \oplus \mathbb{B}_1\end{aligned}$$

Notice at this point it is not necessary to do modular addition.

Step 2: Cut column C_1 into 3 equal width subcolumns. Above the rightmost subcolumn place 3 blocks of $3 \cdot h_1 = 3 \cdot (3 \cdot 4) = 36$ spacer intervals for a total of $4 \cdot (3 \cdot (3 \cdot 4)) = (3 \cdot 4)^2$ spacers. Stack as usual.

This results in column C_2 of height $h_2 = (3 \cdot 4)^2 = 144$.

Observe that C_2 is a disjoint union of four images of C_1.

$$C_2 = C_1 \dot{\cup} T^{36} C_1 \dot{\cup} T^{72} C_1 \dot{\cup} T^{108} C_1$$

Hence define $\mathbb{B}_2 = \{0, 36, 72, 108\} = \{0, 3h_1, 2 \cdot 3h_1, 3 \cdot 3h_1\}$.

Define $\mathbb{A}_2 = \{0, 12, 24\} = \{0, h_1, 2h_1\}$. The levels of C_2 which correspond to W are (see Figure 7)

$$\begin{aligned} \{0, 1, 2,\ 12, 13, 14,\ 24, 25, 26\} &= \{0, 12, 24\} \oplus \{0, 1, 2\} \\ &= \mathbb{A}_1 \oplus \mathbb{A}_2 \end{aligned}$$

That is, Level 0 of column C_1 is divided into Level 0, Level 12 and Level 24 of column C_2.

This gives the factorization of the cyclic group

$$\{0, 1, \cdot, h_2 - 1\} = \big(\mathbb{A}_1 \oplus \mathbb{A}_2\big) \oplus \big(\mathbb{B}_1 \oplus \mathbb{B}_2\big)$$

Step n: Cut column C_{n-1} into three equal width subcolumns. Above the rightmost subcolumn add 3 blocks of $3 \cdot h_{n-1}$ spaces.

This results in column C_n of height $h_n = (4 \cdot 3)^n$

Define

$$\begin{aligned} \mathbb{B}_n &= \{0,\ 3h_{n-1},\ 2 \cdot 3h_{n-1},\ 3 \cdot 3h_{n-1}\} \\ &= \{0,\ 3(4 \cdot 3)^{n-1},\ 2 \cdot 3(4 \cdot 3)^{n-1},\ 3 \cdot 3(4 \cdot 3)^{n-1}\} \end{aligned} \tag{3}$$

Define

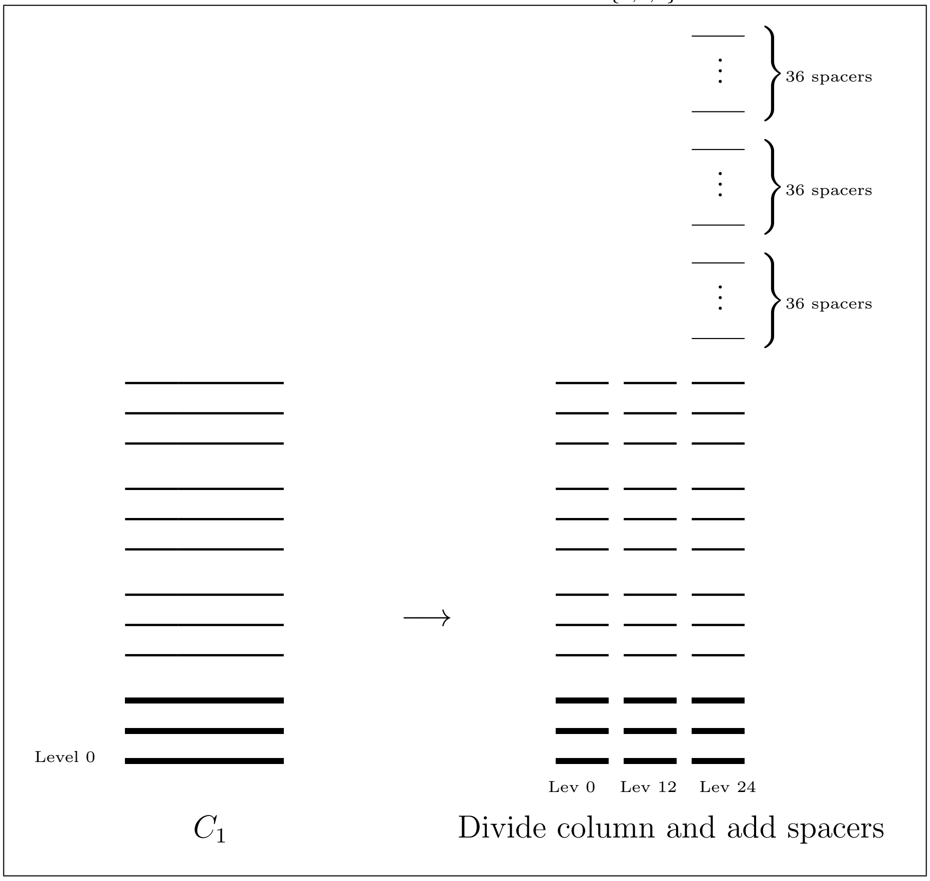
$$\begin{aligned} \mathbb{A}_n &= \{0,\ 3h_{n-1},\ 2 \cdot 3h_{n-1},\ 3 \cdot 3h_{n-1}\} \\ &= \{0,\ (4 \cdot 3)^{n-1},\ 2 \cdot (4 \cdot 3)^{n-1}\} \end{aligned} \tag{4}$$

This gives the factorization

$$\{0, 1, \ldots, h_n - 1\} = \big(\oplus_1^n \mathbb{A}_i\big) \oplus \big(\oplus_1^n \mathbb{B}_i\big)$$

and in the limit we have the exhaustive weakly wandering sequence for $T_{\{0,0,3\}}$.

Figure 7: Step 2 $T_{\{0,0,3\}}$



Theorem 4 *The transformation $T_{\{0,0,3\}}$ is of α-type with $\alpha = \frac{2}{3}$. The set W is exhaustive weakly wandering under the sequence*

$$\oplus_{k=1}^{\infty}\{0,\ 4^{k-1}3^k,\ 2 \cdot 4^{k-1}3^k,\ 3 \cdot 4^{k-1}3^k\}$$

The proof that $\alpha = \frac{2}{3}$ is similar to that of V_3 in Section 3. The difference is that there are more blocks of spacers in C_{n+1} for column C_n to move into. The transformations V_3 and $T_{\{0,0,3\}}$ are not isomorphic but we will not pursue that here.

4.2 Transformation $T_{\{0,1,2\}}$ with $\alpha = 1/3$

We now modify the previous construction. The goal is to change the α-type but keep the same exhaustive weakly wandering sequence. Significantly, the only change in the construction is the distribution of the blocks of spacers. at Step n, the column is cut in three and the same number of spacers is added as in the construction of $T_{\{0,0,3\}}$.

Thus we will have

Theorem 5 *The transformation* $T_{\{0,1,2\}}$ *is of* α*-type with* $\alpha = \frac{1}{3}$*. The set* $W = [0,1)$ *is exhaustive weakly wandering under the sequence*

$$\mathbb{B} = \oplus_{k=1}^{\infty} \mathbb{B}_k$$

defined in Equation 3.

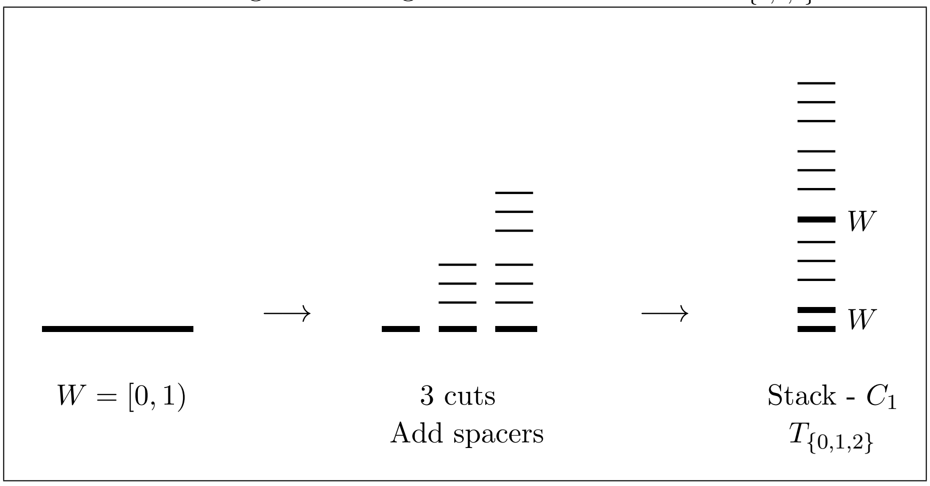
4.2.1 Construction

Start with $W = C_0 = [0,1)$ and $h_0 = 1$.

Step 1: Cut C_0 into three equal width subcolumns. As in the previous example we will add 3 blocks of $3h_0$ spacer intervals for a total of 9 spacers. However, in this construction, we place one block above the middle subcolumn and two blocks above the right subcolumn. Stack right above left subcolumns. See Figure 8.

Column C_1 has height $h_1 = 4 \cdot 3$ which is the same as in the previous example.

Figure 8: Stage 1 for construction of $T_{\{0,1,2\}}$



Step 2: Cut C_1 into three equal width subcolumns. Again we add 3 blocks of $3h_1$ spacer intervals. There are 0 blocks placed over the first subcolumn, 1 block placed over the middle subcolumn and 2 blocks placed over the third subcolum. Stack right over left subcolumns.

Column C_2 has height $h_2 = (4 \cdot 3)^2 = 144$ as before.

Step n: Cut C_{n-1} into three equal width subcolumns. Add 3 blocks of $3h_{n-1}$ spacer intervals in the same manner. That is 1 block over the middle subcolumn and 2 blocks placed over the third subcolum. Stack right over left subcolumns.

Column C_n has height $h_n = (4 \cdot 3)^n$.

4.2.2 Observations for $\alpha = 1/3$

As in the previous example we can identify the large intersections of the C_n

At the first step we see $\mu(TW \cap W) = 1/3$, $\mu(T^3W \cap W) = 1/3$ and $\mu(T^4W \cap W) = 1/3$.

Continuing, $\mu(T^{h_n}C_n \cap C_n) = \frac{1}{3}\mu(C_n)$, $\mu(T^{3h_n}C_n \cap C_n) = \frac{1}{3}\mu(C_n)$, and $\mu(T^{4h_n}C_n \cap C_n) = \frac{1}{3}\mu(C_n)$. And for $h_n < k < h_{n+1}$ $\mu(T^kC_n \cap C_n) \leq \frac{1}{3}\mu(C_{n-1})$ because at least two-thirds of of the column C_n has moved into spacer blocks of Column C_{n+1}.

To complete the proof that $\alpha = \frac{1}{3}$ follows the same approximation arguments as before. That is, first prove the result for sets A consisting of disjoint levels of C_n and then approximate arbitrary sets.

4.2.3 Exhaustive Weakly Wandering Sequence

We now need to show that $\mathbb{B}$ (Formula 3) is exhaustive weakly wandering for $T_{\{0,1,2\}}$.

The issue is that C_1 is not a disjoint union of images of W and C_n is not a disjoint union of images of C_{n-1}.

Begin by examining column C_1 at Step 1, and compare it to the union of the images of W (see Figure 9).

$$C_1 \text{ vs } W \cup T^3W \cup T^6W \cup T^9W$$

For $T_{\{0,1,2\}}$ W consists of 3 levels of C_1. These are levels $\{0, 1, 5\}$. Applying T^3, T^6 and T^9 we get "levels" $\{3, 4, 8\}$, $\{6, 7, 11\}$ and $\{9, 10, 14?\}$.

Two problems immediately become apparent. The first is that there is a level of C_1 which is not contained in the union (this is *Level 2* - the bottom being *Level 0*). The second is that we don't yet know where T^9W (*Level 5*) goes (*i.e.* where is Level 14?) or if it is disjoint from the other images of W. For this we analyze the situation in Step 2.

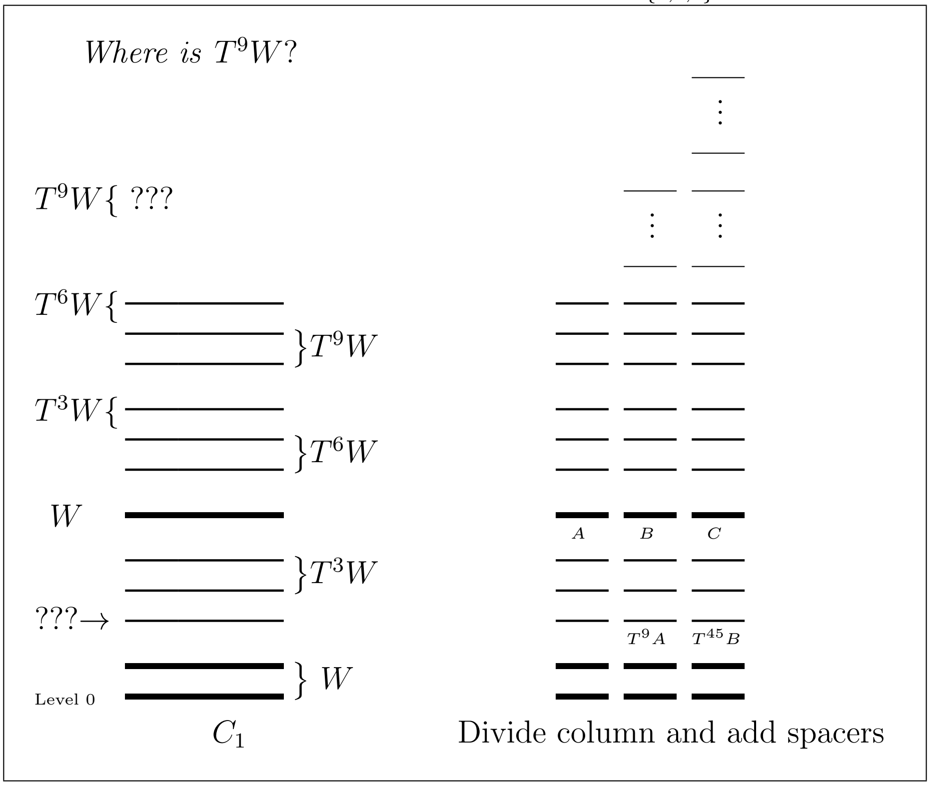
At Step 2, C_1 is divided in three subintervals, and spacers are placed appropriately at the top of the subcolumns. (See right side of Figure 9.) It is now easy to see where the piece of W marked "A" goes. Specifically T^9A maps to a part of Level 2 that wasn't covered.

Now look at the piece of W marked "B". This moves up by T^9 into the spacers. There are a total of 36 spacers above the middle column. Hence $T^{45}B$ again maps to another piece of Level 2. The point is that $45 = 9+36 \in \mathbb{B}_1 \oplus \mathbb{B}_2$. So $T^{45}W$ is one of the images of W that are in $\mathbb{B}$.

Hence at Step 2, two-thirds of Level 2 has been covered. This process continues. That is at Step 3 another two-ninths will be covered and in the limit Level 2 will will be covered completely.

To formalize this use the language of cyclic groups and factorizations.

Figure 9: E.W.W. for $T_{\{0,1,2\}}$



5 Factorizations of Finite Cyclic Groups

We begin with the sets $\mathbb{A}_1$ and $\mathbb{B}_1$ for $T_{\{0,0,3\}}$. We have the factorization.

$$\{0, 1, 2\} \oplus \{0, 3, 6, 9\} = \{0, 1, \ldots 11\}$$

Observe that $\mathbb{B}_1 = \{0, 3, 6, 9\}$ is a subgroup of $\{0, 1, \ldots 11\}$ Modulo 12. In the construction of $T_{\{0,1,2\}}$ placing a block above the middle column corresponds to adding 3 to the third term in $\mathbb{A}_1$. This changes $\mathbb{A}_1$ to $\mathbb{A}_1^* = \{0, 1, 2 + 3\} = \{0, 1, 5\}$ (see Figure 8).

This gives the factorization

$$\{0, 1, 5\} \oplus \{0, 3, 6, 9\} = \{0, 1, \ldots 11\}\ Mod\ 12$$

This works because $\{0, 3, 6, 9\}$ is a subgroup of $\{0, 1, \ldots 11\}$ Mod 12 and adding 3 leaves the coset $2 + \{0, 3, 6, 9\}$ invariant.

This works for all n. That is, adding a block of size $3h_{n-1}$ above the middle column at Step n, changes $\mathbb{A}_n = \{0, h_{n-1}, 2h_{n-1}\}$ to $\mathbb{A}_n^* = \{0, h_{n-1}, 2h_{n-1} + 3h_{n-1}\} = \{0, h_{n-1}, 5h_{n-1}\}$. Using the fact that $\mathbb{B}_n = \{0, 3h_{n-1}, 2 \cdot 3h_{n-1}, 3 \cdot 3h_{n-1}\}$ is a subgroup Modulo h_n we have that

$$\mathbb{A}_n \oplus \mathbb{B}_n = \mathbb{A}_n^* \oplus \mathbb{B}_n\ Mod\ h_n$$

and this gives the factorization of the cyclic groups

$$\{0, 1, \ldots, h_n - 1\} = \left(\oplus_1^n \mathbb{A}_k^* \right) \oplus \left(\oplus_1^n \mathbb{B}_k \right) \ Mod \ h_n$$

From this we can conclude the exhaustiveness $X = \cup_{b \in \oplus_1^\infty \mathbb{B}_i} T^b W$ and the weakly wandering $T^b W \cap T^{b'} W = \emptyset$ for all $b, b' \in \oplus_1^\infty \mathbb{B}_i$.

In particular, suppose Level j of C_n is not contained in $\cup T^b W,\ b \in \oplus_1^n \mathbb{B}$. Then in Column C_{n+1} the Level corresponds to Level j, Level $j + h_n$ and Level $j + h_n + 3h_n$. But $h_n, h_n + 3h_n \in \mathbb{B}_{n+1}$. Hence exactly two-thirds of Level j is covered in the next Step and by induction all of the Level is covered in the limit.

To see that $T^b W \cap T^{b'} W = \emptyset$ for all $b, b' \in \oplus_1^\infty \mathbb{B}_i$, choose n so that $\max(b, b') < \frac{1}{2} h_n$. Recall that $\oplus_1^n \mathbb{A}_i^*$ locates which levels in C_n are part of W. Observe that $\max(a) < \frac{1}{2} h_n$. Hence $T^b W$ and $T^{b'} W$ are contained in C_n - by which we mean the images do not move out of C_n and need to be modded. Hence the images are disjoint.

6 Family of Transformations

We now have all the tools and concepts necessary to construct a sequence satisfying Theorem 1 and the transformations T_α

6.1 Rank One Construction

In this section, we set the notation, review the general rank one construction as used in this paper and then set some restrictions for the actual transformations.

A major difference from the examples that have gone before is that the number of cuts at each step will increase. This is necessary to get different α-type. For example three cuts only enabled $\alpha = 2/3$ and $\alpha = 1/3$.

As usual at the n^{th} stage, column C_{n-1} is cut into c_n subcolumns. Above the i^{th} subcolumn, $s_i^{(n)}$ spacers are placed. The subcolumns are stacked, forming the new column C_n

The construction is therefore completely defined by the infinite sequence of cuts $c_n \geq 2$, and the infinite sequence of sets of spacers $\mathcal{S}_n = \{s_1^{(n)}, \cdots, s_{c_n}^{(n)}\}$ placed above the c_n subcolumns.

Using the above notation, the total number of spacers added at stage n is $|\mathcal{S}_n| = \sum_{i=1}^{c_n} s_i^{(n)}$. The height of column C_n is then $h_n = c_n \cdot h_{n-1} + |\mathcal{S}_n|$.

6.2 Controlling for an exhaustive weakly wandering sequence $\mathbb{B}$

In order to guarantee that all the transformations to be constructed have the same exhaustive weakly wandering sequence, we will put some restrictions on the general rank one construction.

Restrictions.

1. Preset the cuts $c_k = 2^k$.

2. Define the block size, $b_k = c_k \cdot h_{k-1}$, and add the spacers in multiples of the block size (see Section 6.3.1 for an illustration of adding the spacers).

3. Preset the total spacers added at stage k. Specifically, $s_k = 2^{2^k-1} - 1$ is the number of blocks of spacers of size b_k added at stage k.

From the above restrictions, the height of column C_k is also preset to $h_k = \prod_{j=0}^{k} 2^{2^j+j-1}$. Hence the total number of spacers added at Step k is

$$\begin{aligned} |\mathcal{S}_k| &= \left(2^{2^k-1}-1\right)\cdot c_k \cdot h_{k-1} = s_k \cdot b_k \\ &= \left(2^{2^k-1}-1\right)\cdot 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j+j-1} \end{aligned}$$

Condition (1) is necessary to have different α-type. Condition (2), which is related to the factorization, will insure that the same exhaustive weakly wandering sequence works for all the transformations. Condition (3) allows for enough blocks of spacers to control the α-type.

6.3 $\mathbb{B}$ and its structure

The above restrictions completely define a sequence of integers which will be seen to satisfy Theorem 1. We will then proceed to describe the construction of the different T_α.

The Sequence $\mathbb{B}$ as determined by the previous restrictions is

$$\begin{aligned} \mathbb{B} &= \oplus_{n=1}^{\infty}\mathbb{B}_n \\ &= \{0, 2, 16, 18, 32, 34\cdots\} \end{aligned} \tag{5}$$

where

$$\begin{aligned} \mathbb{B}_1 &= \{0, 2\} \\ \mathbb{B}_2 &= \{0, 16, 32, 48, 64, 80, 96, 112\} \\ \mathbb{B}_3 &= \{0, 1024, 2048, 3072, 4096, 5120\cdots 130048\} \\ &\vdots \\ \mathbb{B}_k &= \{0, b_k,\ 2\cdot b_k,\ 3\cdot b_k, \cdots,\ (2^{2^k-1}-1)\cdot b_k\} \\ &\vdots \end{aligned} \tag{6}$$

where the block sizes are

$$b_k = 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j+j-1}$$

6.3.1 Distributing the spacers

Since the number of cuts and the total number of spacers are preset, the variations for the transformations come from the alternate distributions of the blocks of spacers over the subcolumns.

Rule: The basic rule we want to follow is that the total spacers above subcolumn k should be large enough to absorb all the previous subcolumns and spacers.

Here is an illustration. Suppose we were going to cut a column of height $h = 1$ into $c = 4$ pieces. So the block size would be $b = 4 \cdot 1 = 4$.

We start by putting 0 blocks above the first subcolumn, 1 block above the second subcolumn, 2 blocks above the third subcolumn and 4 blocks above the fourth subcolumn. This is a total of 7 blocks. See left-side of Figure 10.

Notationally we represent this as $\mathcal{S} = \{0, 1, 2, 4\} \cdot 4 = \{0, 4, 8, 16\}$.

These 7 blocks can be rearranged, and still satisfy the rule by distributing the spacer blocks as $\mathcal{S} = \{0, 0, 2, 5\} \cdot 4$ and $\mathcal{S} = \{0, 0, 0, 7\} \cdot 4$, See middle and right of Figure 10.

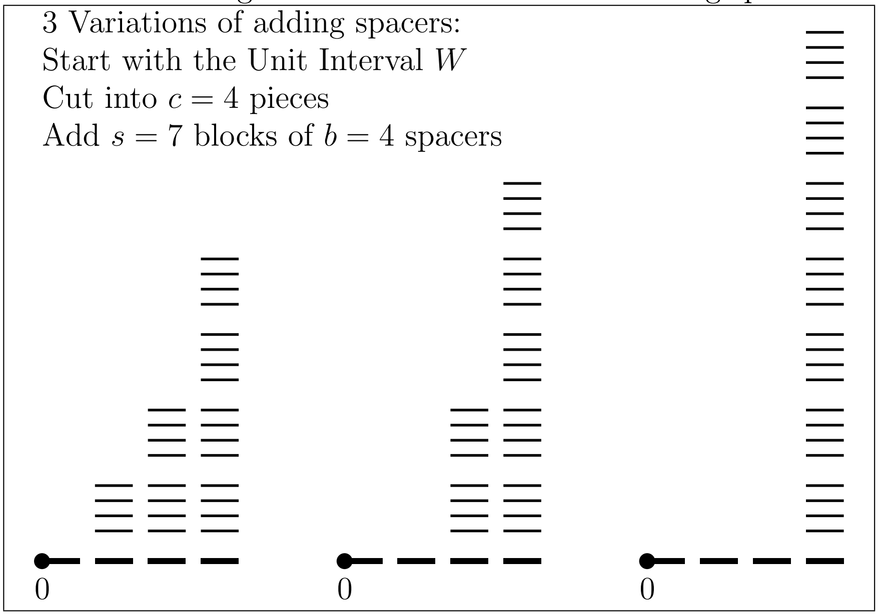
Relating this to factorizations of cyclic groups we have

$$\begin{aligned}\{0, 1, \ldots, 31\} &= \{0, 1, 5, 15\} \oplus \{0, 4, 8, 12, 16, 20, 24, 28\} \; Mod \; 32 \\ &= \{0, 1, 2, 11\} \oplus \{0, 4, 8, 12, 16, 20, 24, 28\} \; Mod \; 32 \\ &= \{0, 1, 2, 3\} \oplus \{0, 4, 8, 12, 16, 20, 24, 28\}\end{aligned}$$

This would be part of a step leading to a common exhaustive weakly wandering sequence.

Relating this to α-type, we see that the left image in Figure 10 could lead to $\alpha = 1/4$; the middle image could lead to $\alpha = 1/2$; and the right image to $\alpha = 3/4$.

Figure 10: Three variations of adding spacers



6.4 Allowed Spacer Block Distributions

Assume now, that any transformation constructed satisfies Restrictions 1, 2 and 3.

Restriction 2, indicates that the spacers are added in certain block sizes. Restriction 3, indicates that the total number of blocks of spacers is fixed at each step.

Now we add a further condition indicating how the blocks of spacers are distributed over the subcolumns.

At Step n we divide column C_{n-1} into $c_n = 2^n$ pieces. The block size is set at $b_n = c_n h_{n-1}$. The total number of blocks of spacers to be added is set at $2^{2^n-1} - 1$. However, we only allow them to be distributed as follows.

Allowed Distributions

$$\begin{aligned}
\mathcal{S} &= \{0, 1, 2, 4, \cdots, 2^{2^n-2}\} \cdot b_n \\
&= \{0, 0, 2, 4, \cdots, 2^{2^n-2+} + 1\} \cdot b_n \\
&= \{0, 0, 0, 4, \cdots, 2^{2^n-2} + 1 + 2\} \cdot b_n \\
&\vdots \\
&= \{0, 0, \cdots, 0, 2^{2^m-1} - 1\} \cdot b_n
\end{aligned} \tag{7}$$

6.5 A Class of Transformations

Let $\mathscr{T}$ denote the class of all transformations constructed as rank one transformations and satisfying Restrictions 1, 2 and 3. Additionally assume that at each step in the construction of an indivdual T that the the spacers are distributed by one of the allowed distributions.

The set $W = [0, 1)$ denotes the common base over which all the transformations are built.

Theorem 6 *Every transformation $T \in \mathscr{T}$ is infinite measure preserving and ergodic. The set W is exhaustive weakly wandering for the sequence $\mathbb{B}$ given in Equation 5.*

Let T^* denote the transformation which uses Distribution 7 at every step. As with the Hajian-Kakutani transformation the Columns C_n are filled up exactly at each step. Hence it is immediate that $\mathbb{B}$ is an exhaustive weakly wandering sequence fot T^*. That is we have factorizations (recall that $\oplus_1^n \mathbb{A}_i$ corresponds to levels of W in Column C_n)

$$\{0, 1, \ldots, h_n - 1\} = \oplus_1^n \mathbb{A}_i \bigoplus \oplus_1^n \mathbb{B}_i$$

and modding is not necessary.

For an arbitrary T, each $\mathbb{A}_i$ is modified to a $\mathbb{A}_i^*$ but we still have factorizations

$$\{0, 1, \ldots, h_n - 1\} = \oplus_1^n \mathbb{A}_i^* \bigoplus \oplus_1^n \mathbb{B}_i \; Mod \; h_n$$

The exhaustiveness and weakly wandering properties follow as before.

6.6 Controlling for α-type

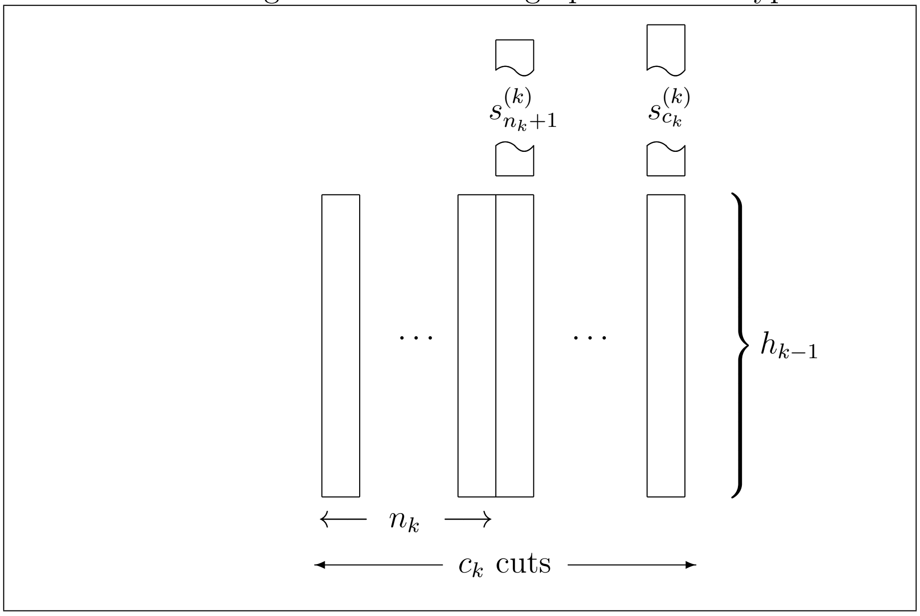
Now we show how to modify the construction in order to get a transformation of α-type for any $0 \leq \alpha \leq 1$.

The following holds for more than just the transformations in $\mathscr{T}$.

Theorem 7 *If in the general construction the following holds (See Figure 11)*

1. $c_k > 1$
2. $n_k < c_k$ *such that* $\lim_{k \to \infty} \frac{n_k}{c_k} = \alpha$
3. $s_i^{(k)} = 0$ *for* $1 \leq i \leq n_k$
4. $s_{j+1}^{(k)} \geq \sum_{i=1}^{j} s_i^{(k)} + j \cdot h_{k-1}$, $j \geq n_k$

Figure 11: Controlling Spacers for α-type



then T is of α-type

The conditions on the spacers guarantee that $\mu(T^{h_k}C_k \cap C_k) = \frac{n_k}{c_k}$ for all k. For $n_k < j < n_{k+1}$, $\mu(T^jC_k \cap C_k) \leq \max(\frac{n_k}{c_k}, \frac{n_{k+1}}{c_{k+1}})$. The same holds for sets A which are disjoint unions of C_k and then for arbitrary sets by an approximation argument.

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Please, cite to this paper as published in
Armen. J. Math., V. **7**, N. 2(2015), pp. 121–145